Dedekind Different and Type Sequence

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Abstract

Let R be a one-dimensional, local, Noetherian domain. We assume R analitycally irreducible and residually rational. Let ω be a *canonical module* of R such that $R \subseteq \omega \subseteq \overline{R}$ and let $\theta_D := R : \omega$ be the *Dedekind different* of R.

Our purpose is to study how θ_D is involved in the type sequence of R and to compare the type sequence of R with the type sequence of θ_D (for the notion of type sequence we refer to [11], [1] and [13]). These relations yield some interesting consequences.

1 Introduction

Let $(R, \mathfrak{m})$ be a one-dimensional, local, Noetherian domain and let $\overline{R}$ be the integral closure of R in its quotient field K. We assume that $\overline{R}$ is a DVR and a finite R-module, which means that R is analitycally irreducible. Let $t \in \overline{R}$ be a uniformizing parameter for $\overline{R}$, so that $t\overline{R}$ is the maximal ideal of $\overline{R}$. We also suppose R to be residually rational, i.e. $R/\mathfrak{m} \simeq \overline{R}/t\overline{R}$.

In our hypotheses there exists a *canonical module* of R unique up to isomorphism, namely a fractional ideal ω such that $\omega : (\omega : I) = I$ for each fractional ideal I of R. We can assume that $R \subseteq \omega \subset \overline{R}$.
The *Dedekind different of* R is the ideal $\theta_D := R : \omega$.

Let $\nu : K \longrightarrow \mathbb{Z} \cup \infty$ be the usual valuation associated to $\overline{R}$. The image $\nu(R) = \{\nu(x),\ x \in R,\ x \neq 0\} \subseteq \mathbb{N}$ is a numerical semigroup of $\mathbb{N}$.

The *multiplicity* of R is the smallest non-zero element e in $\nu(R)$. The *conductor* of $\nu(R)$ is the minimal $c \in \nu(R)$ such that every $m \geq c$ is in $\nu(R)$ and $\gamma := t^c\overline{R}$ is the *conductor ideal* of R. We denote by δ the classical *singularity degree*, that is the number of gaps of the semigroup $\nu(R)$ in $\mathbb{N}$.

We briefly recall the notion of *type sequence* given for rings in [11], recently revisited in [1] and extended to modules in [13].
Let $n = c-\delta$, and call $s_0 = 0, s_1, \dots, s_n = c$ the first $n+1$ elements of $\nu(R)$. Form the chain of ideals $R_0 \supset R_1 \supset R_2 \supset \dots \supset R_n$, where, for each i, $R_i := \{x \in R \ : \ \nu(x) \geq s_i\}$.
Note that $R = R_0$, $R_1 = \mathfrak{m}$, $R_n = \gamma$. Now construct the two chains:

$$R = R : R_0 \subset R : \mathfrak{m} \subset R : R_2 \subset \dots \subset R : R_n = \overline{R}$$
$$\theta_D = \theta_D : R_0 \subset \theta_D : \mathfrak{m} \subset \theta_D : R_2 \subset \dots \subset \theta_D : R_n = \overline{R}$$

For every $i = 1 \dots n$, define

$$r_i = l_R(R : R_i / R : R_{i-1}) = l_R(\omega R_{i-1} / \omega R_i),$$
$$t_i = l_R(\theta_D : R_i / \theta_D : R_{i-1}) = l_R(\omega^2 R_{i-1} / \omega^2 R_i).$$

The *type sequence* of R, denoted by $t.s.(R)$, is the sequence $[r_1, \dots, r_n]$.
The *type sequence* of θ_D, denoted by $t.s.(\theta_D)$, is the sequence $[t_1, \dots, t_n]$.
Observe that r_1 is the *Cohen Macaulay type* of R which is also the minimal number of generators of ω and that t_1 is the *C.M. type* of the R-module θ_D, or the minimal number of generators of ω^2. Moreover, for every i, we have $r_1 \geq r_i \geq 1$ and $t_1 \geq t_i \geq 1$ (see e.g. [13], Prop. 1.6, for all details).

