SQUARE-FREE GRÖBNER DEGENERATIONS

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Abstract. Let I be a homogeneous ideal of $S = K[x_1, \dots, x_n]$ and let $<$ be a term order. We prove that if the initial ideal $J = \mathrm{in}_<(I)$ is radical then the extremal Betti numbers of S/I and of S/J coincide. In particular, $\mathrm{depth}(S/I) = \mathrm{depth}(S/J)$ and $\mathrm{reg}(S/I) = \mathrm{reg}(S/J)$.

1. Introduction

Let S be the polynomial ring $K[x_1, \dots, x_n]$ over a field K equipped with its standard graded structure. Let M be a graded S-module. We denote by $\beta_{ij}(M)$ the (i, j)-th Betti number of M, and by $h^{ij}(M)$ the dimension of the degree j component of its i-th local cohomology module $H^i_{\mathfrak{m}}(M)$ supported on the maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. Furthermore $\mathrm{reg}(M)$ denotes the Castelnuovo-Mumford regularity of M and $\mathrm{depth}(M)$ its depth. Finally recall that a non-zero Betti number $\beta_{i,i+j}(M)$ is called extremal if $\beta_{h,h+k}(M) = 0$ for every $h \geq i$, $k \geq j$ with $(h, k) \neq (i, j)$.

Let I be a homogeneous ideal of S. For every term order $<$ on S we may associate to I a monomial ideal $\mathrm{in}_<(I)$ via the computation of a Gröbner basis. For notational simplicity and when there is no danger of confusion we suppress the dependence of the monomial ideal on $<$ and denote it by $\mathrm{in}(I)$. The ideal $\mathrm{in}(I)$ is called the initial ideal of I with respect to $<$ and $S/\mathrm{in}(I)$ can be realized as the special fiber of a flat family whose generic fiber is S/I. This process is called a Gröbner degeneration. If the ideal I is in generic coordinates (see [Ei94, Chapter 15.9] for the precise definition) the outcome of the Gröbner degeneration is called generic initial ideal of I with respect to $<$ and it is denoted by $\mathrm{gin}(I)$.

It is well known that the homological and cohomological invariants behave well under Gröbner degenerations. Indeed one has:

$$\beta_{ij}(S/I) \leq \beta_{ij}(S/\mathrm{in}(I)) \quad \text{and} \quad h^{ij}(S/I) \leq h^{ij}(S/\mathrm{in}(I)) \text{ for all } i, j$$

and, in particular,

$$\mathrm{reg}(S/I) \leq \mathrm{reg}(S/\mathrm{in}(I)) \quad \text{and} \quad \mathrm{depth}(S/I) \geq \mathrm{depth}(S/\mathrm{in}(I)).$$

Simple examples show that the inequalities are in general strict. On the other hand, Bayer and Stillman [BS87] proved that equality holds is a special and important case:

Theorem 1.1 (Bayer-Stillman). *Let $<$ be the degree reverse lexicographic order. Then for every homogeneous ideal* I *of* S *one has:*

$$\mathrm{reg}(S/I) = \mathrm{reg}(S/\mathrm{gin}(I)) \quad \textit{and} \quad \mathrm{depth}(S/I) = \mathrm{depth}(S/\mathrm{gin}(I)).$$

Furthermore Bayer, Charalambus and Popescu [BCP99] generalized Theorem 1.1 proving that S/I and $S/\mathrm{gin}(I)$ have the same extremal Betti numbers (positions and values).

Algebras with straightening laws (ASL for short) were introduced by De Concini, Eisenbud and Procesi in [DEP82, Ei80] and, more or less simultaneously and in a slightly different way, by Baclawski [Ba81]. Actually they appear under the name of "ordinal Hodge

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algebras" in [DEP82] while the terminology "ASL" is used by Eisenbud in his survey [Ei80] and later on by Bruns and Vetter in [BV88]. This notion arose as an axiomatization of the underlying combinatorial structure observed by many authors in classical algebras appearing in invariant theory, commutative algebra and algebraic geometry. For example, coordinate rings of flag varieties, their Schubert subvarieties and various kinds of rings defined by determinantal equations are ASL. Roughly speaking an ASL is an algebra A whose generators and relations are governed by a finite poset H in a special way. Any ASL A has a discrete counterpart A_D defined by square-free monomials of degree 2. Indeed it turns out that A_D can be realized as Gröbner degeneration of A and ASL's can be characterized via Gröbner degenerations [Co07, Lemma 5.5].

In [DEP82] it is proved, in the general setting of Hodge algebras, that A is Gorenstein or Cohen-Macaulay if A_D is Gorenstein or Cohen-Macaulay and the authors mention that: "The converse is false: It is easy for A to be Gorenstein without A_D being so, and presumably the same could happen for Cohen-Macaulayness". Within the general framework of Hodge algebras discussed in [DEP82], a Cohen-Macaulay algebra with a non-Cohen-Macaulay associated discrete algebra has been provided by Hibi in [Hi86, Example pg. 285]. In contrast, a Cohen-Macaulay ASL with non-Cohen-Macaulay discrete counterpart could not be found. In this paper we show that such an example does not exist because $\operatorname{depth} A = \operatorname{depth} A_D$ for any ASL A (Corollary 3.9).

The question whether some homological invariants of an ASL depend only on its discrete counterpart leads quickly to consider possible generalizations. Herzog was guided by these considerations to conjecture the following:

Conjecture 1.2 (Herzog). *Let* I *be a homogeneous ideal of a standard graded polynomial ring* S*, and* $<$ *a term order on* S*. If* $\operatorname{in}(I)$ *is square-free, then the extremal Betti numbers of* S/I *and those of* $S/\operatorname{in}(I)$ *coincide (positions and values).*

In other words, Herzog's intuition was that a square-free initial ideal behaves, with respect to the homological invariants, as the reverse lexicographic generic initial ideal. However the stronger statement asserting that $\operatorname{gin}(I) = \operatorname{gin}(\operatorname{in}(I))$ if $\operatorname{in}(I)$ is square-free turned out to be false, see Example 3.1. The above conjecture, in various forms, has been discussed in several occasions by Herzog and his collaborators. It appeared in print only recently in [CDG18b, Conjecture 1.7] and in the introduction of [HR18]. In this paper we solve positively Herzog's conjecture. Indeed we establish a stronger result:

Theorem 1.3. *Let* I *be a homogeneous ideal of a standard graded polynomial ring* S *such that* $\operatorname{in}(I)$ *is a square-free monomial ideal for some term order. Then*

$$h^{ij}(S/I) = h^{ij}(S/\operatorname{in}(I)) \quad \forall\, i, j.$$

Since the extremal Betti numbers of S/I can be characterized in terms of the values of $h^{ij}(S/I)$, Herzog's conjecture follows from Theorem 1.3. We note that Theorem 1.3 holds for more general gradings, see Remark 2.5.

