Invariants associated with ideals in one-dimensional local domains. (Journal of Algebra, 316 (1), 2007, 32-53)

Anna Oneto[a] and Elsa Zatini[b*]

[a] *Ditpem, Università di Genova, P.le Kennedy, Pad. D - I 16129 Genova (Italy);*

[b] *Dima, Università di Genova, Via Dodecaneso 35 - 16146 Genova (Italy)*

Abstract. Let R be a one-dimensional local Noetherian domain with maximal ideal $\mathfrak{m}$, quotient field K and residue field $R/\mathfrak{m} := k$. We assume that the integral closure $\overline{R}$ of R in its quotient field K is a DVR and a finite R-module. We assume also that the field k is isomorphic to the residue field of $\overline{R}$. For I a proper ideal of R, denote the *inverse* of I by I^*; that is, I^* is the set $(R :_K I)$ of elements of K that multiply I into R. We investigate two numerical invariants associated to a proper ideal I of R that have previously come up in the literature from various points of view. The two invariants are: (1) the difference between the composition lengths of I^*/R and R/I, and (2) the difference between the product, when the composition length of R/I is multiplied by the composition length of $\mathfrak{m}^*/R$, and the length of I^*/R. We show that these two differences can be expressed in terms of the type sequence of R, a finite sequence of positive integers related to the natural valuation inherited from $\overline{R}$.

1 Introduction.

We begin by giving the setting of the paper.

Setting 1.1 Let $(R, \mathfrak{m})$ be a one-dimensional local Noetherian domain with quotient field K and residue field k. We assume throughout that the normalization $\overline{R}$ of R is a DVR and a finite R-module, i.e. R is analytically irreducible. Let $t \in \overline{R}$ be a uniformizing parameter for $\overline{R}$, so that $t\overline{R}$ is the maximal ideal of $\overline{R}$. We also suppose that the field k is isomorphic to the residue field $\overline{R}/t\overline{R}$, i.e. R is residually rational.

A fractional ideal ω of R is called a *canonical ideal* of R provided that for any nonzero fractional ideal I we have $I = (\omega :_K (\omega :_K I))$, where for two fractional ideals J, L we denote $(J :_K L) = \{a \in K \mid aL \subseteq J\}$. Throughout the paper

*Corresponding author.

E-mail addresses: oneto@dimet.unige.it (A.Oneto), zatini@dima.unige.it (E.Zatini).

we make heavy use of the canonical ideal. We notice in the next section, after Notation 2.2, that in our setting a canonical ideal ω exists and we can assume that $R \subseteq \omega \subseteq \overline{R}$.

The theorem below is well known:

Theorem 1.2 ([3], [10], [12, Theorem 13.1]) *With $R, \mathfrak{m}, K$ as in Setting 1.1, the following statements are equivalent:*

(1) R is Gorenstein.

(2) $\omega = R$.

(3) The composition length of $\mathfrak{m}^*/R$ is 1, where $\mathfrak{m}^* := (R :_K \mathfrak{m})$.

(4) The composition length of $\overline{R}/\mathfrak{C}$ is twice that of $\overline{R}/R$, where $\mathfrak{C} := (R :_K \overline{R})$ is the conductor ideal.

(5) For every nonzero proper ideal I of R, the composition length of R/I equals the composition length of I^*/R, where $I^* := (R :_K I)$ is the inverse of I.

In this paper we consider two numerical invariants related to properties (1)−(5) of Gorenstein rings from the theorem above.

Notation 1.3 We write $\ell_R(M)$ for the composition length of a module M. The *Cohen-Macaulay type* of R, which we denote by r, is $\ell_R(\mathfrak{m}^*/R)$. For I a proper ideal of R, we define the invariants $a(I)$ and $b(I)$ as follows:

$$a(I) := \ell_R(I^*/R) - \ell_R(R/I)$$

$$b(I) := r\ell_R(R/I) - \ell_R(I^*/R).$$

In view of the properties (1)−(5) above, these invariants measure how far R is from being Gorenstein. For a Gorenstein ring and I a proper ideal, (1.2.3) and (1.2.5) imply that $r = 1$, $a(I) = 0$, and so $b(I) = 0$. In 1963, R. Berger conjectured that $a(I)$ might always be non-negative [3]. Counterexamples were given by J. Jäger in 1977; in particular,

$$R = k[[t^9, t^{15}, t^{17}, t^{23}, t^{25}, t^{29}, t^{31}]],\ I = (t^{38}, t^{44}, t^{50}) \Longrightarrow a(I) = -1,$$

as he shows in [10]. We show in Theorem 3.16.5 that in our setting "almost Gorenstein" rings do satisfy $a(I) \geq 0$, for all reflexive ideals I.

We recall the definition.

Definition 1.4 [2, Definition-Proposition 20.] *Let $(R, \mathfrak{m})$ be a one-dimensional local Cohen-Macaulay ring with finite integral closure and with a canonical ideal ω such that $R \subseteq \omega \subseteq \overline{R}$. Let r be the Cohen-Macaulay type of R from (1.3). The ring R is called almost Gorenstein if one of the equivalent conditions (1) and (2) below holds:*

(1) $\mathfrak{m} = \mathfrak{m}\,\omega$.

(2) $r - 1 = 2\ell_R(\overline{R}/R) - \ell_R(\overline{R}/\mathfrak{C})$.

In this article we prove that properties similar to (1)−(5) of Theorem 1.2 characterize almost Gorenstein rings in our setting. We give part of the characterization below:

Theorem 1.5 *Let $(R, \mathfrak{m})$ be as in Setting 1.1 and let ω be a canonical ideal of R with $R \subseteq \omega \subseteq \overline{R}$. Let r be the Cohen-Macaulay type of R. Then R is almost Gorenstein if and only if $a(I) = r - 1 - \ell_R(I^{**}/I)$ for every non-principal ideal I contained in R.*

The inequalities $a(I) \leq 2\ell_R(\overline{R}/R) - \ell_R(\overline{R}/\mathfrak{C})$ and $b(I) \geq 0$ hold for every nonzero ideal I (see Remark 3.1). In [10] Jäger finds another upper bound for $a(I)$, namely $a(I) \leq (r-1)\ell_R(R/I)$.

In Theorem 3.11, we obtain expressions for the invariants $a(I)$ and $b(I)$ in terms of the type sequence $[r_1, ..., r_n]$, defined in (2.5), where $n = \ell_R(R/\mathfrak{C})$. These expressions yield new lower and upper bounds and vanishing conditions for the invariants. For example we obtain the inequality

$a(I) \leq (r-1)\ell_R(R/I^{**}) - \ell_R(I^{**}/I)$,

which improves the inequality of Jäger referred to above. Another consequence of the expression for $a(I)$ in Theorem 3.11 is that we get a new sufficient condition for $a(I)$ to be positive. Also, when I is an integrally closed ideal or when $\omega \subseteq (I :_K I)$, we see that $a(I) \geq r - 1 \geq 0$. Moreover, if R is almost Gorenstein, then $a(I) = r - 1$ for every non-principal reflexive ideal I.

Regarding the invariant $b(I)$, on the other hand, more attention has been reserved for the particular case of $I = \mathfrak{C}$, the conductor ideal of R. A general structure theorem for rings satisfying the equality $b(\mathfrak{C}) = 0$ or $b(\mathfrak{C}) = 1$ is given in the 1992 article of W. Brown and H. Herzog [4].

In their 1997 paper [5], M. D'Anna and D. Delfino find the upper bound $b(\mathfrak{C}) \leq (r-1)(\ell_R(R/\mathfrak{C}) - 1)$. In a series of papers, they attack the problem of classifying rings according to the value of the quantity $b(\mathfrak{C})$ with other authors [6], [7], [8]. In the present authors' earlier work with F. Odetti [15], the lower bound of $(r-1)\ell_R(\omega^*/\mathfrak{C})$ is given for $b(\mathfrak{C})$. From the expressions in Theorem 3.11, we get the following bounds for $b(I)$:

$b(I) \leq (r-1)(\ell_R(R/I) - 1) + \ell_R(I^{**}/I) + d(I)$,

$b(I) \geq r\ \ell_R(I^{**}/I) \geq 0$, which hold for every proper ideal I, and

$b(I) \geq (r-1)\ell_R((I^{**}+\omega^*)/I) + \ell_R(I^{**}/I)$, valid, for instance, when $I \subseteq \omega^*$, as well as a necessary and sufficient condition on the vanishing of $b(I)$:

$(VC) \qquad b(I) = 0 \iff I^{**} = I,\ d(I) = 0 \text{ and } r_i = r \text{ for all } i \notin V^I$,

where the terms V^I and $d(I)$ are defined in (3.5). The set V^I is a subset of $\mathbb{N}$ and consists of indices associated to the values of I (considering the usual valuation for the DVR $\overline{R}$); the non-negative invariant $d(I)$ is the difference between certain composition lengths associated with I.