We show in Prop. 3.4 that, if $s_i \in \nu(\theta_D)$, then the correspondent $r_i + 1$ is 1. Hence, denoting by p the number of 1 's in the type sequence of R, we get (see Prop. 3.7) the inequalities

$$\delta \leq (c-\delta)r_1 - p(r_1 - 1) \ \leq \ (c-\delta)r_1 - l_R(\theta_D / \gamma)(r_1 - 1)$$

which improve the well known formula $\delta \leq (c-\delta)r_1$ (see Remark 3.12).

A ring R is called *almost Gorenstein ring* if its type sequence is of the kind $[r_1, 1, \dots, 1]$; in the general case we focus our attention to the last i such that $r_i > 1$, and we show its special meaning related to the blowing up of the canonical module and to the Dedekind different (Prop.4.3).

We compare the two type sequences in several cases. For instance, in a ring R of CM type 2 they can be completely determined by using the Dedekind different (Prop. 4.10). Under suitable hypotheses we have that $r_i \leq t_i$, although this is not always true. We conjecture however that $r_1 \leq t_1$ always holds and we can prove this inequality in the following cases:

- R is almost Gorenstein (see Prop. 5.1);
- R has C.M. type 2, 3, $e-1$ (see Prop. 4.10, Corollary 3.9, Prop.4.9);
- $\theta_D = \gamma$ (see Prop. 4.8);
- R satisfies the inequality $l_R(R/\theta_D)(r_1 - 2) \leq 2\delta - c$ (see Prop. 4.11).

In section 5 some results are achieved for minimal and maximal type sequences. In particular in Prop. 5.1, we prove that R is a *almost Gorenstein ring*, (that is $t.s.(R)$ is minimal), if and only if $t.s.(\theta_D)$ is also minimal.
On the other side we prove in Prop. 5.4, that the $t.s.(R)$ is maximal, i.e. of the kind $[e-1, \dots., e-1, e-1-a]$ for some $a < e-2$ or of the kind $[e-1, \dots., e-1,\ 1]$ if and only if $t.s.(\theta_D)$ is maximal, i.e. of the kinds $[e, e, \dots., e, e-a]$, $[e, e, \dots., e,\ 1]$ respectively.

2 Preliminaries and remarks on the canonical module

A fractional ideal of the value semigroup $\nu(R)$ is a subset $H \subseteq \mathbb{Z}$ such that $H + \nu(R) \subseteq H$. We denote by $c(H)$ the *conductor* of H, which is the smallest integer $j \in H$ such that $j + \mathbb{N} \subseteq H$. The number $\delta(H) := \#[\mathbb{Z}_{\geq h_0} \setminus H]$ where $h_0 = min\{h \in H\}$ is the number of gaps of H. For any fractional ideal I of R, $\nu(I)$ is a fractional ideal of $\nu(R)$. Further we set:

$$c(I) := c(\nu(I)), \qquad \delta(I) := \delta(\nu(I)), \qquad c := c(R), \qquad \delta := \delta(R).$$

We point out the useful fact that, given two fractional ideals $I_1, I_2, I_2 \subseteq I_1$, the length of the R-module I_1/I_2 can be computed by means of valuations: $l_R(I_1/I_2) = \#[\nu(I_1) \setminus \nu(I_2)]$, (see [11], Proposition 1).

Now we collect some of the properties of the canonical module which are important in this context.

First we recall the following well-known:

Proposition 2.1 (see [8], [10], [12]) *Let ω be a canonical module of R such that $R \subseteq \omega \subseteq \overline{R}$ and let ω^{**} be its bidual, i.e. $\omega^{**} = R : (R : \omega)$. Then:*

1) $\omega : \omega = R$.
2) $l_R(I/J) = l_R(\omega : J / \omega : I)$.
3) $c(\omega) = c$ and $\nu(\omega) = \{j \in \mathbb{Z} \mid c - 1 - j \notin \nu(R)\}$.
4) $\omega : \overline{R} = \gamma$.
5) $\omega \subseteq \omega^{**} = \omega : \omega\, \theta_D$.
6) R is Gorenstein $\iff \omega = R \iff \theta_D = R \iff \omega = \omega^{**}$.
Hence: R not Gorenstein $\implies \gamma \subseteq \theta_D \subseteq \mathfrak{m}$.
7) If $S \supseteq R$ is an overring birational to R, then $\omega : S$ is a canonical module for S.