It turns out that, in many respects, the relationship between I and a square-free initial ideal $\operatorname{in}(I)$ (when it exists) is tighter than the relation between I and its degree reverse lexicographic generic initial ideal $\operatorname{gin}(I)$. For example Herzog and Sbarra proved in [HS02] that if K has characteristic 0 then the assertion of Theorem 1.3 with $\operatorname{in}(I)$ replaced by $\operatorname{gin}(I)$ holds if and only if S/I is sequentially Cohen-Macaulay. Furthermore, a consequence of Theorem 1.3 is that, if $\operatorname{in}(I)$ is a square-free monomial ideal, then S/I satisfies Serre's condition (S_r) if and only if $S/\operatorname{in}(I)$ does (see Corollary 2.11); the latter statement is false for generic initial ideals. As a further remark note that Conjectures 1.13 and 1.14 in [CDG18b] are indeed special cases of Theorem 1.3.

The paper is structured as follows. Section 2 is devoted to prove Theorem 1.3, and to draw some immediate consequences. In Section 3 we discuss properties of ideals admitting a monomial square-free initial ideal and some consequences of Theorem 1.3 on them. We discuss as well three families of ideals with square-free initial ideals that appeared in the literature: ideals defining ASL's (3.1), Cartwright-Sturmfels ideals (3.2) and Knutson ideals (3.3). Finally, in Section 4, we discuss some open questions.

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2. The main result

The goal of the section is to prove Theorem 1.3. We use the notation of the introduction. In the examples appearing in this paper, the term orders will always refine the order of the variables $x_1 > \ldots > x_n$. Furthermore, we will write "lex" for the lexicographic term order, and "degrevlex" for the degree reverse lexicographic term order.

The main ingredient of the proof is Proposition 2.4 that can be regarded as a characteristic free version of [KK18, Theorem 1.1]. One key ingredient in the proof of Proposition 2.4 is the notion of cohomologically full singularities that was introduced and studied in the recent preprint [DDM18]. Let us recall the definition:

Definition 2.1. A Noetherian local ring $(A_1, \mathfrak{n}_1)$ is cohomologically full if, for any surjection of local rings $\phi : (A_2, \mathfrak{n}_2) \to (A_1, \mathfrak{n}_1)$ such that $\overline{\phi} : A_2/\sqrt{(0)} \to A_1/\sqrt{(0)}$ is an isomophism, the induced map on local cohomology $H^i_{\mathfrak{n}_2}(A_2) \to H^i_{\mathfrak{n}_2}(A_1)$ is surjective for all $i \in \mathbb{N}$.

The next proposition is the analog of [KK18, Proposition 5.1].

Proposition 2.2. *Let $(R, \mathfrak{t})$ be an Artinian local ring, and $(A, \mathfrak{n})$ be a Noetherian local flat R-algebra such that the special fiber $A/\mathfrak{t}A$ is cohomologically full. Let N be a finitely generated R-module, and $N = N_0 \supseteq N_1 \supseteq \ldots \supseteq N_q \supseteq N_{q+1} = 0$ a filtration of submodules such that $N_j/N_{j+1} \cong R/\mathfrak{t}$ for all $j = 0 \ldots, q$. Then, for all $i \in \mathbb{N}$ and $j = 0 \ldots, q$, the following complex of A-modules is exact:*

$$0 \to H^i_{\mathfrak{n}}(N_{j+1} \otimes_R A) \to H^i_{\mathfrak{n}}(N_j \otimes_R A) \to H^i_{\mathfrak{n}}((N_j/N_{j+1}) \otimes_R A) \to 0$$

Proof. Notice that the surjection $A \xrightarrow{\phi} A/\mathfrak{t}A$ yields an isomorphism between $A/\sqrt{(0)}$ and $(A/\mathfrak{t}A)/\sqrt{(0)}$. Since the tensor product is right-exact, we have a surjection of A-modules

$$N_j \otimes_R A \xrightarrow{\beta} (N_j/N_{j+1}) \otimes_R A \cong A/\mathfrak{t}A.$$

Denoting by β' the composition of β with the isomorphism $(N_j/N_{j+1}) \otimes_R A \cong A/\mathfrak{t}A$, choose $x \in N_j \otimes_R A$ such that $\beta'(x) = 1$, and set $\alpha : A \to N_j \otimes_R A$ the multiplication by x. Then $\beta' \circ \alpha : A \to A/\mathfrak{t}A$ equals ϕ. Therefore, being $A/\mathfrak{t}A$ cohomologically full, the induced map of A-modules

$$H^k_{\mathfrak{n}}(\beta' \circ \alpha) = H^k_{\mathfrak{n}}(\beta') \circ H^k_{\mathfrak{n}}(\alpha) : H^k_{\mathfrak{n}}(A) \to H^k_{\mathfrak{n}}(A/\mathfrak{t}A)$$

is surjective for all $k \in \mathbb{N}$, so that $H^k_{\mathfrak{n}}(\beta) : H^k_{\mathfrak{n}}(N_j \otimes_R A) \to H^k_{\mathfrak{n}}((N_j/N_{j+1}) \otimes_R A)$ is surjective as well. Since A is a flat R-algebra, for each $j = 0, \ldots, q$ we have a short exact sequence of A-modules

$$0 \to N_{j+1} \otimes_R A \to N_j \otimes_R A \xrightarrow{\beta} (N_j/N_{j+1}) \otimes_R A \to 0.$$

Passing to the long exact sequence on local cohomology

$$\ldots \to H^{i-1}_{\mathfrak{n}}(N_j \otimes_R A) \xrightarrow{H^{i-1}_{\mathfrak{n}}(\beta)} H^{i-1}_{\mathfrak{n}}((N_j/N_{j+1}) \otimes_R A) \to$$

$$H^i_{\mathfrak{n}}(N_{j+1} \otimes_R A) \to H^i_{\mathfrak{n}}(N_j \otimes_R A) \xrightarrow{H^i_{\mathfrak{n}}(\beta)} H^i_{\mathfrak{n}}((N_j/N_{j+1}) \otimes_R A) \to \ldots,$$

being each $H^k_{\mathfrak{n}}(\beta)$ surjective, we get the thesis. □

The following is, essentially, already contained in [Ly83].

Proposition 2.3. *Let* $J \subseteq S = K[x_1, \ldots, x_n]$ *be a square-free monomial ideal. Then* $(S/J)_{\mathfrak{m}}$ *is cohomologically full.*

Proof. If K has positive characteristic, then $(S/J)_{\mathfrak{m}}$ is F-pure by [HR76, Proposition 5.38]. Hence, in characteristic zero $(S/J)_{\mathfrak{m}}$ is of F-pure type (and therefore of F-injective type). Then $(S/J)_{\mathfrak{m}}$ is Du Bois by [Sc09, Theorem 6.1]. So in each case, we conclude by [MSS17, Lemma 3.3, Remark 3.4]. □

In the proposition below, $(R, \mathfrak{t})$ is a homomorphic image of a Gorenstein local ring, $P = R[x_1, \ldots, x_n]$ is a standard graded polynomial ring over R and A is a graded quotient of P. Denote by $\mathfrak{n}$ the unique homogeneous maximal ideal $\mathfrak{t}P + (x_1, \ldots, x_n)$ of P.

Proposition 2.4. *With the notation above, assume furthermore that* A *is a flat* R*-algebra. If* $(A/\mathfrak{t}A)_{\mathfrak{n}}$ *is cohomologically full, then* $\mathrm{Ext}^i_P(A, P)$ *is a free* R*-module for all* $i \in \mathbb{Z}$.