These bounds for $b(I)$ extend the bounds obtained in [5] and [15] for $b(\mathfrak{C})$, which were mentioned above. The condition (VC) for $I = \mathfrak{C}$ yields that $b(\mathfrak{C}) = 0$ if and only if the type sequence is constant and equals $[r, r, ..., r]$.

In Section 2 we state preliminaries and notation; this includes properties of the canonical ideal and the definition of the type sequence. In Section 3, we undergo a thorough analysis of $a(I)$ and $b(I)$ as outlined above, and we obtain the quoted theorem, which establishes equivalences to the almost Gorenstein property. In Section 4 we give an example of application of the preceding results, specializing to the case where $I = \mathfrak{C}$. Under the same setting, these methods can be developed to classify all the domains having $b(\mathfrak{C}) \leq 3(r-1)$ (see [17]).

2 Preliminaries and notation.

Setting 2.1 Let $(R, \mathfrak{m})$ be a one-dimensional local Noetherian domain with residue field k and quotient field K. We assume throughout that the normalization $\overline{R}$ of R in K is a DVR and a finite R-module, i.e., R is analytically irreducible. Let $t \in \overline{R}$ be a uniformizing parameter for $\overline{R}$, so that $t\overline{R}$ is the maximal ideal of $\overline{R}$. We also suppose that the field k is isomorphic to the residue field $\overline{R}/t\overline{R}$, i.e., R is residually rational. We denote the usual valuation on K associated to $\overline{R}$ by v; that is, $v : K \longrightarrow \mathbb{Z} \cup \infty$, and $v(t) = 1$. In particular, $v(R) := \{v(a) \mid a \in R,\ a \neq 0\} \subseteq \mathbb{N}$ is the *numerical semigroup* of R. Then, since the conductor $\mathfrak{C} := (R :_K \overline{R})$ is an ideal of both R and $\overline{R}$, there exists a positive integer c so that $\mathfrak{C} = t^c\overline{R}$, $\ell_R(\overline{R}/\mathfrak{C}) = c$ and $c \in v(R)$. Furthermore, $(R :_K \mathfrak{C}) = \overline{R}$. We list the elements of $v(R)$ in order of size: $v(R) := \{s_i\}_{i \geq 0}$, where $s_0 = 0$ and $s_i < s_{i+1}$, for every $i \geq 0$. Let n be the positive integer so that $s_n = c$. For every $i \geq 0$, let R_i denote the ideal of elements whose values are bounded by s_i, that is,

$$R_i := \{a \in R \mid v(a) \geq s_i\}.$$

Notation 2.2 We assume Setting 2.1. The following is a list of symbols and relations to be used in the sequel. Some are repeated from above.

- $t \in \overline{R}$ is such that $t\overline{R}$ is the maximal ideal of $\overline{R}$ and $v(t) = 1$.
- $v(R) = \{v(a) \mid a \in R,\ a \neq 0\} =: \{s_i\}_{i \geq 0}$, where $0 = s_0 < s_1 < ....$
- $R_i := \{a \in R \mid v(a) \geq s_i\}$.
- $\mathfrak{C} := (R :_K \overline{R}) = t^c\overline{R}$, then $(R :_K \mathfrak{C}) = \overline{R}$.
- $\delta := \ell_R(\overline{R}/R)$, the *singularity degree* of R.
- $c := \ell_R(\overline{R}/\mathfrak{C})$.
- n is such that $s_n = c$, $\mathfrak{C} = R_n$, $n = \ell_R(R/\mathfrak{C}) = c - \delta$.
- $r := \ell_R(\mathfrak{m}^*/R)$, the *Cohen-Macaulay type* of R.

For fractional ideals I, J:

- $(I : J) := (I :_K J) = \{a \in K \mid aJ \subseteq I\}$.
- $I^* := (R : I)$.
- $\mathfrak{C}_I := (I : \overline{R})$, the largest $\overline{R}$-ideal contained in I.

Let I be a proper ideal of R and let $y \in I$ be such that $I\overline{R} = y\overline{R}$. Then:

- $a(I) := \ell_R(I^*/R) - \ell_R(R/I)$.
- $b(I) := r\ell_R(R/I) - \ell_R(I^*/R)$
- $e(I) := v(y)$, the *multiplicity* of I, so that $t^{e(I)}\overline{R} = I\overline{R}$.
- $e := e(\mathfrak{m})$, the *multiplicity* of R.
- $c(I) := \ell_R(\overline{R}/\mathfrak{C}_I)$, so that $t^{c(I)}\overline{R} = \mathfrak{C}_I$; $c \leq c(I)$ since $\mathfrak{C}_I \subseteq \mathfrak{C}$.
- n_I is such that $s_{n_I} = c(I)$, $\mathfrak{C}_I = R_{n_I}$, $n_I = \ell_R(R/\mathfrak{C}_I) = c(I) - \delta$.

- $h(I)$ is such that $s_{h(I)} = e(I)$, the first element of $v(I)$ and of $v(I^{**})$. Then $h(I) = |v(R) \cap [0, e(I) - 1]|$.
- $\overline{I} := I\overline{R} \cap R$, the *integral closure* of I.

From the definition of $\overline{I}$, it follows that

(2.2.1) $\qquad e(I) = e(\overline{I}) \qquad$ and $\qquad R_{h(I)} = \overline{I}$.

For a one-dimensional Cohen-Macaulay ring R with total ring of fractions K, a fractional ideal ω is a *canonical ideal* provided that ω contains a nonzero divisor and for every fractional ideal I which contains a nonzero divisor we have $I = (\omega : (\omega : I))$. For a one-dimensional local Cohen-Macaulay ring R a canonical ideal exists if and only if the completion $\widehat{R}_{\mathfrak{p}}$ is a Gorenstein ring for every minimal prime ideal $\mathfrak{p}$ of the completion $\widehat{R}$ of R with respect to its maximal ideal [9, Satz 6.21]. In our Setting 2.1 the completion $\widehat{R}$ of R with respect to its maximal ideal is reduced [12, Theorem 10.2], hence R has a canonical ideal ω, which is unique up to isomorphism [9, Satz 2.8]. The hypothesis R analytically irreducible assures that we can assume

$$R \subseteq \omega \subseteq \overline{R}$$

[10, Korollar 1].

By [13, Proposition 1], with this setting, given a pair of fractional nonzero ideals $I \supseteq J$, the hypothesis R residually rational allows us to compute the length of the R-module I/J by means of valuations:

(2.2.2) $$\ell_R(I/J) = |v(I) \setminus v(J)|.$$

In the following proposition we recall some well-known properties of the canonical ideal.

Proposition 2.3 *Let ω be a canonical ideal for R such that $R \subseteq \omega \subseteq \overline{R}$. Then:*

(1) $(\omega : \omega) = R$.

(2) $\ell_R(I/J) = \ell_R((\omega : J)/(\omega : I))$ *and* $\ell_R(J^*/I^*) = \ell_R((\omega I)/(\omega J))$ *for every pair of fractional ideals* $J \subseteq I$.

(3) R is Gorenstein if and only if $\omega^* = R$.
If R is not Gorenstein, then $\mathfrak{C} \subseteq \omega^* \subseteq \mathfrak{m}$.

(4) $v(\omega) = \{j \in \mathbb{Z} \mid c - 1 - j \notin v(R)\}$.
In particular $c - 1 \notin v(\omega)$ and $c + \mathbb{N} \subseteq v(\omega)$.

(5) For every fractional ideal I, $s \in v(I\omega)$ if and only if $c - 1 - s \notin v(R : I)$.

Proof. Item 1 and the first equality of (2) are in [9, Bemerkung 2.5]. It follows that $\ell_R(J^*/I^*) = \ell_R((\omega : \omega J)/(\omega : \omega I)) = \ell_R((\omega I)/(\omega J))$; hence (2) is clear. Since the assumption $R \subseteq \omega \subseteq \overline{R}$ implies that $\mathfrak{C} = (R : \overline{R}) \subseteq \omega^* \subseteq R$, part (3) is easily derived, recalling that R is Gorenstein if and only if $\omega = R$ [12, Theorem 13.1].

For items 4 and 5 see [10, Satz 5] and [15, Lemma 2.3]. $\diamond$

Remark 2.4 Let I be a proper ideal of R. The integral closure $\overline{I}$ and the bidual I^{**} of I satisfy the following relations:

(2.4.1) $\quad I \subseteq I^{**} \subseteq \overline{I}, \quad I^{**} \subseteq \omega I = \omega I^{**}, \quad e(I^{**}) = e(I)$, since $e(I) = e(\overline{I})$.

To see the non-obvious relations, $I^{**}= (R:(R:I)) \subseteq (\omega : (R:I)) = I\,\omega \subset I\overline{R}$, hence $I^{**} \subseteq \overline{I}$, and $\ell_R((\omega I^{**})/(\omega I)) = \ell_R(I^*/I^{***}) = 0$.

We note also:

(2.4.2) The condition $\omega \subseteq (I:I)$, i.e. $\omega I = I$, implies that $I = I^{**}$.

Now we recall the notion of *type sequence*, first introduced by Matsuoka in his 1971 paper [13] and recently revisited in [1].