Lemma 2.2 *Let I be a fractional ideal of R.*

i) If $I \supseteq \gamma$ and $\nu(I) \subseteq \nu(\omega)$, then there exists a unit $u \in \overline{R}$ such that $uI \subseteq \omega$.
If $\nu(I) = \nu(\omega)$, then $uI = \omega$.

ii) There exists a unit $u \in \overline{R}$ such that $ut^{c-c(I)}I \subseteq \omega$.

<u>Proof</u>. *i)* We note that $I \supseteq \gamma \implies \omega : I \subseteq \overline{R} \implies (\omega : I)\overline{R} \subseteq \overline{R}$. The hypotheses $I \supseteq \gamma$ and $\nu(I) \subseteq \nu(\omega)$ imply that $c(I) = c$, hence $I : \overline{R} = \gamma$ and $l_R(\overline{R}/(\omega : I)\overline{R}) = l_R(I : \overline{R} / \omega : \overline{R}) = 0$. From the equality $\overline{R} = (\omega : I)\overline{R}$ we deduce that $\omega : I$ contains a unit u of $\overline{R}$ and $uI \subseteq \omega$. The second assertion is now immediate, since $l_R(\omega / uI) = \#[\nu(\omega) \setminus \nu(I)] = 0$.

ii) We can apply item *i)* to the fractional ideal $t^{c-c(I)}I$, because the conditions $t^{c-c(I)}I \supseteq \gamma$ and $\nu(t^{c-c(I)}I) \subseteq \nu(\omega)$ are satisfied. □

A strict connection between the value sets of θ_D and ω^2 is remarked by D'Anna in [5], Lemma 3.2. Part iii) of next lemma is a slight generalization of it.

Lemma 2.3 *Let I be a fractional ideal of R. Let $h, s \in \mathbb{Z}$, $h \geq 1$. Then:*

i) $\nu(\omega : I) = \nu(\omega) - \nu(I)$.

ii) $\nu(\omega : I) = \{y \in \mathbb{Z} \mid c - 1 - y \notin \nu(I)\}$.

iii) $s \in \nu(R : \omega^{h-1} I) \Longleftrightarrow c - 1 - s \notin \nu(\omega^h I)$.

In particular: $s \in \nu(\theta_D) \Longleftrightarrow c - 1 - s \notin \nu(\omega^2)$.

Proof. *i)* The proof given in [13], Prop. 2.4, works also under our assumptions.

ii) $\subseteq$ Using *i)*, we see that $y \in \nu(\omega : I) \Longrightarrow c - 1 - y \notin \nu(I)$, since $c - 1 \notin \nu(\omega)$.

$\supseteq$ Let $y \in \mathbb{Z}$ be such that $c - 1 - y \notin \nu(I)$, and let $z \in \nu(I)$. Again by *i)* we can prove that $y + z \in \nu(\omega)$. Now $c - 1 - (y + z) = (c - 1 - y) - z \notin \nu(R) \Longrightarrow y + z \in \nu(\omega)$.

iii) Observe that $R : \omega^{h-1} I = \omega : \omega^h I$, then apply *ii)*. $\square$

Lemma 2.4 *Let I be a fractional ideal of R and let $J := I : \omega$. Then*

i) J is a reflexive R-module, i.e. $J = R : (R : J)$.

ii) If J is not invertible, then $\mathfrak{m} : \mathfrak{m} \subseteq J : J$.

In particular, θ_D is reflexive and $\mathfrak{m} : \mathfrak{m} \subseteq \theta_D : \theta_D$.

Proof. *i)* The inclusion $J \subseteq R : (R : J)$ always holds.