Proof. Let $X = \mathrm{Spec}(A_{\mathfrak{n}})$ and $Y = \mathrm{Spec}(R)$. Then $f : X \to Y$ is a flat, essentially of finite type morphism of local schemes which is embeddable in a Gorenstein morphism. Furthermore, if $y \in Y$ is the closed point, the fiber X_y is the affine scheme $\mathrm{Spec}(A_{\mathfrak{n}}/\mathfrak{t}A_{\mathfrak{n}})$. In [KK18, Corollary 6.9] it is proved that, if in addition the schemes are essentially of finite type over $\mathbb{C}$ and X_y has Du Bois singularities, then $h^{-i}(\omega^\bullet_{X/Y})$ is flat over Y for any $i \in \mathbb{Z}$, where $\omega^\bullet_{X/Y}$ denotes the relative dualizing complex of f (see [StacksProj, Section 45.24] for basic properties of relative dualizing complexes). We note that the proof of [KK18, Corollary 6.9] holds as well by replacing the assumption that X_y has Du Bois singularities with the (weaker) assumption that X_y has cohomologically full singularities. In fact, the Du Bois assumption is used only in the proof of [KK18, Proposition 5.1], that as we noticed in Proposition 2.2 holds true even under the assumption that the special fiber is cohomologically full.

So, also under our assumptions, we have that $h^{-i}(\omega^\bullet_{X/Y})$ is flat over Y for any $i \in \mathbb{Z}$. But $\omega^\bullet_{X/Y}$ is the sheafication of $R\,\mathrm{Hom}(A_{\mathfrak{n}}, P_{\mathfrak{n}})[n]$, so $\mathrm{Ext}^{n-i}_{P_{\mathfrak{n}}}(A_{\mathfrak{n}}, P_{\mathfrak{n}}) \cong \mathrm{Ext}^{n-i}_P(A, P)_{\mathfrak{n}}$ is a flat R-module. So $\mathrm{Ext}^{n-i}_P(A, P)$ is a flat R-module, and therefore $\mathrm{Ext}^{n-i}_P(A, P)_j$ is a flat R-module for all $j \in \mathbb{Z}$. Being $\mathrm{Ext}^{n-i}_P(A, P)$ finitely generated as P-module, $\mathrm{Ext}^{n-i}_P(A, P)_j$ is actually a finitely generated flat, and so free, R-module for any $j \in \mathbb{Z}$. In conclusion, $\mathrm{Ext}^{n-i}_P(A, P)$ is a direct sum of free R-modules and hence it is a free R-module itself. □

We are ready to prove Theorem 1.3:

Proof of Theorem 1.3. Set $J = \mathrm{in}(I)$ and let $w = (w_1, \ldots, w_n) \in \mathbb{N}^n$ be a weight such that $J = \mathrm{in}_w(I)$ (see, for example, [St95, Proposition 1.11]). Let t be a new indeterminate over K, $R = K[t]_{(t)}$ and $P = R[x_1, \ldots, x_n]$. Provide P with the grading given by $\deg(x_i) = 1$ and $\deg(t) = 0$. By considering the w-homogenization $\mathrm{hom}_w(I) \subseteq P$, set $A = P/\mathrm{hom}_w(I)$. It is well known that the inclusion $R \to A$ is flat, $A/(t) \cong S/J$ and $A \otimes_R K(t) \cong (S/I) \otimes_K K(t)$. In particular, by Proposition 2.3, the special fiber $(A/(t))_{\mathfrak{n}}$, where $\mathfrak{n}$ is the unique homogeneous maximal ideal of P, is cohomologically full. By Proposition 2.4, hence, $\mathrm{Ext}^i_P(A, P)$ is a free R-module for any $i \in \mathbb{Z}$. So $\mathrm{Ext}^i_P(A, P)_j$ is a finitely generated free

R-module for any $j \in \mathbb{Z}$. Say $\mathrm{Ext}^i_P(A,P)_j \cong R^{r_{i,j}}$. Since $t \in P$ is a nonzero divisor on A we have the short exact sequence

$$0 \to A \xrightarrow{\cdot t} A \to A/(t) \to 0.$$

Applying $\mathrm{Hom}_S(-,S)$ to it, for all $i \in \mathbb{Z}$ we get:

$$0 \to \mathrm{Coker}(\alpha_{i-1,t}) \to \mathrm{Ext}^i_S(A/(t),S) \to \mathrm{Ker}(\alpha_{i,t}) \to 0,$$

where $\alpha_{k,t}$ is the multiplication by t on $\mathrm{Ext}^k_S(A,S)$. Notice that there are natural isomorphisms $\mathrm{Ext}^k_S(A,S) \cong \mathrm{Ext}^k_P(A,P)$, so $\mathrm{Ker}(\alpha_{k,t}) = \mathrm{Ker}(\mu_{k,t})$ and $\mathrm{Coker}(\alpha_{k,t}) = \mathrm{Coker}(\mu_{k,t})$ where $\mu_{k,t}$ is the multiplication by t on $\mathrm{Ext}^k_P(A,P)$. We thus have, for all $i \in \mathbb{Z}$, the short exact sequence:

$$0 \to \mathrm{Coker}(\mu_{i-1,t}) \to \mathrm{Ext}^i_S(A/(t),S) \to \mathrm{Ker}(\mu_{i,t}) \to 0.$$

For any $j \in \mathbb{Z}$, the above short exact sequence induces a short exact sequence of K-vector spaces

$$0 \to \mathrm{Coker}(\mu_{i-1,t})_j \to \mathrm{Ext}^i_S(A/(t),S)_j \to \mathrm{Ker}(\mu_{i,t})_j \to 0.$$

Since $\mathrm{Ext}^k_P(A,P)_j \cong R^{r_{k,j}}$, we get

$$\mathrm{Ker}(\mu_{i,t})_j = 0 \quad \text{and} \quad \mathrm{Coker}(\mu_{i-1,t})_j \cong K^{r_{i-1,j}}.$$

Therefore $\mathrm{Ext}^i_S(A/(t),S)_j \cong K^{r_{i-1,j}}$. On the other hand

$$(\mathrm{Ext}^i_S(S/I,S)_j) \otimes_K K(t) \cong \mathrm{Ext}^i_{S\otimes_K K(t)}((S/I)\otimes_K K(t), S\otimes_K K(t))_j \cong$$
$$\mathrm{Ext}^i_{P\otimes_R K(t)}(A\otimes_R K(t), P\otimes_R K(t))_j \cong (\mathrm{Ext}^i_P(A,P)_j)\otimes_R K(t) \cong K(t)^{r_{i-1,j}}.$$

Therefore $\mathrm{Ext}^i_S(S/J,S)_j \cong K^{r_{i-1,j}} \cong \mathrm{Ext}^i_S(S/I,S)_j$ for all $i,j \in \mathbb{Z}$. By Grothendieck local duality we have:

$$h^{ij}(S/I) = \dim_K \mathrm{Ext}^{n-i}_S(S/I,S)_{j-n} \quad \text{and} \quad h^{ij}(S/J) = \dim_K \mathrm{Ext}^{n-i}_S(S/J,S)_{j-n}$$

for all $i,j \in \mathbb{Z}$, so we conclude. □

Remark 2.5. *The proof of Theorem 1.3 works also for more general gradings. Assume* $S = K[x_1,\dots,x_n]$ *is equipped with a* $\mathbb{Z}^m$*-graded structure such that* $\deg(x_i) \in \mathbb{N}^m \setminus \{0\}$*. Let* $I \subseteq S$ *be a* $\mathbb{Z}^m$*-graded ideal such that* $\mathrm{in}(I)$ *is square-free, then*

$$\dim_K H^i_{\mathfrak{m}}(S/I)_\nu = \dim_K H^i_{\mathfrak{m}}(S/\mathrm{in}(I))_\nu \quad \forall\, i \in \mathbb{N},\ \nu \in \mathbb{Z}^m.$$