Definition 2.5 *The ideals* R_i *defined in (2.1) give a strictly increasing sequence*
$R = R_0 \supset R_1 = \mathfrak{m} \supset R_2 \supset \ldots \supset R_n = \mathfrak{C} \supset R_{n+1} \supset \ldots$,
which induces the chain of duals:
$R \subset (R:R_1) \subset \ldots \subset (R:R_n) = \overline{R} \subset (R:R_{n+1}) = t^{-1}\overline{R} \subset \ldots.$
We put $r_i := l_R((R:R_i)/(R:R_{i-1})),\ i \geq 1,$ *and we call the finite sequence of integers* $[r_1, \ldots, r_n]$ *the type sequence of* R.

Example 2.6 Let $R = k[[t^5, t^8, t^{11}]]$, where k is a field and t an indeterminate. Then $\overline{R} = k[[t]]$, and $v(R) = \{0, 5, 8, 10, 11, 13, 15, 16, 18 \rightarrow\}$, so that
$\mathfrak{C} = t^{18}k[[t]],\ c = 18,\ n = 8,\ \delta = 10,\ e = 5.$
$v(\omega) = \{0, 3, 5, 8, 10, 11, 13, 14, 15, 16, 18 \rightarrow\}$. Hence
$\omega = R + t^3R$ and $r = l_R(\omega/(\mathfrak{m}\omega)) = 2$, by (2.3.2).
The type sequence is $[2, 1, 1, 1, 2, 1, 1, 1]$.
Consider now the proper ideal $I = (t^{10}, t^{13})$.
$v(I) = \{10, 13, 15, 18, 20, 21, 23, 24, 25, 26, 28 \rightarrow\}$, so that $e(I) = 10$,
$\overline{I} = t^{10}k[[t]] \cap R = (t^{10}, t^{11}, t^{13}),\ \mathfrak{C}_I = t^{28}k[[t]],\ c(I) = 28,\ n_I = 18,\ h(I) = 3,$
$v(I^*) = \{-5, -2, 0, 3, 5, 6, 8, 9, 10, 11, 12...\}$, hence
$a(I) = \ell_R(I^*/R) - \ell_R(R/I) = 8 - 8 = 0,$
$b(I) = r\ell_R(R/I) - \ell_R(I^*/R) = 8.$
$v(I^{**}) = \{10, 13, 15, 18, 20, 21, 23 \rightarrow\}$, hence $I^{**} = (t^{10}, t^{13}, t^{27})$; furthermore
$v(\omega I) = \{10, 13, 15, 16, 18, 20, 21, 23 \rightarrow\}$, and the inclusions $I \subseteq I^{**} \subseteq \omega I$ of (2.4.1) are strict.

We list some properties of type sequences, which are useful in the sequel.

Proposition 2.7 *Let* $r_i,\ n,\ c,\ \delta$ *be as above. Then:*

(1) The first element of the type sequence is the Cohen-Macaulay type r *of* R.

(2) $1 \leq r_i \leq r$ *for every* $i \geq 1$ *and* $r_i = 1$ *for every* $i > n$.

(3) $\delta = \sum_1^n r_i$.

(4) $2\delta - c = \ell_R(\omega/R) = \sum_1^n (r_i - 1) = \sum_1^\infty (r_i - 1)$.

(5) The elements of $v(\omega^*)$ *give rise to 1's in the type sequence:*
$s_i \in v(\omega^*) \implies r_{i+1} = 1.$

(6) $r_i = \ell_R((\omega R_{i-1})/(\omega R_i))$, *for every* i.

Proof. Items 1, 3, 4 follow directly from Definition 2.5. Property (2) follows from the next lemma. Item 5 is proved in [15, Proposition 3.4]. Item 6 is immediate, by (2.5) and (2.3.2). $\diamond$

Lemma 2.8 [10, Satz 2]. *Let* $(R, \mathfrak{m})$ *be a local Cohen-Macaulay ring of dimension one. Let* M, N, I, *be fractional ideals such that* $I \subseteq N$. *Then*

$$\ell_R((M : I)/(M : N)) \leq \ell_R((M : \mathfrak{m})/M) \cdot \ell_R(N/I).$$

Definition 2.9 *With Setting 2.1 and Notation 2.2, the ring* R *is said to have maximal length if* $r(c - \delta) = \delta$, *that is,* $rn = \delta$.

Remarks 2.10 (1) Using (2.7.4), we recover immediately the cases of minimal and maximal type sequence (see Definitions 1.4 and 2.9):

- R is almost Gorenstein if and only if the *type sequence* is $[r, 1, \ldots, 1]$.
- R is of maximal length if and only if the *type sequence* is constant: $[r, r, \ldots, r]$.

(2) By Equality (2.7.4), we have that $r - 1 \leq 2\delta - c$.

Next we include some relations involving the conductor of a proper ideal, the invariants r_i defined in (2.5) and some quantities from (2.2).

Proposition 2.11 *Let* I *be a proper ideal of* R *with conductor* $\mathfrak{C}_I = t^{c(I)}\overline{R} \subseteq I$. *Then:*

(1) $(R : \mathfrak{C}_I) = t^{c-c(I)}\overline{R}$, $c \leq c(I)$, *and* $v(R : \mathfrak{C}_I) = \mathbb{Z}_{\geq c-c(I)}$.

(2) $\sum_{i=1}^{n_I} r_i = \ell_R((R : \mathfrak{C}_I)/R) = c(I) - c + \delta$.

(3) $\sum_{i=1}^{n_I}(r_i - 1) = 2\delta - c$.

(4) $\ell_R(I^*/R) = \sum_{i=1}^{n_I} r_i - \ell_R((R : \mathfrak{C}_I)/I^*)$.

Proof. Using assertion (1), which is immediate, we obtain (2):

$$\ell_R((R : \mathfrak{C}_I)/R) = \ell_R((t^{c-c(I)}\overline{R})/\overline{R}) + \ell_R(\overline{R}/R) = c(I) - c + \delta.$$

Formula (3) comes directly from (2). From (2) and from the inclusions

$$\begin{array}{ccc} R & \subseteq & \overline{R} \\ \cap\!\!\!| & & \cap\!\!\!| \\ I^* & \subseteq & (R : \mathfrak{C}_I) \end{array}$$

we deduce equality (4). $\diamond$

3 Invariants a(I) and b(I).

The aim of the section is to express the invariants $a(I)$ and $b(I)$ defined in (1.3) in terms of the type sequence of R. The particular description given in Theorem 3.11 allows us to get bounds and vanishing conditions, improving results of several authors. First we collect some remarks concerning $a(I)$ and $b(I)$.

Throughout this section we let R denote a local ring as in Setting 2.1 and we use Notation 2.2.

Remarks 3.1 (1) We give the values of $a(I)$ and $b(I)$ in some special cases:
$I = \mathfrak{C} \implies a(\mathfrak{C}) = 2\delta - c, \quad b(\mathfrak{C}) = r(c - \delta) - \delta;$
$I = \mathfrak{m} \implies a(\mathfrak{m}) = r - 1, \quad b(\mathfrak{m}) = 0;$
$I = (f)$, a principal ideal with $v(f) = s \implies a(I) = 0, \quad b(I) = (r - 1)s$.
The statements for $\mathfrak{C}$ are immediate from (2.2). For $I = (f)$, it suffices to note that $l_R(I^*/R) = l_R((f^{-1}R)/R) = l_R(R/(f)) = l_R(\overline{R}/(f\overline{R})) = s$.

(2) In [10, Hilfssatz 1] it is shown that, for every proper ideal I,
$a(I) = a(\mathfrak{C}) - \ell_R((\omega I)/I) \leq a(\mathfrak{C})$.
As a consequence we have the following:
(a) $a(I) = 0$ for every proper ideal $I \iff R$ is Gorenstein.
(b) $a(\mathfrak{m}) = a(\mathfrak{C}) \iff R$ is almost Gorenstein (see Definition 1.4).

(3) There is a simple formula relating our invariants, which comes directly from the definitions: $a(I) + b(I) = (r - 1)\ell_R(R/I)$.

(4) The invariant $b(I)$ satisfies $b(I) \geq 0$ for every ideal I.
This fact follows by applying Jäger's inequality in Lemma 2.8 above with $M = N = R$.

(5) Let I, J be two proper ideals such that $J \subseteq I$. Then:
(a) $a(J) - a(I) = \ell_R(J^*/I^*) - \ell_R(I/J)$.
(b) $b(J) - b(I) = r\ell_R(I/J) - \ell_R(J^*/I^*) \geq 0$. In particular:
(c) $a(I) = a(I^{**}) - \ell_R(I^{**}/I)$.
(d) $b(I) = 0$ for every ideal I containing $\mathfrak{C}$ if and only if R is a ring of maximal length (see Definition 2.9).
A direct calculation gives assertion (a), hence (b) follows from equality (3). The positivity of $b(J) - b(I)$ is again a consequence of Jäger's result (2.8).