To prove $\supseteq$, observe that $x(R : J) \subseteq R \Longrightarrow x(R : J)\ \omega \subseteq \omega \Longrightarrow x\omega \subseteq \omega : (R : J) = \omega : (\omega : J\omega) = J\omega \subseteq I \Longrightarrow x \in J$.

ii) It suffices to note that J not invertible $\Longrightarrow J(R : J) \neq R \Longrightarrow J(R : J) \subseteq \mathfrak{m} \Longrightarrow J : J = R : J(R : J) \supseteq R : \mathfrak{m} = \mathfrak{m} : \mathfrak{m}$. $\square$

In the last part of this section we point out how θ_D brings some relations with the bidual ω^{**} and the blowing up of the canonical module.

Denote by $B := \cup_{n=0,\ldots,\infty}\ \omega^n : \omega^n$ the *blowing up of the canonical module* of R (independent on the choice of ω). This overring has been studied recently in relation to almost Gorenstein rings (see [2], ch.3, [5], ch.3).

Remark 2.5 *The ring B satisfies the following properties:*

i) For $m >> 0$, $B = \omega^m : \omega^m = \omega^m$. (See [5], 3) .

ii) B is a reflexive R-module.
In fact $B = (\omega^m : \omega^{m-1}) : \omega$ and we can apply Lemma 2.4.

iii) $\gamma \subseteq R : B \subseteq \theta_D$.

iv) $\omega(R : B) = \omega : B = R : B$.
In fact $\omega(R : B) = \omega : (\omega : (\omega(R : B))) = \omega : B = \omega : \omega^{m+1} = R : \omega^m = R : B$.

v) $\theta_D : \theta_D \subseteq B$.
In fact $B = R:(R:B) = R:\omega(R:B) = \theta_D : (R:B) \supseteq \theta_D : \theta_D$.

Proposition 2.6 *The following facts hold:*

i) $\omega \subseteq \omega^{**} \subseteq \omega^2 \subseteq B \subseteq \overline{R}$.

ii) $l_R(\theta_D/\gamma) = l_R(\overline{R}/\omega^2)$.

iii) $l_R(\omega^2/\omega^{**}) = l_R(\omega\,\theta_D/\theta_D)$.

iv) If R is not Gorenstein, then:

$c(\omega^2) \leq c(\omega^{**}) \leq c-e$.

$c(\omega^2) = c-e \iff e \in \nu(\theta_D)$.

Proof. *i)* $\omega^{**} = R:(R:\omega) = \omega : \omega(\omega : \omega^2) \subseteq \omega : (\omega : \omega^2) = \omega^2$.

ii) Since $\omega : \gamma = \overline{R}$ and $\omega : \theta_D = \omega : (\omega : \omega^2) = \omega^2$, using the second property in Prop. 2.1, we get the thesis.

iii) is immediate by Prop. 2.1.

iv) $j \geq c-e \Longrightarrow c-1-j \leq e-1 \Longrightarrow$ either $c-1-j=0$ or $c-1-j \notin \nu(R)$. Hence $j \in \nu(\omega) \cup \{c-1\} \subseteq \nu(\omega^{**})$.

Finally observe that $e \in \nu(\theta_D) \iff c-1-e \notin \nu(\omega^2)$ by Lemma 2.3. □

Since a ring is Gorenstein if and only if $B = \omega$, it is now natural to set a characterization for the condition $B = \omega^2$. The condition is always verified by almost Gorenstein rings (see [2], Prop. 28). We point out that there exist not almost Gorenstein rings with $B = \omega^2$, for instance the semigroup ring $R = \mathbb{C}[[t^h]]$, $h \in \nu(R) = \{0, 7, 8, 9, 11, 13, \rightarrow\}$.

Proposition 2.7 *The following conditions are equivalent:*

i) ω^{**} *is a ring.*
ii) $\omega^{**} = \omega^2$.
iii) $\omega\,\theta_D = \theta_D$.
iv) $\theta_D : \theta_D = B$.
v) $R:B = \theta_D$.
vi) $B = \omega^2$.