Remark 2.6. *Theorem 1.3 holds as well under the assumption that* $S/\mathrm{in}(I)$ *is cohomologically full. Examples of cohomologically full rings arise form known ones via flat extensions. For example, for a sequence* $\mathfrak{a} = \mathfrak{a}_1,\dots,\mathfrak{a}_n$ *of positive integers one can consider the* K*-algebra map* $\phi_{\mathfrak{a}} : S \to S$ *defined by* $\phi_{\mathfrak{a}}(x_i) = x_i^{\mathfrak{a}_i}$*, which is indeed a flat extension. Since cohomologically fullness is preserved under flat extensions (see* [DDM18, Lemma 3.4]*) one has that* $S/\phi_{\mathfrak{a}}(J)$ *is cohomologically full if* S/J *is cohomologically full. In particular, by Proposition 2.3 we have that* $S/\phi_{\mathfrak{a}}(J)$ *is cohomologically full if* J *is a square-free monomial ideal. Hence the conclusion of Theorem 1.3 holds also if* $\mathrm{in}(I) = \phi_{\mathfrak{a}}(J)$ *where* J *is square-free and* $\mathfrak{a}$ *is any sequence of positive integers.*

Since the extremal Betti numbers of S/I can be described in terms of $h^{ij}(S/I)$ (cf. [Ch07]), we have:

Corollary 2.7. *Let* $I \subseteq S$ *be a homogeneous ideal such that* $\mathrm{in}(I)$ *is square-free. Then the extremal Betti numbers of* S/I *and those of* $S/\mathrm{in}(I)$ *coincide (positions and values). In particular,* $\mathrm{depth}\, S/I = \mathrm{depth}\, S/\mathrm{in}(I)$ *and* $\mathrm{reg}\, S/I = \mathrm{reg}\, S/\mathrm{in}(I)$.

Remark 2.8. *One could wonder if* $S/\sqrt{\mathrm{in}(I)}$ *is Cohen-Macaulay whenever* S/I *is Cohen-Macaulay (independently from the fact that* $\mathrm{in}(I)$ *is square-free). In* [Va09] *it is proved that, if* S/I *is Cohen-Macaulay,* $\mathrm{Proj}\, S/\mathrm{in}(I)$ *cannot be disconnected by removing a closed subset of codimension larger than* 1 *(a necessary condition for the Cohen-Macaulayness of* $S/\sqrt{\mathrm{in}(I)}$*). However, we present two determinantal examples, one for lex and the other for revlex, such that* S/I *is Cohen-Macaulay but* $S/\sqrt{\mathrm{in}(I)}$ *not.*

(1) *Let* $S = K[x_1,\ldots,x_7]$ *and* I *be the ideal of* 2*-minors of :*

$$\begin{pmatrix} x_1+x_2 & x_5 & x_4 \\ -x_5+x_6 & x_3+x_7 & x_5 \\ x_4+x_7 & x_1-x_3 & x_5+x_7 \end{pmatrix}$$

It turns out that S/I *is a* 3*-dimensional Cohen-Macaulay domain. With respect to lex, one has:*

$$\mathrm{in}(I) = (x_1x_5, x_4x_5, x_1x_6, x_1x_4, x_1x_7, x_1^2, x_3x_4, x_2x_5, x_1x_3, x_2x_6x_7, x_2x_4x_7, x_3x_5^2, x_2x_3x_6).$$

One can check that $\mathrm{depth}\, S/\sqrt{\mathrm{in}(I)} = 2$.

(2) *Let* $S = K[x_1,\ldots,x_9]$ *and* I *the ideal of 2-minors of :*

$$\begin{pmatrix} x_3+x_7 & x_6 & x_1 & x_5 \\ x_9 & x_4+x_5 & x_7 & x_1+x_2 \\ x_3 & x_3 & x_7 & x_7-x_8 \end{pmatrix}$$

It turns out that S/I *is a* 3*-dimensional Cohen-Macaulay ring. With respect to revlex one has:*

$$\mathrm{in}(I) = (x_1x_7, x_1x_3, x_3x_7, x_4x_7, x_4x_8, x_3x_4, x_2x_7, x_3x_5, x_2x_3, x_6x_7, x_3^2, x_1^2, x_4x_5, x_1x_4, x_7x_8,$$
$$x_7^2, x_3x_9, x_5x_6x_9, x_4x_6x_9, x_2x_6x_9, x_1x_6x_9, x_1x_8^2, x_1x_5x_9, x_1x_6x_8, x_5x_6x_8, x_2x_6x_8, x_2x_5x_8^2x_9).$$

One can check that $\mathrm{depth}\, S/\sqrt{\mathrm{in}(I)} = 2$.

Corollary 2.7 and Theorem 1.1 show that a square-free initial ideal and the revlex generic initial ideal have important features in common. Next we show that from other perspectives a square-free initial ideal (when it exists!) behaves better than the revlex generic initial ideal. We recall first some definitions. Let $A = S/J$ where $J \subseteq S$ is a homogeneous ideal.

(1) A is Buchsbaum if for any homogeneous system of parameters $f_1,\ldots,f_d$ of A,
$$(f_1,\ldots,f_{i-1}) : f_i = (f_1,\ldots,f_{i-1}) : \mathfrak{m} \quad \forall\ i = 1,\ldots,d.$$
(2) A is generalized Cohen-Macaulay if $H^i_{\mathfrak{m}}(A)$ has finite length for all $i < \dim A$.
(3) For $c \in \mathbb{N}$, A is Cohen-Macaulay in codimension c if $A_{\mathfrak{p}}$ is Cohen-Macaulay for any prime ideal $\mathfrak{p}$ of A such that $\mathrm{height}\,\mathfrak{p} \leq \dim A - c$.
(4) For $r \in \mathbb{N}$, A satisfies Serre's (S_r) condition if $\mathrm{depth}\, A_{\mathfrak{p}} \geq \min\{r, \mathrm{height}\,\mathfrak{p}\}$ for any prime ideal $\mathfrak{p}$ of A.

Remark 2.9. *If* $J \subseteq S$ *is an ideal and* $\mathfrak{p} \subseteq S$ *is a prime ideal of height* h *containing* J*, then for all* $k \in \mathbb{N}$*:*

$$\begin{aligned} \mathrm{depth}(S_{\mathfrak{p}}/JS_{\mathfrak{p}}) \geq k &\iff H^i_{\mathfrak{p}S_{\mathfrak{p}}}(S_{\mathfrak{p}}/JS_{\mathfrak{p}}) = 0 \ \ \forall\ i < k \\ &\iff \mathrm{Ext}^{h-i}_{S_{\mathfrak{p}}}(S_{\mathfrak{p}}/JS_{\mathfrak{p}}, S_{\mathfrak{p}}) = 0 \ \ \forall\ i < k \\ &\iff (\mathrm{Ext}^{h-i}_S(S/J,S))_{\mathfrak{p}} = 0 \ \ \forall\ i < k. \end{aligned}$$

Recall that S/J *is pure if* $\dim S/\mathfrak{p} = \dim S/J$ *for all associated prime ideal* $\mathfrak{p}$ *of* J*. Since*
$$\dim \mathrm{Ext}^k_S(S/J,S) = \sup\{n - \mathrm{height}\,\mathfrak{p} :\ \mathfrak{p} \in \mathrm{Spec}\, S,\ (\mathrm{Ext}^k_S(S/J,S))_{\mathfrak{p}} \neq 0\},$$

from the equivalences above can deduce that:

(1) S/J *is pure* $\iff \dim \operatorname{Ext}_S^{n-i}(S/J,S) < i \ \ \forall\, i < \dim S/J$.