(6) Consider for $i \in \mathbb{N}$ the invariants r_i introduced in (2.5). By definition we have that $\sum_{h=1}^{i} r_h = \ell_R((R : R_i)/R)$. Therefore,
(a) $a(R_i) = \sum_{h=1}^{i}(r_h - 1)$; in particular, $a(R_i) = 2\delta - c$, for every $i \geq n$.
(b) $b(R_i) = \sum_{h=1}^{i}(r - r_h)$; in particular, $b(\mathfrak{C}) = \sum_{h=1}^{n}(r - r_h)$.
If $i \geq n$, then $b(R_i) = b(\mathfrak{C}) + (i - n)(r - 1)$. In fact,
$$b(R_i) = \sum_{h=1}^{n}(r - r_h) + \sum_{h=n+1}^{i}(r - r_h) = b(\mathfrak{C}) + (i - n)(r - 1), \text{ by (2.7.2).}$$

In the second part of the next proposition we improve the inequality $a(I) \leq (r - 1)l_R(R/I)$ for Arf rings. The term *Arf ring* originates with Lipman in [11], where the precise definition can be found. For the purposes of this article and this setting, the definition of *Arf* can be taken to be the characterization given by D'Anna and Delfino in [5, Proposition 1.15]. For each i with $1 \leq i \leq n$, let $\mathfrak{C}_i := ((R : R_i) : \overline{R})$ be the conductor of the ring $(R : R_i)$. Then the ring R is an Arf ring if and only if

(3.2) $\qquad \ell_R((R : R_i)/\mathfrak{C}_i) = \ell_R(R/\mathfrak{C}) - i$ for each i with $1 \le i \le n$.

Furthermore D'Anna and Delfino show for an Arf ring R

(3.3) $\qquad \ell_R(\overline{R}/\mathfrak{C}_i) = c - s_i \quad$ [5, Lemma 2.5].

Proposition 3.4 *The following facts hold.*

(1) R is Arf if and only if $\ell_R((R : R_i)/R) = s_i - i$ for every $1 \le i \le n$.

(2) If R is Arf, then for every proper ideal I of R

$$a(I) \le (r-1)\ell_R(R/I) - (e\,h(I) - e(I)).$$

Proof. We have $\overline{R} \supseteq (R : R_i) \supseteq R$ and $(R : R_i) \supseteq \mathfrak{C}_i$, and so $\ell_R((R : R_i)/R) = \ell_R(\overline{R}/R) - \ell_R(\overline{R}/\mathfrak{C}_i) + \ell_R((R : R_i)/\mathfrak{C}_i)$. Thus (1) holds.

We now assume R is Arf. Using the definition from (2.2), the item 1 above and $\ell_R(R/R_i) = i$, we have

$a(R_i) = \ell_R((R : R_i)/R) - \ell_R(R/R_i) = s_i - 2i.$

On the other hand, every Arf ring has maximal embedding dimension, or, equivalently, maximal Cohen-Macaulay type $r = e-1$ [11, Theorem 2.2]. Thus, using (3.1.4), we obtain, for all $i \ge 0$,

$$\begin{aligned} 0 \le b(R_i) &= r\ell_R(R/R_i) - \ell_R((R : R_i)/R) \\ &= (e-1)i - (s_i - i) = ei - s_i. \end{aligned}$$

Now, to show the inequality in item 2, we consider the ideals $I \subseteq R_{h(I)} = \overline{I}$, as in Notation 2.2. By (3.1.5.b), $b(I) - b(R_{h(I)}) \ge 0$; hence

$b(I) \ge b(R_{h(I)}) = e\,h(I) - s_{h(I)} = e\,h(I) - e(I) \ge 0,$

by the argument above, where $i = h(I)$, and so $s_i = e(I)$. We use Remark 3.1.3 to obtain the desired inequality. $\diamond$

We need now to introduce a new invariant $d(I)$ for every proper ideal I. It will be very useful in the next computations.

Notation 3.5 For I a proper ideal of R, let $n_I, \mathfrak{C}_I$ be as in (2.2). We set:

- $V^I := \{h+1 \mid h \in \mathbb{N} \text{ and } s_h \in v(I^{**})\}$.
- $d(I) := \ell_R((R : \mathfrak{C}_I)/I^*) - \sum r_h \mid h \in V^I \cap [1, n_I]$.

Remarks 3.6 (1) The number $h(I)+1$ (see Notation 2.2) is the first element in V^I, since $s_{h(I)} = e(I) = e(I^{**})$ as in (2.4.1). Also $n_I + 1 \in V^I$ since $s_{n_I} = c(I) \in v(I^{**})$.

(2) Note that $d(I)$ is an invariant for isomorphism classes, namely $d(I) = d(uI)$ for every unit $u \in \overline{R}$, since lengths can be computed using values as remarked in (2.2.2).

(3) The cardinality of the set V^I defined in (3.5) has a precise meaning in terms of lengths: $|V^I \cap [1, n_I]| = \ell_R(I^{**}/\mathfrak{C}_I)$, and $|V^I \cap [1, n]| = \ell_R((I^{**} + \mathfrak{C})/\mathfrak{C})$. Moreover, $|\mathbb{N} \setminus V^I| = \ell_R(R/I^{**})$.

(4) In a ring of maximal length $r_h = r$ for all $h \leq n$; hence for every proper ideal I we have $d(I) = \ell_R((R:\mathfrak{C}_I)/I^*) - r\ell_R(I^{**}/\mathfrak{C}_I)$.

(5) The inequalities: $|V^I \cap [1,n_I]| \leq \sum_{h \in V^I \cap [1,n_I]} r_h \leq r|V^I \cap [1,n_I]|$, valid by virtue of (2.7.2), imply that

$$\ell_R((R:\mathfrak{C}_I)/I^*) - r\ell_R(I^{**}/\mathfrak{C}_I) \leq d(I) \leq \ell_R((R:\mathfrak{C}_I)/I^*) - \ell_R(I^{**}/\mathfrak{C}_I).$$

Proposition 3.7 *Let R be as in Setting 2.1. For every proper ideal I we have the following relations:*

(1) $d(I) = \ell_R((\omega I)/I^{**}) - \sum (r_h - 1) \mid h \in V^I$.

In particular, if I is a principal ideal, then $d(I) = \sum (r_h - 1) \mid h \notin V^I$.

(2) $d(I^{**}) = d(I)$.

(3) If $I \subseteq \omega^*$, *then* $d(I) = \ell_R((\omega I)/I^{**})$.

(4) If $\omega \subseteq (I:I)$, *then* $d(I) = 0$.

(5) $d(I) = \sum r_h - \ell_R(I^*/R^*_{h(I)}) \mid h > h(I), h \notin V^I$.

(6) If I is integrally closed, then $d(I) = 0$.

Proof. Since $R \subseteq \omega \subseteq \overline{R}$, we have $\mathfrak{C}_I \subseteq \mathfrak{C}_I\omega \subseteq \mathfrak{C}_I\overline{R} = \mathfrak{C}_I$; thus $\mathfrak{C}_I\omega = \mathfrak{C}_I$. Now, by (2.3.2),

$$\ell_R((R:\mathfrak{C}_I)/I^*) = \ell_R((\omega I)/(\omega\mathfrak{C}_I)) = \ell_R((\omega I)/\mathfrak{C}_I).$$

Also $\mathfrak{C}_I \subseteq I \subseteq I^{**} \subseteq \omega I$, using (2.4.1). This implies

$$\begin{aligned} d(I) &= \ell_R((\omega I)/\mathfrak{C}_I) - \textstyle\sum r_h \mid h \in V^I \cap [1,n_I], \text{ from (3.5)},\\ &= \ell_R((\omega I)/I^{**}) + \ell_R(I^{**}/\mathfrak{C}_I) - \textstyle\sum r_h \mid h \in V^I \cap [1,n_I]\\ &= \ell_R((\omega I)/I^{**}) + |V^I \cap [1,n_I]| - \textstyle\sum r_h \mid h \in V^I \cap [1,n_I], \text{ using (3.6.3)},\\ &= \ell_R((\omega I)/I^{**}) - \textstyle\sum (r_h - 1) \mid h \in V^I \cap [1,n_I]\\ &= \ell_R((\omega I)/I^{**}) - \textstyle\sum (r_h - 1) \mid h \in V^I, \end{aligned}$$

where the last equality holds because $r_h = 1$ for all $h > n_I$, since $n_I \geq n$, by (2.7.2). If I is principal, then $d(I) = \ell_R(\omega/R) - \sum_{h \in V^I}(r_h - 1) = \sum_{h \notin V^I}(r_h - 1)$, by (2.7.4). Thus item 1 holds.

For item 2, recall that $\omega I = \omega I^{**}$, by (2.4.1), and that $V^I = V^{I^{**}}$ from the definition in (3.5). Now apply (1).

The assumption $I \subseteq \omega^*$ in (3) implies that $I^{**} \subseteq \omega^{***} = \omega^*$. Hence assertion (3) follows from (1) using (2.7.5).

To prove (4), observe that the inclusion $\omega \subseteq (I:I)$ implies $\omega I = I = I^{**}$, by (2.4), and also $I \subseteq \omega^*$; hence the conclusion by part (3).