Proof. *i)* $\Longrightarrow$ *ii)*. In this hypothesis: $\omega \subseteq \omega^{**} \subseteq \omega^2 \subseteq \omega\,\omega^{**} = \omega^{**}$.

ii) $\Longrightarrow$ *iii)* is immediate by Prop. 2.6.

iii) $\Longrightarrow$ *iv)* $\omega\,\theta_D = \theta_D \Longrightarrow \omega^m\theta_D = \theta_D \Longrightarrow B \subseteq \theta_D : \theta_D$ and the other inclusion always holds (see Remark 2.5).

iv) $\Longrightarrow$ *v)* $\theta_D : \theta_D = B \Longrightarrow B\,\theta_D \subseteq R \Longrightarrow \theta_D \subseteq R:B$ and the other inclusion always holds (see Remark 2.5).

v) $\Longrightarrow$ *vi)* $\theta_D = \omega : \omega^2 = R:B = \omega : B\,\omega = \omega : B \Longrightarrow$
$\omega : (\omega : \omega^2) = \omega : (\omega : B)$.

vi) $\Longrightarrow$ *i)* $\omega^3\theta_D = \omega^2\,\theta_D \subseteq \omega \Longrightarrow \omega^2 \subseteq \omega : \omega\,\theta_D = \omega^{**} \Longrightarrow \omega^{**} = B$. □

3 Type-sequences and length.

The number p of 1's in $t.s.(R)$, is related to the length of the $R/\mathfrak{m}$-algebra R/θ_D and is involved in other interesting inequalities. First we show (Prop. 3.4) how elements of $\nu(\theta_D)$ give rise to 1's in $t.s.(R)$, and in $t.s.(\theta_D)$. From this we get $\delta \leq (c-\delta)r_1 - p(r_1-1) \leq (c-\delta)r_1 - l_R(\theta_D/\gamma)(r_1-1)$ (Prop. 3.7) and we state other bounds.

Proposition 3.1 (see [5]) *Let* $\nu(R) = \{s_0 = 0, s_1, ...s_n = c, \rightarrow\}$, $n = c-\delta$, *and let* $t.s.(R) = [r_1, ..., r_n]$ *and* $t.s.(\theta_D) = [t_1, ..., t_n]$ *be the type sequences of* R *and* θ_D *respectively. Then:*

i) $c(\theta_D : R_i) = c(R : R_i) = c - s_i$, *for each* $i = 0, ..., n$.

ii) $\nu(\theta_D : R_i)_{<c-s_i} = \{c-1-b,\ b \in \mathbb{Z}_{\geq s_i} \backslash \nu(\omega^2 R_i)\}$, *for each* $i = 0, ..., n$.

iii) Let $n_i := c(R : R_i) - \delta(R : R_i)$ *and let* $m_i := c(\theta_D : R_i) - l_R(\overline{R}/\theta_D : R_i)$, *then:*

1. $r_{i+1} = s_{i+1} - s_i + n_{i+1} - n_i, \qquad i = 0, ..., n-1.$
2. $t_{i+1} = s_{i+1} - s_i + m_{i+1} - m_i, \qquad i = 0, ..., n-1.$
3. $\sum_{i=1}^{n} r_i = \delta.$
4. $\sum_{i=1}^{n} t_i = \delta + l_R(R/\theta_D).$

iv) Denoting by ω_i *the canonical module* $\omega : (R : R_i)$ *of the overring* $R : R_i$ *obtained by duality, we have:* $r_i = l_R(\omega_i/\omega_{i-1})$.

Proof. By Lemma 2.3 we have that: $x \in \nu(\theta_D : R_i) \iff c-1-x \notin \nu(\omega^2 R_i)$.

$i)$ If $j \geq c - s_i \implies c-1-j < s_i \implies c-1-j \notin \nu(\omega^2 R_i) \implies j \in \nu(\theta_D : R_i) \subseteq \nu(R : R_i)$. Moreover $s_i \in \nu(\omega R_i) \implies c - s_i - 1 \notin \nu(R : R_i)$ by Lemma 2.3.

$ii)$ follows from the above considerations.