(2) S/J *is generalized Cohen-Macaulay* $\iff \dim \operatorname{Ext}_S^{n-i}(S/J,S) \leq 0 \ \ \forall\, i < \dim S/J$.

(3) S/J *is Cohen-Macaulay in codimension* $c \iff \dim \operatorname{Ext}_S^{n-i}(S/J,S) < c \ \ \forall\, i < \dim S/J$.

(4) S/J *satisfies* (S_r) *for* $r \geq 2 \iff \dim \operatorname{Ext}_S^{n-i}(S/J,S) \leq i - r \ \ \forall\, i < \dim S/J$.

For the latter equivalence, the assumption $r \geq 2$ *is needed to guarantee the purity of* S/J*, see* [Sc79, Lemma (2.1)] *for details.*

Remark 2.10. *Given a homogeneous ideal* $J \subseteq S$*, if* $A = S/J$ *is Buchsbaum then it is generalized Cohen-Macaulay, but the converse does not hold true in general. If* $H_{\mathfrak{m}}^i(A)$ *is concentrated in only one degree for all* $i < \dim A$*, however,* A *must be Buchsbaum by* [Sc82, Theorem 3.1]*. If* J *is a square-free monomial ideal and* A *is generalized Cohen-Macaulay, then it turns out that* $H_{\mathfrak{m}}^i(A) = (H_{\mathfrak{m}}^i(A))_0$ *for all* $i < \dim A$*. In particular, for a square-free monomial ideal* $J \subseteq S$ *we have that* S/J *is Buchsbaum if and only if* S/J *is generalized Cohen-Macaulay.*

Putting together the remarks above and Theorem 1.3 we get:

Corollary 2.11. *Let* $I \subseteq S$ *be a homogeneous ideal such that* $\operatorname{in}(I)$ *is square-free. Then*

(i) S/I *is generalized Cohen-Macaulay if and only if* S/I *is Buchsbaum if and only if* $S/\operatorname{in}(I)$ *is Buchsbaum;*

(ii) *For any* $r \in \mathbb{N}$*,* S/I *satisfies the* (S_r) *condition if and only if* $S/\operatorname{in}(I)$ *satisfies the* (S_r) *condition;*

(iii) *For any* $c \in \mathbb{N}$*,* S/I *is Cohen-Macaulay in codimension* c *if and only if* $S/\operatorname{in}(I)$ *is Cohen-Macaulay in codimension* c*.*

3. Ideals with square-free initial ideals

In this section, we will outline some properties and non-properties of ideals with square-free initial ideals. After that we will show three special classes of these ideals appeared in the literature, pointing at some consequences of Theorem 1.3 for each of them.

We first observe that by [BCP99] two ideals with the same degrevlex generic initial ideal have the same extremal Betti numbers. Hence a stronger version of Herzog's conjecture 1.2 would be the assertion that if I is an ideal with a square-free initial ideal J then the corresponding degrevlex generic initial ideals coincide. The following example shows that the stronger statement is actually false.

Example 3.1. *Let* $S = K[x_{ij} : 1 \leq i, j, \leq 4]$ *and* $<$ *be the degrevlex order associated to the total order* $x_{11} > x_{12} > x_{13} > x_{14} > x_{21} > \cdots > x_{44}$*. Let* I *be the ideal of* 2*-minors of* (x_{ij})*. Then* $J = \operatorname{in}(I) = (x_{ij}x_{hk} : i < h$ *and* $j > k)$ *is square-free and quadratic and* $\operatorname{gin}(I)$ *and* $\operatorname{gin}(J)$ *differ already in degree* 2*. Indeed, degree* 2 *part of* $\operatorname{gin}(I)$ *is* $(x_{ij} : i = 1,2$ *and* $j = 1,2,3,4)^2$ *and the degree* 2 *part of* $\operatorname{gin}(J)$ *is obtained from that of* $\operatorname{gin}(I)$ *by replacing* $x_{23}x_{24}, x_{24}^2$ *with* $x_{11}x_{31}, x_{12}x_{31}$*.*

When K has positive characteristic then S/J is F-pure for any square-free monomial ideal $J \subseteq S$ (cf. [HR76, Proposition 5.38]). However, S/I may fail F-purity when I has a square-free initial ideal:

Example 3.2. *Let* $S = K[x_1, \ldots, x_5]$ *where* K *has characteristic* $p > 0$*, and* I *the ideal generated by the* 2*-minors of the matrix:*

$$\begin{pmatrix} x_4^2 + x_5^a & x_3 & x_2 \\ x_1 & x_4^2 & x_3^b - x_2 \end{pmatrix}.$$

Note that, if $\deg(x_4) = a$*,* $\deg(x_1) = \deg(x_3) = 1$*,* $\deg(x_2) = b$ *and* $\deg(x_5) = 2$*, the ideal* I *is homogeneous. Singh proved in* [Si99, Theorem 1.1] *that, if* $a - a/b > 2$ *and* $\mathrm{GCD}(p, a) = 1$*, then* S/I *is not* F*-pure. However, considering lex as term order, one has*

$$\mathrm{in}(I) = (x_1x_3,\ x_1x_2,\ x_2x_3).$$

On the other hand we have:

Proposition 3.3. *Let* $I \subseteq S$ *be a homogeneous ideal with a with square-free initial ideal. Then* S/I *is cohomologically full.*

Proof. As in the proof of Theorem 1.3, setting $J = \mathrm{in}(I)$, take a weight $w = (w_1, \dots, w_n) \in \mathbb{N}^n$ such that $J = \mathrm{in}_w(I)$. Let t be a new indeterminate over K, $R = K[t]_{(t)}$ and $P = R[x_1, \dots, x_n]$. By considering the w-homogenization $\mathrm{hom}_w(I) \subseteq P$, set $A = P/\mathrm{hom}_w(I)$. Since $A/(t) \cong S/J$ is cohomologically full and t is a non-zero divisor on A, then A is cohomologically full by [DDM18, Theorem 3.1]. So, $A_t \cong (S/I) \otimes_K K(t)$ is cohomologically full by [DDM18, Lemma 3.4], and therefore (again by [DDM18, Lemma 3.4]), S/I is cohomologically full. $\square$

Corollary 3.4. *Let* $I \subseteq S$ *be a homogeneous ideal with a square-free initial ideal. If* $J \subseteq S$ *is a homogeneous ideal such that* $\sqrt{J} = I$*, then*

$$\operatorname{depth} S/J \leq \operatorname{depth} S/I \quad \text{and} \quad \operatorname{reg}(S/J) \geq \operatorname{reg}(S/I).$$

Proof. It follows by Proposition 3.3. $\square$

Given an ideal $I \subseteq S$, the cohomological dimension of I is defined as:

$$\mathrm{cd}(S, I) = \max\{i \in \mathbb{N} : H^i_I(S) \neq 0\}.$$

One may ask what is the relationship between the cohomological dimension of I and that of in(I). In general, without further assumptions on in(I), they are unrelated.