After writing $\ell_R((R:\mathfrak{C}_I)/I^*) = \ell_R((R:\mathfrak{C}_I)/R^*_{h(I)}) - \ell_R(I^*/R^*_{h(I)})$, formula (5) becomes clear, since $\ell_R((R:\mathfrak{C}_I)/R^*_{h(I)}) = \sum r_h \mid h \in (h(I), n_I]$, by definition of the invariants r_h, and $(h(I), n_I] \setminus (V^I \cap [1,n_I]) = \{h \mid h > h(I), h \notin V^I\}$.

Since $I = \overline{I}$ means $I = R_{h(I)}$, the set $\{h > h(I), h \notin V^I\}$ is empty; hence the equality (5) readily implies (6). ◇

The basic idea for the next theorem comes from (2.3.5), which establishes a duality between the valuations of ωI and those of I^*.

Theorem 3.8 *Let R be as in Setting 2.1. For every proper ideal I we have:*

(1) $\ell_R(I^{**}/\mathfrak{C}_I) \leq \sum_{h \in V^I \cap [1,n_I]} r_h \leq \ell_R((R:\mathfrak{C}_I)/I^*)$.

(2) $\ell_R(I^*/R) \leq \sum_{h \notin V^I} r_h = \ell_R(R/I^{**}) + \sum_{h \notin V^I} (r_h - 1)$.

Proof. The proof is substantially the same as in [16, Proposition 4.2]; some changes are necessary, because we don't assume that I is a reflexive ideal containing the conductor $\mathfrak{C}$.

The first inequality of item 1 is immediate from (3.6.3). To prove the second inequality, suppose that $h \in V^I \cap [1, n_I]$. That is, by the definition in (3.5), $s_{h-1} \in v(I^{**})$ and $1 \leq h \leq n_I$. Choose $x_{h-1} \in I^{**}$ so that $v(x_{h-1}) = s_{h-1}$. Recall that $s_{h-1} < s_h \leq c(I)$, since $h-1 < h \leq n_I$. Now
$r_h = \ell_R((\omega R_{h-1})/(\omega R_h))$, by (2.7.6),
$= |(v(\omega R_{h-1}) \setminus v(\omega R_h))|$, by (2.2.2),
$= |(v(\omega x_{h-1}) \setminus v(\omega R_h))|$, since $R_{h-1} = \{a \in R \mid v(a) = s_{h-1}\} \cup R_h$.
Now $x_{h-1} \in I^{**}$, and so $v(x_{h-1}\omega) \subseteq v(\omega I^{**}) = v(\omega I)$, by (2.4.1).

Consider $Y = \bigcup_{h \in V^I \cap [1,n_I]} (v(x_{h-1}\omega + \omega R_h) \setminus v(\omega R_h))$, a disjoint union by definition. We define
$\varphi : Y \longrightarrow \mathbb{Z}_{\geq c-c(I)} \setminus v(I^*)$ via, for $y \in Y, \varphi(y) = c - 1 - y$;
this is well defined by (2.3.5). By (2.11.1), $\mathbb{Z}_{\geq c-c(I)} = v(R:\mathfrak{C}_I)$, and the result follows.

For part (2), use the second inequality in (1) combined with (2.11.4), to get:
$\ell_R(I^*/R) \leq \sum_{h \in [1,n_I]} r_h - \sum_{h \in V^I \cap [1,n_I]} r_h = \sum_{h \notin V^I} r_h$
$= \ell_R(R/I^{**}) + \sum_{h \notin V^I} (r_h - 1)$. ◇

Corollary 3.9 *For every proper ideal I we have:*

(1) $d(I) \geq 0$.

(2) $\ell_R((\omega I)/I) \geq \sum (r_h - 1) \mid h \in V^I$.

(3) $\ell_R(I/\mathfrak{C}_I) \leq \ell_R((R:\mathfrak{C}_I)/I^*)$.
Equality holds $\iff$ *I is reflexive, $d(I) = 0$ and $r_h = 1 \ \forall \ h \in V^I \cap [1, n_I]$.*

(4) Assume R is almost Gorenstein. Then $d(I) = 0$ if I is non-principal, $d(I) = r - 1$ otherwise.

Proof. The positivity of $d(I)$ is a consequence of the last inequality in (3.8.1).
For assertion (2), by combining (3.1.5) and part (2) of Theorem 3.8, we get

$a(I) \leq a(I^{**}) = \ell_R(I^*/R) - \ell_R(R/I^{**}) \leq \sum_{h \notin V^I}(r_h - 1)$.

Using now (3.1.2), we conclude that:

$$\ell_R((\omega I)/I) = 2\delta - c - a(I)$$
$$\geq \sum_{h=1}^{\infty}(r_h - 1) - \sum_{h \notin V^I}(r_h - 1) = \sum_{h \in V^I}(r_h - 1).$$

To prove (3), using $I \subseteq I^{**}$ and (3.8.1), consider the following chain of inequalities:

$$\ell_R(I/\mathfrak{C}_I) \leq \ell_R(I^{**}/\mathfrak{C}_I) \leq \sum_{h \in V^I \cap [1, n_I]} r_h \leq \ell_R((R : \mathfrak{C}_I)/I^*).$$

For the last statement, we note that the strict inclusion $(\mathfrak{m} : I) \subset (R : I)$ implies the existence of an element $x \in K$ such that $xI \subseteq R$, but $xI \not\subseteq \mathfrak{m}$, then $xI = R$, so I is a principal ideal. Therefore, the assumption I non-principal insures that $(R : I)I \subseteq \mathfrak{m}$.

Now, if R is almost Gorenstein and I is non-principal, then $\omega I = I^{**}$. In fact, as observed in (2.4), the inclusion $I^{**} \subseteq \omega I$ always holds. On the other hand, $(R : I)I\omega \subseteq \mathfrak{m}\omega = \mathfrak{m} \subset R$ implies $I\omega \subseteq I^{**}$. The conclusion $d(I) = 0$ follows from (3.7.1), combined with the fact that $d(I)$ is non-negative, as stated in (1). The case I principal comes directly from (3.7.1), because $r_h = 1$ for all $h \notin V^I, h \neq 1$. ◇

The next theorem extends to any birational overring S of R the formulas proved in [16] in the case of the blowing-up Λ of R along a proper ideal. We remark also that for $S = \overline{R}$ the first inequality $\ell_R(S/R) \leq r\ \ell_R(R/(R : S))$ becomes the well-known relation $\delta \leq r(c - \delta)$.

Theorem 3.10 *Let R be as in Setting 2.1. Let S be an overring of R with $R \subseteq S \subseteq \overline{R}$, and let $I := (R : S)$ be the conductor ideal of S into R. We have the following relations:*

(1) $\ell_R(S/R) = \sum_{h \notin V^I} r_h - \ell_R(S^{**}/S) - d(I) \leq r\ \ell_R(R/I)$.

(2) $\ell_R(S/R) = \sum_{h \leq h(I)} r_h - \ell_R(S^{**}/S) + \ell_R(S^{**}/R^*_{h(I)})$.

Proof. The hypothesis $R \subseteq S \subseteq \overline{R}$ ensures that the conductor $\mathfrak{C}_I$ of I equals the conductor $\mathfrak{C}$ of R. In fact, $\mathfrak{C}$ is an $\overline{R}$-ideal contained in I, so $\mathfrak{C} \subseteq \mathfrak{C}_I$ by the maximality of $\mathfrak{C}_I$ with respect to this property. Since the other inclusion obviously holds, we have $\mathfrak{C} = \mathfrak{C}_I$. Then the proof of [16, Theorem 4.4] works also in this general case, and we may omit the proof. ◇

From Theorem 3.8 we deduce the following two formulas which relate the invariants $a(I)$ and $b(I)$ with the type sequence.

Theorem 3.11 *For every proper ideal I of R we have:*

(1) $a(I) = \sum_{h \notin V^I}(r_h - 1) - \ell_R(I^{**}/I) - d(I)$.

(2) $b(I) = \sum_{h \notin V^I}(r - r_h) + r\ell_R(I^{**}/I) + d(I)$.

Proof. Using Notation 1.3 and (3.5) we can write:

$a(I) + d(I) + \ell_R(I^{**}/I) =$

$= \ell_R(I^*/R) - l_R(R/I) + \ell_R((R : \mathfrak{C}_I)/I^*) - \sum_{h\in V^I\cap[1,n_I]} r_h + \ell_R(I^{**}/I).$

By (2.11.4), $\ell_R(I^*/R) = \sum_{i\in[1,n_I]} r_i - \ell_R((R : \mathfrak{C}_I)/I^*)$ and, since $I \subseteq I^{**} \subseteq R$, we have $\ell_R(R/I) - \ell_R(I^{**}/I) = \ell_R(R/I^{**})$. Thus

$a(I) + d(I) + \ell_R(I^{**}/I) = \sum_{h\notin V^I} r_h - \ell_R(R/I^{**})$

$= \sum_{h\notin V^I}(r_h - 1)$, by (3.6.3).