Example 3.5. *Recall that for any ideal* $J \subseteq S$ *of height* h *and generated by* r *polynomials* $h \leq \mathrm{cd}(S, J) \leq r$ *and furthermore,* $\mathrm{cd}(S, J) = \mathrm{cd}(S, \sqrt{J})$*.*

(1) *For any ideal* $I \subseteq S$ *of height* h*, the generic initial ideal* $\mathrm{gin}(I)$ *w.r.t. degrevlex has cohomological dimension* h*: in fact* $\sqrt{\mathrm{gin}(I)} = (x_1, \dots, x_h)$*. However, there are many ideals* I *of height* h *for which* $\mathrm{cd}(S, I) > h$*. Hence there are ideals for which*

$$\mathrm{cd}(S, I) > \mathrm{cd}(S, \mathrm{in}(I)).$$

holds.

(2) *In* [Va09, Example 2.14] *has been considered the ideal*

$$I = (x_1x_5 + x_2x_6 + x_4^2,\ x_1x_4 + x_3^2 - x_4x_5,\ x_1^2 + x_1x_2 + x_2x_5) \subseteq S = K[x_1, \dots, x_6].$$

It turns out that I *is a height* 3 *complete intersection and* S/I *is a normal domain (notice that in* [Va09, Example 2.14] *there is a typo in the last equation). By considering lex, we have:*

$$\sqrt{\mathrm{in}(I)} = (x_1, x_2, x_3) \cap (x_1, x_3, x_6) \cap (x_1, x_2, x_5) \cap (x_1, x_4, x_5).$$

However $S/\sqrt{\mathrm{in}(I)}$ *has depth* 2*, so* $\mathrm{cd}(S, \sqrt{\mathrm{in}(I)}) = 6 - 2 = 4$ *by* [Ly83]*. Therefore* $\mathrm{cd}(S, \mathrm{in}(I)) = 4$*, but* $\mathrm{cd}(S, I) = 3$*. So, there are ideals for which*

$$\mathrm{cd}(S, I) < \mathrm{cd}(S, \mathrm{in}(I)).$$

holds.

Proposition 3.6. *If* $I \subseteq S$ *is a homogeneous ideal and* $\mathrm{in}(I)$ *is square-free, then*

$$\mathrm{cd}(S, I) \geq \mathrm{cd}(S, \mathrm{in}(I)).$$

Furthermore, if K *has positive characteristic, then* $\mathrm{cd}(S, I) = \mathrm{cd}(S, \mathrm{in}(I))$.

Proof. Under this assumption S/I is cohomologically full by Proposition 3.3, so $\mathrm{cd}(S, I) \geq n - \mathrm{depth}(S/I)$ by [DDM18, Proposition 2.5]. However, $\mathrm{depth}(S/I) = \mathrm{depth}(S/\mathrm{in}(I))$ by Corollary 2.7, and $\mathrm{cd}(S, \mathrm{in}(I)) = n - \mathrm{depth}(S/\mathrm{in}(I))$ by [Ly83].

If K has positive characteristic, $\mathrm{cd}(S, I) \leq n - \mathrm{depth}(S/I)$ by [PS73, Proposition 4.1 and following remark]. $\square$

Example 3.7. *If* K *has characteristic 0, the inequality in Proposition 3.6 may be strict. For example if* $S = K[X]$ *where* $X = (x_{ij})$ *denotes an* $r \times s$ *generic matrix, the ideal* $I \subseteq S$ *generated by the size* t *minors of* X *has cohomological dimension* $rs - t^2 + 1$ *by a result of Bruns and Schwänzl in* [BS90]. *However, Sturmfels proved in* [St90] *that, if* $<$ *is lex refining* $x_{11} > x_{12} > \ldots > x_{1s} > x_{21} > \ldots > x_{r1} > \ldots > x_{rs}$, *then* $\mathrm{in}(I)$ *is square-free and* $S/\mathrm{in}(I)$ *is Cohen-Macaulay of dimension* $rs - (r - t + 1)(s - t + 1)$. *In particular, once again by* [Ly83], $\mathrm{cd}(S, \mathrm{in}(I)) = (r - t + 1)(s - t + 1) < \mathrm{cd}(S, I)$.

3.1. **ASL: Algebras with straightening laws.** Let $A = \oplus_{i \in \mathbb{N}} A_i$ be a graded algebra and let $(H, \prec)$ be a finite poset set. Let $H \to \cup_{i>0} A_i$ be an injective function. The elements of H will be identified with their images. Given a chain $h_1 \preceq h_2 \preceq \cdots \preceq h_s$ of elements of H the corresponding product $h_1 \cdots h_s \in A$ is called standard monomial. One says that A is an algebra with straightening laws on H (with respect to the given embedding H into $\cup_{i>0} A_i$) if three conditions are satisfied:

(1) The elements of H generate A as a A_0-algebra.
(2) The standard monomials are A_0-linearly independent.
(3) For every pair h_1, h_2 of incomparable elements of H there is a relation (called the straightening law)
$$h_1 h_2 = \sum_{j=1}^{u} \lambda_j h_{j1} \cdots h_{j\nu_j}$$
where $\lambda_j \in A_0 \setminus \{0\}$, the $h_{j1} \cdots h_{j\nu_j}$ are distinct standard monomials and, assuming that $h_{j1} \preceq \cdots \preceq h_{j\nu_j}$, one has $h_{j1} \prec h_1$ and $h_{j1} \prec h_2$ for all j.

It then follows from the three axioms that the standard monomials form a basis of A over A_0 and that the straightening laws are indeed the defining equations of A as a quotient of the polynomial ring $A_0[H] = A_0[h : h \in H]$. That is, the kernel I of the canonical surjective map $A_0[H] \to A$ of A_0-algebras induced by the function $H \to \cup_{i>0} A_i$ is generated by the straightening laws regarded as elements of $A_0[H]$, i.e.,

$$A = A_0[H]/I \quad \text{with} \quad I = (h_1 h_2 - \sum_{j=1}^{u} \lambda_j h_{j1} \cdots h_{j\nu_j} : h_1 \not\prec h_2 \not\prec h_1).$$

Remark 3.8. *Equipping the polynomial ring* $A_0[H]$ *with the* $\mathbb{N}$*-graded structure induced by assigning to* h *the degree of its image in* $\cup_{i>0} A_i$, *then the ideal* I *is homogeneous.*

The ideal $J = (h_1 h_2 : h_1 \not\prec h_2 \not\prec h_1)$ of $A_0[H]$ defines a quotient $A_D = A_0[H]/J$ which is an ASL as well, called the discrete ASL associated to H. In [DEP82] it is proved that A_D is the special fiber of a flat family with general fiber A. Indeed, at least when A_0 is a field, one can obtain the same result by observing that with respect to (weighted) degrevlex associated to a total order on H that refines the given partial order $\prec$ one has $J = \mathrm{in}(I)$. More precisely it has been observed in [Co07, Lemma 5.5] that ASL's can also be defined via Gröbner degenerations.

As we have already said in the introduction, various kinds of generic determinantal rings are indeed ASL's. Here it is important to note that the ASL presentation is not, in general, the minimal presentation of the algebra. For example, for generic determinantal rings the poset consists of all the subdeterminants of the matrix. Nevertheless the non-minimal presentation is good enough to prove, via deformation to the discrete counterpart, that these classical algebras are Cohen-Macaulay. As a consequence of Theorem 1.3 and Remark 2.5 we have:

Corollary 3.9. *Let A be an ASL over a field K and let A_D be the corresponding discrete counterpart. Then* $\operatorname{depth} A = \operatorname{depth} A_D$*. In particular, A is Cohen-Macaulay if and only if A_D is Cohen-Macaulay.*

Remark 3.10. *In* [Mi10]*, Miyazaki proved that if A is a Cohen-Macualay ASL and A_D is Buchsbaum then A_D is Cohen-Macualay.*

Remark 3.11. *Let $S = K[x_1, \dots, x_n]$ and let H be the set of square-free monomials different from 1 ordered by division: for $m_1, m_2 \in H$ one sets $m_1 \preceq m_2$ if and only if $m_2 | m_1$. Then S can be regarded as an ASL over H with straightening law:*

$$m_1 m_2 = \mathrm{GCD}(m_1, m_2)\mathrm{LCM}(m_1, m_2).$$

This induces an ASL structure on every Stanley-Reisner ring $K[\Delta]$ associated with a simplicial complex Δ on n vertices. Here the underlying poset is given by the non-empty faces of Δ ordered by reverse inclusion. Hence the discrete ASL associated to $K[\Delta]$ is $K[\mathrm{sd}(\Delta)]$ where $\mathrm{sd}(\Delta)$ is the barycentric subdivision of Δ. So, denoting by N the number of square-free monomials different from 1 of S and by $S' = K[x_m : m \neq 1$ is a square-free monomial of $S]$, by Theorem 1.3 and by Grothendieck duality we have

$$\dim \operatorname{Ext}_S^{n-i}(K[\Delta], S) = \dim \operatorname{Ext}_{S'}^{N-i}(K[\mathrm{sd}(\Delta)], S') \quad \forall\ i \in \mathbb{Z}$$

(Krull dimensions). This was already known: in fact, the Krull dimensions of the deficiency modules of a Stanley-Reisner ring are topological invariants by the work of Yanagawa [Ya11]*.*

A further application is the Eisenbud-Green-Harris conjecture for standard graded ASL's. This conjecture, originally stated in [EGH93], has been generalized in various forms, for example:

Conjecture 3.12. *If $I \subseteq S$ is a height h quadratic homogeneous ideal, then the Hilbert function of S/I is less than or equal (componentwise) to that of S/J, where J is the sum of $(x_1^2, \dots, x_h^2)$ and a Lex ideal in the first* $\operatorname{projdim} S/I$ *variables.*

Theorem 3.13. *Conjecture 3.12 holds for standard graded ASL's.*

Proof. Let A be a standard graded ASL, say $A = S/I$ where I is the ideal generated by the straightening laws. By taking the right term order, $\mathrm{in}(I)$ is a square-free quadratic monomial ideal. Furthermore, $\operatorname{depth} S/\mathrm{in}(I) = \operatorname{depth} S/I$ by Corollary 2.7. So the conclusion follows at once from the fact that S/I and $S/\mathrm{in}(I)$ have the same Hilbert function and from the main result of [CCV14]. □

3.2. **Cartwright-Sturmfels ideals.** Cartwright-Sturmfels ideals were introduced and studied by Conca, De Negri, Gorla in a series of recent papers [CDG15, CDG17, CDG18, CDG18b]. We recall briefly their definition and main properties. Given positive integers $d_1, \dots, d_m$ one considers the polynomial ring $S = K[x_{ij} : 1 \leq i \leq m$ and $1 \leq j \leq d_i]$ with $\mathbb{Z}^m$-graded structure induced by assignment $\deg(x_{ij}) = e_i \in \mathbb{Z}^m$. The group $G = \mathrm{GL}_{d_1}(K) \times \dots \times \mathrm{GL}_{d_m}(K)$ acts on S as the group of multigraded K-algebra automorphisms.

The Borel subgroup $B = B_{d_1}(K) \times \cdots \times B_{d_m}(K)$ of the upper triangular invertible matrices acts on S by restriction. An ideal J is Borel-fixed if $g(J) = J$ for all $g \in B$. A multigraded ideal $I \subset S$ is Cartwright-Sturmfels if its multigraded Hilbert function coincides with that of a Borel-fixed radical ideal. The main properties of Cartwright-Sturmfels ideals are:

(1) If I is Cartwright-Sturmfels then all its initial ideals are square-free.
(2) The set of Cartwright-Sturmfels ideals is closed under multigraded linear sections and multigraded projections.

Examples of Cartwright-Sturmfels ideals are:

(3) The ideal of 2-minors and the ideal of maximal minors of matrices of distinct variables with graded structure given by rows or columns, [CDG15, CDG17, CDG18].
(4) Binomial edge ideals, [CDG18b].
(5) Ideals of multigraded closure of linear spaces, [CDG18b].

So, multigraded linear sections and multigraded projections of the above ideals Cartwright-Sturmfels as well. Note that Conjecture [CDG18b, 1.14] turns out to be a special case of the multigraded version of Theorem 1.3, see Remark 2.5.

3.3. **Knutson ideals.** For this subsection, either $K = \mathbb{Q}$ or $K = \mathbb{Z}/p\mathbb{Z}$. Fix also $f \in S = K[x_1, \ldots, x_n]$ such that $\mathrm{in}(f)$ is a square-free monomial for some term order $<$.

Let $\mathcal{C}_f$ be the smallest set of ideals of S satisfying the following conditions:

(1) $(f) \in \mathcal{C}_f$;
(2) If $I \in \mathcal{C}_f$, then $I : J \in \mathcal{C}_f$ for any ideal $J \subseteq S$; in particular the associated primes of I are in $\mathcal{C}_f$.
(3) If $I, J \in \mathcal{C}_f$, then $I + J \in \mathcal{C}_f$ and $I \cap J \in \mathcal{C}_f$.

Example 3.14. *If* $f = x_1 \cdots x_n$*, then* $\mathcal{C}_f$ *is the set of all the square-free monomial ideals of* S*.*

The class of ideals $\mathcal{C}_f$ is introduced and studied by Knutson in [Kn09]. For this reason, we will call the elements of $\mathcal{C}_f$ Knutson ideals associated with f. In positive characteristic, if $I \in \mathcal{C}_f$ then $\phi_f(I) \subseteq I$ where $\phi_f : S \to S$ is a special splitting (associated to the polynomial f) of the Frobenius morphism $F : S \to S$. In particular, if $I \in \mathcal{C}_f$, then S/I is F-pure (in positive characteristic). Knutson proved in [Kn09] that for every $I \in \mathcal{C}_f$ the initial ideal $\mathrm{in}(I)$ is square-free and hence every ideal in $\mathcal{C}_f$ is radical. Moreover, one can infer that $\mathrm{in}(I) \neq \mathrm{in}(J)$ whenever $I, J \in \mathcal{C}_f$ are different. So $\mathcal{C}_f$ is always a finite set.