This proves part (1).

Using the relation $a(I) + b(I) = (r - 1)\ell_R(R/I)$ from (3.1.3) and part (1), we can write:

$b(I) = (r-1)[\ell_R(R/I^{**})+\ell_R(I^{**}/I)]-[\sum_{h\notin V^I} r_h-\ell_R(R/I^{**})-\ell_R(I^{**}/I)-d(I)]$

$= r\ell_R(R/I^{**}) + r\ell_R(I^{**}/I) - \sum_{h\notin V^I} r_h + d(I),$

and so part (2) follows by (3.6.3). $\diamond$

Example 3.12 For the ideal $I = (t^{10}, t^{13})$ in the ring R of Example 2.6, we have $v(I^{**}) = \{s_3, s_5, s_6, s_8, s_{10}, s_{11}, s_{13} \rightarrow\}$; then

$V^I = \{4, 6, 7, 9, 11, 12, 14 \rightarrow\}$ and $V^I \cap [1, n] = \{4, 6, 7\}$.

Recall that the type sequence is $[2, 1, 1, 1, 2, 1, 1, 1]$.

Since $\sum(r_i - 1) \mid i \in V^I = 0$ and $v(\omega I)\setminus v(I^{**}) = \{16\}$, by (3.7.1) we obtain that

$d(I) = \ell_R((\omega I)/I^{**}) = 1.$

Also, $\sum(r_i - 1) \mid i \notin V^I = 2$ and $\ell_R(I^{**}/I) = 1$, since $v(I^{**})\setminus v(I) = \{27\}$; hence equalities in (3.11) are verified.

From Theorem 3.11, we immediately get interesting lower and upper bounds.

Corollary 3.13 *For a proper ideal I of R, set $q := |\{i \notin V^I \mid r_i = 1\}|$. The following inequalities hold:*

(1) $a(I) \leq (r - 1)[\ell_R(R/I^{**}) - q] - \ell_R(I^{**}/I) \leq (r - 1)\ell_R(R/I^{**}) - \ell_R(I^{**}/I).$

$a(I) \geq r - 1 - \ell_R(I^{**}/I) - d(I).$

In particular, if I is such that $\omega \subseteq (I : I)$, then $a(I) \geq r - 1$.

(2) $b(I) \leq (r - 1)(\ell_R(R/I) - 1) + \ell_R(I^{**}/I) + d(I).$

$b(I) \geq r\ell_R(I^{**}/I) + (r - 1)q.$

(3) (Vanishing condition)

$b(I) = 0 \iff I = I^{**},\ r_h = r$ *for every* $h \notin V^I$, *and* $d(I) = 0$.

Proof. To prove part (1), note that

$\sum_{h\notin V^I}(r_h - 1) = \sum(r_h - 1) \mid h \notin V^I, r_h \neq 1$, and

$|\{h \notin V^I \mid r_h \neq 1\}| = \ell_R(R/I^{**}) - q.$

Now the inequalities of (1) come directly from (3.11.1), recalling that $1 \leq r_h \leq r$ for all h, by (2.7.2) and $d(I) \geq 0$, by (3.9.1). When $\omega \subseteq (I : I)$, we have $d(I) = 0$, by (3.7.4) and $I^{**} = I$, by (2.4.2); hence $a(I) \geq r - 1$.

Since $a(I) + b(I) = (r-1)\ell_R(R/I)$, as observed in (3.1.3), statement (2) follows easily from (1).

Assertion (3) is an immediate consequence of (3.11.2). $\diamond$

By involving the inverse of the canonical ideal, we make the bounds in Corollary 3.13 more explicit:

Corollary 3.14 *If I satisfies the condition $v(\omega^* \cap I^{**}) = v(\omega^*) \cap v(I^{**})$, then:*

(1) $a(I) \leq (r-1)\ell_R(R/(I^{**} + \omega^*)) - \ell_R(I^{**}/I)$.

(2) $b(I) \geq (r-1)\ell_R((I^{**} + \omega^*)/I) + \ell_R(I^{**}/I)$.

Proof. Set $H := \{h+1 \mid s_h \in v(\omega^*) \setminus v(\omega^* \cap I^{**})\}$, so $r_i = 1$ for every $i \in H$ by (2.7.5). Also, $|H| = \ell_R(\omega^*/(I^{**} \cap \omega^*)) = \ell_R((I^{**} + \omega^*)/I^{**})$. Then

$$\begin{aligned}\textstyle\sum_{h \notin V^I}(r_h - 1) &= \textstyle\sum(r_h - 1) \mid h \notin V^I \cup H \\ &= \textstyle\sum(r_h - 1) \mid h \notin V^I \cup H, h \in [1,n], \text{ since } r_i = 1 \text{ for } i \geq n \\ &\leq (r-1)N, \text{ where } N := |[1,n] \setminus (V^I \cup H)|.\end{aligned}$$

Now, $N = n - |(V^I \cap [1,n]) \cup (H \cap [1,n])|$.

The assumption $v(\omega^* \cap I^{**}) = v(\omega^*) \cap v(I^{**})$ insures that $H \cap V^I = \emptyset$, and so

$$\begin{aligned}N &= n - |V^I \cap [1,n]| - |H \cap [1,n]| \\ &= \ell_R(R/\mathfrak{C}) - \ell_R((I^{**} + \mathfrak{C})/\mathfrak{C}) - \ell_R((I^{**} + \omega^*)/(I^{**} + \mathfrak{C})) \\ &= \ell_R(R/(I^{**} + \omega^*)).\end{aligned}$$

Thus (3.11.1) implies the inequality of (1).

As in the preceding corollary, we derive (2) from (1). $\diamond$

Remarks 3.15 (1) The upper bounds found in Corollary 3.13.1 improve the result $a(I) \leq (r-1)\ell_R(R/I)$ obtained by Jäger [10, Korollar 3], while the first inequality in (3.13.2) generalizes the upper bound $b(\mathfrak{C}) \leq (r-1)(\ell_R(R/\mathfrak{C}) - 1)$, already known for the conductor ideal [5, Proposition 2.1].

(2) The condition $v(\omega^* \cap I^{**}) = v(\omega^*) \cap v(I^{**})$ in Corollary 3.14 is satisfied for instance when $I \subseteq \omega^*$; in fact, $I \subseteq \omega^* \implies I^{**} \subseteq \omega^{***} = \omega^*$. In particular it holds for $I = \mathfrak{C}$, since $\mathfrak{C} \subseteq \omega^*$, so inequality (3.14.2) extends the lower bound $b(\mathfrak{C}) \geq (r-1)\ell_R(\omega^*/\mathfrak{C})$, stated in [15, Theorem 3.7].

(3) For $I = \mathfrak{C}$, using the second inequality in (3.13.2) we obtain

$b(\mathfrak{C}) \geq (r-1)q \geq (r-1)\ell_R(\omega^*/\mathfrak{C})$, where $q = |\{i \in [1,n] \mid r_i = 1\}|$.

Finally we obtain a characterization of the almost Gorenstein property, defined in (1.4), in terms of the invariant $a(I)$ (see next (1) $\iff$ (5)), which is just the analogue of a theorem stated by E. Matlis for Gorenstein rings [12, Theorem 13.1]. We recall that a fractional ideal I is said to be *reflexive* if it satisfies the condition $I = I^{**}$.

Theorem 3.16 *Here "ideal" means "fractional ideal". Let R be as in Setting 2.1 and let ω be a canonical ideal for R such that $R \subseteq \omega \subseteq \overline{R}$. The following facts are equivalent:*

(1) R is almost Gorenstein.

(2) $\ell_R(I\omega/I) = r - 1$ *for every principal ideal* I.

(3) $I\omega = I^{**}$ *for every non-principal ideal* I.

(4) For every pair of reflexive ideals I, J*, such that* $J \subseteq I$,

$$\ell_R(I/J) = \ell_R(J^*/I^*) + h(r-1), \quad \textit{where}$$

$$\begin{aligned} h &= \ \ 0 \iff I, J \textit{ are either both non-principal or both principal,} \\ h &= \ \ 1 \iff I \textit{ is non-principal and } J \textit{ is principal,} \\ h &= -1 \iff I \textit{ is principal and } J \textit{ is non-principal.} \end{aligned}$$

(5) $a(I) = (r-1) - \ell_R(I^{**}/I)$ *for every non-principal ideal* $I \subseteq R$.

Proof. To see the equivalence (1) $\iff$ (2), we observe that $\ell_R(I\omega/I) = \ell_R(\omega/R)$ for every principal ideal I. Since $\ell_R(\omega/R) = 2\delta - c$ by (2.7.4), the equivalence is immediate from Definition 1.4.

For (1) $\implies$ (3), see the proof of item 4 in Corollary 3.9, which is valid also for fractional ideals.

For the converse (3) $\implies$ (1), it suffices to put $I = \mathfrak{m}$ in (3), consequently $\mathfrak{m}\omega = \mathfrak{m}$. Therefore, R is almost Gorenstein by (1.4).

Now we show (1) $\implies$ (4). From the diagram

$$\begin{array}{ccc} I\omega & \supseteq & I \\ \cup\!\vert & & \cup\!\vert \\ J\omega & \supseteq & J \end{array}$$

we see that $\ell_R(I/J) = \ell_R((I\omega)/(J\omega)) - \ell_R((I\omega)/I) + \ell_R((J\omega)/J)$. Since

$$\ell_R(J^*/I^*) = \ell_R((I\omega)/(J\omega))$$

by (2.3.2), the conclusion follows by using items 2 and 3.

To prove the implication (4) $\implies$ (1), put $I = \mathfrak{m}$, $J = \mathfrak{C}$, and consequently $h = 0$, in the formula of item 4. Clearly

$$\ell_R(\mathfrak{m}/\mathfrak{C}) = c - \delta - 1, \quad \ell_R(\mathfrak{C}^*/\mathfrak{m}^*) = \delta - r,$$

and so we obtain $2\delta - c = r - 1$, which means R almost Gorenstein by (1.4).

It remains to prove that condition (5) is equivalent to the others. If R is an almost Gorenstein ring, then equality (5) holds for every non-principal ideal $I \subseteq R$, by Theorem 3.11.1, because in this hypothesis $r_h = 1$ for all $h \neq 1$, by (2.10.1), and $d(I) = 0$ by (3.9.4). Conversely, equality (5), with $I = \mathfrak{C}$, gives immediately that $r - 1 = 2\delta - c$. $\diamond$

4 The conductor case.

In the special case of the conductor ideal $\mathfrak{C}$, the description of the invariant $b(\mathfrak{C})$ in terms of type sequence given in (3.1.6.b),

$$b(\mathfrak{C}) = \sum_{h=1}^{n} (r - r_h),$$

is useful for the classification of one-dimensional analytically irreducible local rings having $b(\mathfrak{C})$ small enough. Results related to this problem that are already in the literature can be found in [5], [6], [7], [8], [18].

Delfino gives a characterization of rings satisfying the condition $b < r-1$ and a complete description of the value set of rings satisfying the condition $b \leq r$, under the additional assumption $r = e-1$ in [7, Corollary 2.11 and Corollary 2.14]. See also Proposition 2.7 from [5] for a further generalization. In the quoted paper [7] more attention is devoted to the invariant $\ell_R(R/(\mathfrak{C} + xR))$, where xR is a minimal reduction of $\mathfrak{m}$. In particular, it is proved that $b = r-1 \Longrightarrow \ell_R(R/(\mathfrak{C}+xR)) = 1$ or 2 [7, Proposition 2.4], and that $b = r-1$ and $\ell_R(R/(\mathfrak{C}+xR)) = 2 \Longrightarrow r = e-2$ [7, Corollary 2.13]. In [8] the authors show the inequality $r\ell_R(R/(\mathfrak{C}+xR)) \leq b+e-1$, which is improved by means of the type sequence in statement (4.3.1).

We fix the setting and notation for this section as follows:

Setting/Notation 4.1 We assume the setting of (2.1) and the notation of (2.2) and (2.5) as well as the following:

- $b := b(\mathfrak{C}) = r\ell_R(R/\mathfrak{C}) - \ell_R(\overline{R}/R)$.
- $x \in \mathfrak{m}$ is such that $v(x)$ is the multiplicity e; $\ell_R(R/xR) = e$ [11, Ch.1].
- $p \in \mathbb{N}$ is such that $c - e \leq pe < c$. $\quad (p{=}0 \iff c = e)$.
- $i_0 \in [1, n]$ is such that $s_{i_0-1} = min\{y \in v(R) \mid y \geq c-e\}$.
  $(i_0 = 1 \iff c = e)$.
- $B := [i_0, n], \quad A := [1, n] \setminus B$.

Lemma 4.2 *With notation as in (4.1),*

(1) $|B| = \ell_R((\mathfrak{C} :_R \mathfrak{m})/\mathfrak{C}) = \ell_R(R/(\mathfrak{C}+xR)) \geq e-r \geq 1$.

(2) $\sum_{h\in B} r_h \leq e-1$.

Proof. The following two observations are apparent from (2.2):

(i) $v(\mathfrak{C} :_R \mathfrak{m}) \setminus v(\mathfrak{C}) = \{s_i \in v(R) \mid c-e \leq s_i < c\}$.

(ii) The set $\{s_i \in v(R) \mid c-e \leq s_i < c\}$ is in 1-1 correspondence with the interval $[i_0-1, n-1]$.

Then

$$\begin{aligned}\ell_R((\mathfrak{C} :_R \mathfrak{m})/\mathfrak{C}) &= |v(\mathfrak{C} :_R \mathfrak{m}) \setminus v(\mathfrak{C})|, \text{ by (2.2.2)}\\ &= |[i_0-1, n-1]| = |B|, \text{ by (i) and (ii) above,}\end{aligned}$$

and so the first equality of item 1 is proved.

Claim: For x as in (4.1), $x(\mathfrak{C} :_R \mathfrak{m}) = xR \cap \mathfrak{C}$.

Proof of Claim: For "$\subseteq$", let $r \in (\mathfrak{C} :_R \mathfrak{m})$; now $x \in \mathfrak{m}$, and so $xr \in \mathfrak{C}$. For "$\supseteq$", using (4.1) and (2.2), $v(x) = e$ and $x\overline{R} = t^e\overline{R} = \mathfrak{m}\overline{R}$. If $r \in R$ with $xr \in \mathfrak{C}$, then $r\mathfrak{m} \subseteq rx\overline{R} \subseteq \mathfrak{C}\overline{R} = \mathfrak{C}$. Thus $r \in (\mathfrak{C} :_R \mathfrak{m}), xr \in x(\mathfrak{C} :_R \mathfrak{m})$, and the claim holds.

We obtain the equalities

$$\ell_R(R/(\mathfrak{C} :_R \mathfrak{m})) = \ell_R(xR/x(\mathfrak{C} :_R \mathfrak{m})) = \ell_R(xR/(xR\cap\mathfrak{C})) = \ell_R((\mathfrak{C}+xR)/\mathfrak{C}),$$

and using the following diagram

$$\begin{array}{ccc} (\mathfrak{C} :_R \mathfrak{m}) & \subseteq & R \\ \cup\!\mid & & \cup\!\mid \\ \mathfrak{C} & \subseteq & \mathfrak{C} + xR, \end{array}$$

we see immediately that $\ell_R((\mathfrak{C} :_R \mathfrak{m})/\mathfrak{C}) = \ell_R(R/(\mathfrak{C} + xR))$. Finally,

$x^{-1}\mathfrak{m}\mathfrak{C} \subseteq R \Longrightarrow (\mathfrak{C} + xR)\mathfrak{m} \subseteq xR$, so that $\mathfrak{C} + xR \subseteq (xR : \mathfrak{m})$. Hence

$\ell_R((\mathfrak{C} + xR)/xR) \leq \ell_R((xR : \mathfrak{m})/xR)$

$= \ell_R(R : \mathfrak{m})/R) = r$, since $(xR : \mathfrak{m}) = x(R : \mathfrak{m})$, and

$\ell_R(R/(\mathfrak{C} + xR)) = \ell_R(R/xR) - \ell_R((\mathfrak{C} + xR)/xR) \geq e - r$.

This completes the proof of (1).

We now prove part (2). Since $\omega R_{i_0-1} \subseteq \omega$ and $c - 1 \notin v(\omega)$, by (2.3.4), we have that $v(\omega R_{i_0-1})_{<c} \subseteq [c-e, c-2]$, so $|v(\omega R_{i_0-1})_{<c}| \leq e - 1$. Thus

$\sum_{h\in B} r_h = \sum_{h=i_0}^{n} r_h = \ell_R((\omega R_{i_0-1})/(\omega R_n))$, by (2.7.6)

$= \ell_R((\omega R_{i_0-1})/\mathfrak{C}) = |v(\omega R_{i_0-1})_{<c}| \leq e - 1$. ⋄

Next we give two formulas relating $b = b(\mathfrak{C})$ to the type sequence. They are important for further calculations. Inequality (2) improves [8, Theorem 2.3].

Theorem 4.3 *With the notation and setting as in (4.1), the following inequalities hold:*

(1) $b + e - 1 \geq b + \sum_{h\in B} r_h = \sum_{h\in A}(r - r_h) + r\ell_R(R/(\mathfrak{C} + xR))$.

(2) $b \geq (r-1)(e-r-1) + \sum_{h\in A}(r - r_h)$.

Proof. (1) The first inequality follows from (4.2.2). To complete the proof of (1), we see that

$b = \sum_{h=1}^{n}(r - r_h)$, by (3.1.6.b)

$= \sum_{h\in A}(r - r_h) + \sum_{h\in B}(r - r_h)$ from (4.1)

$= \sum_{h\in A}(r - r_h) + r\ell_R(R/(\mathfrak{C} + xR)) - \sum_{h\in B} r_h$, by Lemma 4.2, and so item 1 holds.