In [Kn09, Section 7], Knutson has shown that many interesting ideals belong to $\mathcal{C}_f$ for a suitable choice of f, for example ideals defining matrix Schubert varieties and ideals defining Kazhdan-Lusztig varieties. Below we present an example of a Knutson prime ideal that is not Cohen-Macaulay. The interest lais in the fact that we expect that a prime ideal with a square-free initial ideal, under some additional assumption, should be Cohen-Macaulay (see Section 4). This example originated from a discussion we had with Jenna Rajchgot at MSRI in 2012; we thank her for pointing us that the example we discussed there is actually a Knutson ideal:

Example 3.15. *Let* $S = K[x_1, \ldots, x_5]$*, and* $f = gh$ *where* $g = x_1x_4x_5 - x_2x_4^2 - x_3x_5^2$ *and* $h = x_2x_3 - x_4x_5$*. By considering lex,* $\mathrm{in}(f) = x_1x_2x_3x_4x_5$*. We have that* (g, h) *is a height 2 complete intersection in* $\mathcal{C}_f$*. One can check that*

$$\mathfrak{p} = (g, h, \quad x_1x_3x_4 - x_3^2x_5 - x_4^3, \quad x_1x_2x_5 - x_2^2x_4 - x_5^3)$$

is a height 2 prime ideal. Since it contains (g, h)*, must be associated to it, so* $\mathfrak{p} \in \mathcal{C}_f$ *and one can check that* $S/\mathfrak{p}$ *is not Cohen-Macaulay.*

Next, we show how to derive the solution of a conjecture stated in [BV15] in the case of Knutson ideals. Given an ideal $I \subseteq S$, the graph with vertex set $\mathrm{Min}(I)$ and edges $\{\mathfrak{p}, \mathfrak{p}'\}$

if $\operatorname{height}(\mathfrak{p}+\mathfrak{p}') = \operatorname{height}(I)+1$ is called the dual graph of I. The ideal I is called Hirsch if the diameter of the dual graph is bounded above from height(I). In [BV15, DV17] has been explained why Hirsch ideals are a natural class to consider, and there have also been provided several examples of such ideals. In particular, in [BV15, Conjecture 1.6] has been conjectured that, if I is quadratic and S/I is Cohen-Macaulay, then I is Hirsch.

Proposition 3.16. *Let $I \subseteq S$ be a homogeneous Knutson ideal such that S/I satisfies (S_2) (e.g. S/I is Cohen-Macaulay). If either $\operatorname{in}(I)$ is quadratic or $\operatorname{height}(I) \leq 3$, then I is Hirsch.*

Proof. By the assumption in(I) is the Stanley-Reisner ideal I_Δ of a simplicial complex Δ on n vertices. Since S/I satisfies (S_2), S/I_Δ satisfies (S_2) as well by Corollary 2.11. In other words, Δ must be a normal simplicial complex. If I_Δ is quadratic (that is Δ is flag), then I_Δ is Hirsch by [AB14]. If $\operatorname{height}(I_\Delta) \leq 3$, then it is simple to check that I_Δ is Hirsch (see [Ho16, Corollary A.4] for the less trivial case in which $\operatorname{height}(I_\Delta) = 3$). In each case, we conclude because, under the assumptions of the theorem, by [DV17, Theorem 3.3] the diameter of the dual graph of I is bounded above from that of in(I). □

4. Questions and answers

The first version of this paper, posted on arXiv on May 30 2018, contains in Section 4 five questions. We received several comments concerning these questions and it turns out that only Question 4.2 is still open. We reproduce below the questions with the original numbering and the relative answers.

Question 4.1. *Let $I \subseteq S$ be a prime ideal with a square-free initial ideal. Does S/I satisfy Serre's condition (S_2)?*

Notice that, if $\mathfrak{p} \subseteq S$ is a prime ideal with a square-free initial ideal, then $\operatorname{depth} S/\mathfrak{p} \geq 2$ (provided $\dim S/\mathfrak{p} \geq 2$) by [KS95, Va09]. However the answer is negative: the ideal in 3.15 does not satisfy Serre's condition (S_2).

Question 4.2. *Let $I \subseteq S$ be a homogeneous prime ideal with a square-free initial ideal such that $\operatorname{Proj} S/I$ is nonsingular. Is S/I Cohen-Macaulay and with negative a-invariant?*

A similar question appeared in [Va18, Problem 3.6]. For ASL's, a question similar to Question 4.2 already caught some attention in the eighties: Buchweitz proved a related statement (unpublished, see [DEP82]), that using Theorem 1.3 can be expressed as follows:

Theorem 4.3 (Buchweitz)**.** *Let A be an ASL domain over a field of characteristic 0 such that $\operatorname{Proj} A$ has rational singularities. If A is Cohen-Macaulay, then A has rational singularities.*

A result of Brion [Br03] on multiplicity free irreducible varieties implies S/I is Cohen-Macaulay for every prime Cartwright-Sturmfels ideal I. Similarly one can ask:

Question 4.4. *Let $I \subseteq S$ be a Knutson prime ideal. Is S/I Cohen-Macaulay?*

Jenna Rajchgot informed us that the ideal Example 3.15 is indeed a Knutson prime ideal $\mathfrak{p} \subseteq S$ such that $S/\mathfrak{p}$ is not Cohen-Macaulay. This provides a negative answer to Question 4.4.

Question 4.5. *Let $I \subseteq S$ be an ideal such that $\operatorname{in}(I)$ is a square-free monomial ideal for degrevlex. If $\operatorname{char}(K) > 0$, is it true that S/I is F-pure?*

In Example 3.2 S/I is not F-pure and I is an ideal with a square-free initial ideal. However, in that case $\text{in}(I)$ is a square-free monomial ideal only for lex, so it does not provide a negative answer to the above question. Also, the question whether ASL's over fields of positive characteristic are F-pure was already raised in [DEP82], and as explained in 3.1 ASL's have square-free quadratic degenerations with respect to degrevlex term orders.

Combining results on Cartwright-Sturmfels ideals recalled in 3.2 with a result of Othani [Oh13], we obtain a negative answer to Question 4.5. Othani [Oh13] proves that, if I is the binomial edge ideal of the 5-cycle, the ring S/I is not F-pure in characteristic 2. As recalled in 3.2 binomial edge ideals, being Cartwright-Sturmfels, have a square-free monomial ideal for every term order. As far as we know it might be true that for a Cartwright-Sturmfels ideal I the rign S/I is of F-pure type in characteristic 0.

Question 4.6. *Let $I \subseteq S$ be a prime ideal such that $\text{in}(I)$ is a square-free monomial ideal for degrevlex. Is it true that S/I is normal?*

Without assuming that the term order is of degrevlex type, it is easy to see that the answer to the above question is negative (consider for example $(x_1x_2x_3 + x_2^3 + x_3^3) \subseteq K[x_1, x_2, x_3]$). On the other hand, Eisenbud in [Ei80] conjectured a positive answer to Question 4.6 for ASL's. Hibi informed us that a 3-dimensional standard graded ASL which is a non-normal (Gorenstein) domain is described in [HW85]. This provides a negative answer to Question 4.6.

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