From the last equality and (4.2.2) we deduce part (2). In fact, recalling that $\ell_R(R/(\mathfrak{C} + xR)) \geq e - r$, we obtain

$b \geq \sum_{h\in A}(r - r_h) + r(e - r) - (e - 1)$, as desired. ⋄

Formula (4.3.1) suggests that the composition length of $R/(\mathfrak{C} + xR)$ is especially important in this context. The next lemma describes in detail the case $\ell_R(R/(\mathfrak{C} + xR)) = 1$; the cases of length ≥ 2 are treated in [17].

Lemma 4.4 *With the notation of (4.1), the following facts are equivalent:*

(1) $\ell_R(R/(\mathfrak{C} + xR)) = 1$.

(2) $v(R) = \{0, e, ..., pe, c \rightarrow\}$.

(3) The type sequence of R, defined in (2.5), is $[e-1, ..., e-1, r_n]$.

If R satisfies these equivalent conditions, then:

$\delta = c - p - 1, \quad b = e(p+1) - c \leq r - 1, \quad r = e - 1, \quad r_n = e - 1 - b.$

Proof. Clearly (1) holds $\iff \mathfrak{C}+xR=\mathfrak{m} \iff$ (2) holds, since $v(x)=e$, by (4.1). Also (1) $\Longrightarrow r=e-1$, by (4.2.1), and (2) $\Longrightarrow p=n-1$, i.e., $\delta=c-p-1$.

To see (2) $\Longrightarrow$ (3), first we note that

$(R:R_{n-1})=(R:(x^pR+\mathfrak{C}))=(x^{-p}R)\cap\overline{R}=x^{-p}(R\cap x^p\overline{R})$.

Now recalling Definition 2.5 we obtain:

$$\sum_{h=1}^{n-1} r_h=\ell_R((R:R_{n-1})/R)=\ell_R((R\cap x^p\overline{R})/x^pR)=\ell_R(R_{n-1}/x^pR)$$
$$=\ell_R(R/x^pR)-\ell_R(R/R_{n-1})=ep-p=r(n-1).$$

Hence $r_h=r$ for each $h=1,...,n-1$. Since $b=\sum_{h=1}^{n}(r-r_h)$, by (3.1.6.b), it follows immediately that $b=r-r_n$. Therefore, $b<r$ and the type sequence is $[e-1,....,e-1,e-1-b]$.

The assumption in (3) $r_h=e-1$, for $h\in[1,n]$, implies that $s_h=he$, by Proposition 4.9 of [15]. Hence (3) $\Longrightarrow$ (2) follows easily. ⋄

Lemma 4.5 *(1) If* $0\le b<r-1$, *then* $e-r=\ell_R(R/(\mathfrak{C}+xR))=1$.

(2) If $b=r-1>0$, *then there are two possibilities:*

(i) $e-r=\ell_R(R/(\mathfrak{C}+xR))=2$ *or* *(ii)* $e-r=\ell_R(R/(\mathfrak{C}+xR))=1$.

Proof. If $b<r-1$, from (4.3.2) we get $(r-1)(e-r-2)<0$, so $e-r<2$. Analogously, $b=r-1>0 \Longrightarrow e-r\le 2$. Now, in both cases we obtain

$r\ell_R(R/(\mathfrak{C}+xR))\le e+r-2$, by (4.3.1).

It follows that

$\ell_R(R/(\mathfrak{C}+xR))=1$, when $e=r+1$,

$\ell_R(R/(\mathfrak{C}+xR))\le 2$, when $e=r+2$.

By (4.4), $e-r=2 \Longrightarrow \ell_R(R/(\mathfrak{C}+xR))>1$, and so we are done. ⋄

By combining the above two lemmas, we deduce immediately the statements of the next theorem, which are partially already known (see [4], [7], [8], [6]). Nevertheless, in our setting, they give a complete characterization of all rings having $\ell_R(R/(\mathfrak{C}+xR))=1$.

Theorem 4.6 *Let R have the setting and notation of (4.1) and suppose that R is not Gorenstein. Let* $ts(R)$ *denote the type sequence of R. Then:*

(1) The following facts are equivalent:

(a) $b<r-1$.

(b) $v(R)=\{0,e,..,pe,c\rightarrow\}$ *with* $pe+2<c\le(p+1)e$.

(c) $ts(R)=[e-1,...,e-1,r_n]$, $r_n>1$.

If these conditions hold, then:

$\ell_R(R/(\mathfrak{C}+xR))=1,\quad c=(p+1)e-b,\quad r=e-1,\quad r_n=e-1-b.$

(2) The following facts are equivalent:

(d) $b=r-1$ *and* $\ell_R(R/(\mathfrak{C}+xR))=1$.

(e) $v(R)=\{0,e,..,pe,pe+2=c\rightarrow\}$.

(f) $ts(R)=[e-1,...,e-1,1]$.

Proof. For part (1), we begin by proving that (a) $\Longrightarrow$ (b). By (4.5.1), $\ell_R(R/(\mathfrak{C}+xR)) = 1$. Applying (4.4), we get

$v(R) = \{0, e, 2e, ..., pe, c \rightarrow\}$, with $(p+1)e \geq c$, by (4.1), and also

$r = e-1, \quad b = (p+1)e - c$.

Clearly the hypothesis $b < r-1$ gives $pe+2 < c$, and so (b) holds.

Applying again (4.4), we see that (b) implies that the type sequence of R is $[e-1, ..., e-1, r_n]$, and also $b = (p+1)e - c$.

Then the hypothesis $c > pe+2$ gives $b < e-2$; thus $r_n = e-1-b > 1$ and the proof of (b) $\Longrightarrow$ (c) is complete.

If (c) holds, then by (4.4), $r - r_n = b$, and so $r - b = r_n > 1$, i.e. the inequality of (a) holds.

Part (2) follows immediately by applying Lemma 4.4. $\diamond$

Note. A natural continuation is to classify singularities having $b \geq r-1$ and $\ell_R(R/(\mathfrak{C}+xR)) \geq 2$. This can be done, using the methods in this paper, until b reaches $3(r-1)$; for the proofs we refer to the separate paper [17], in preparation.

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References

[1] V. Barucci, D. E. Dobbs, M. Fontana, Maximality Properties in Numerical Semigroups and Applications to One-Dimensional Analytically Irreducible Local Domains, Mem. Amer. Math. Soc. vol. 125, n. 598 (1997).

[2] V. Barucci, R. Fröberg, One-Dimensional Almost Gorenstein Rings, Journal of Algebra 188 (1997) 418-442.

[3] R. Berger, Differentialmoduln eindimensionaler lokaler Ringe, Math. Zeitschr. 81, 326-354 (1963).

[4] W. C. Brown, J. Herzog, One Dimensional Local rings of Maximal and Almost Maximal Length, Journal of Algebra 151, 332-347 (1992).

[5] M. D'Anna, D. Delfino, Integrally closed ideals and type sequences in one-dimensional local rings, Rocky Mountain J. Math. 27, (4) (1997) 1065-1073.

[6] M. D'Anna, V. Micale, Construction of one-dimensional rings with fixed value of $t(R)\lambda_R(R/C) - \lambda_R(\overline{R}/R)$, International Journal of Commutative rings 2 (1) (2002).

[7] D. Delfino, On the inequality $\lambda(\overline{R}/R) \leq t(R)\lambda(R/C)$ for one-dimensional local rings, Journal of Algebra 169 (1994), 332-342.

[8] D. Delfino, L. Leer, R. Muntean, A length inequality for one-dimensional local rings, Comm. Algebra 28 (2000), 2555-2564.

[9] J. Herzog, E. Kunz, Der kanonische Modul eines Cohen-Macaulay Rings, Lecture Notes in Math. vol. 238, Springer, Berlin, (1971).

[10] J. Jäger, Längenberechnung und kanonische Ideale in eindimensionalen Ringen, Arch. Math. 29 (1977), 504-512.

[11] J. Lipman, Stable ideals and Arf rings, Amer. J. Math. 93 (1971), 649-685.

[12] E. Matlis, 1-Dimensional Cohen-Macaulay Rings, Springer-Verlag (1973).

[13] T. Matsuoka, On the degree of singularity of one-dimensional analytically irreducible noetherian rings, J. Math. Kyoto Univ. 11 (1971), 485-494.

[14] A. Oneto, E. Zatini, Type-sequences of modules, J. Pure Appl. Algebra, 160 (2001), 105-122.

[15] F. Odetti, A. Oneto, E. Zatini, Dedekind Different and Type Sequence, Le Matematiche, Vol.LV (2000), Fasc.II, 467-484.

[16] A. Oneto, E. Zatini, An application of type sequences to the blowing-up, Beiträge zur Algebra und Geometrie, 46 (2005), No. 2, 471-489.

[17] A. Oneto, E. Zatini, Classification of one-dimensional analytically irreducible local domains by a length inequality. In preparation.

[18] D. P. Patil, G. Tamone, On the length equalities for one-dimensional rings, J. Pure Appl. Algebra, 205 (2006), 266-278.