$iii)$ For the first equality see [5]. The second one is analogous:
by definition and item $i)$, $m_{i+1} = c - s_{i+1} + l_R(\overline{R}/\theta_D : R_{i+1})$ and $m_i = c - s_i + l_R(\overline{R}/\theta_D : R_i)$. Since $l_R(\overline{R}/\theta_D : R_i) - l_R(\overline{R}/\theta_D : R_{i+1}) = l_R(\theta_D : R_{i+1}/\theta_D : R_i) = t_{i+1}$, we get the thesis by subtraction. The other equalities are immediate by definition.

$iv)$ Apply Prop. 2.1, 7): $\omega_i = \omega : (R : R_i) = \omega : (\omega : \omega R_i) = \omega R_i$. □

Proposition 3.2 *Let* $t.s.(R) = [r_1, ..., r_n]$ *and* $t.s.(\theta_D) = [t_1, ..., t_n]$. *Let* $x_{i-1} \in \mathfrak{m}$ *be such that* $\nu(x_{i-1}) = s_{i-1} < c$. *Then:*

i) $r_i = 1 \iff x_{i-1} \in Ann_R\big(\omega/(x_{i-1}R + \omega R_i)\big)$.

ii) $r_i = 1 \implies t_i = 1$.

Proof. $i)$ Since $R_{i-1} = x_{i-1}R + R_i$, we have $\omega R_{i-1} = x_{i-1}\omega + \omega R_i$. Then $r_i = l_R(\omega R_{i-1}/\omega R_i) = 1 \iff \omega R_{i-1} = x_{i-1}R + \omega R_i \iff$ $x_{i-1} \in Ann_R\big(\omega/(x_{i-1}R + \omega R_i)\big)$.

$ii)$ By hypothesis $\omega R_{i-1} = x_{i-1}R + \omega R_i \implies \omega^2 R_{i-1} = x_{i-1}\omega + \omega^2 R_i$, hence by $i)$, $\omega^2 R_{i-1} = x_{i-1}R + \omega^2 R_i \implies t_i = l_R(\omega^2 R_{i-1}/\omega^2 R_i) = 1$. □

Lemma 3.3 ([5], Lemma 4.1) *Let $z_1, ..., z_r$ be any minimal set of generators of ω. Then, if $x_i \in R$ and $\nu(x_i) = s_i$, the R-module $\omega R_i/\omega R_{i+1}$ is generated by $x_i z_1 + \omega R_{i+1}, ..., x_i z_r + \omega R_{i+1}$.*

Proposition 3.4 *Let $t.s.(R) = [r_1, ..., r_n]$ and $t.s.(\theta_D) = [t_1, ..., t_n]$ be the type sequences of R and θ_D respectively. Then:*

$$s_i \in \nu(\theta_D) \implies r_{i+1} = t_{i+1} = 1.$$

Proof. $r_{i+1} = l_R(\omega R_i/\omega R_{i+1})$. Let $\omega = (1, z_2, ..., z_r)$ and let $x_i \in \theta_D$ be such that $\nu(x_i) = s_i < c$. Then $\omega R_i =< x_i, ..., x_i z_r >$ mod ωR_{i+1}, by Lemma 3.3. Thus $x_i \in R : \omega \implies x_i z_j \in R_{i+1} \subseteq \omega R_{i+1}$ for all $j > 1$ (since $\nu(x_i z_j) > i) \implies r_{i+1} = 1$ and by Prop. 3.2 $t_{i+1} = 1$. □

Notation 3.5 *We put:*

$p := \#\ [i \in \{1, ..., c-\delta\} \mid r_i = 1]$

$\sigma := l_R(\omega/R) - l_R(R/\theta_D) = 2\delta - c - l_R(R/\theta_D)$

The invariant σ has been introduced in [9]. It is known that $\sigma(R) \geq 0$, when $r_1 \leq 3$ or R is smoothable, but there are examples with $\sigma < 0$ (see 4.12).

Lemma 3.6 *The following facts hold:*

i) $l_R(\theta_D/\gamma) \leq p$.

ii) $c - \delta - p \leq l_R(R/\theta_D) \leq c - \delta$.

iii) $3\delta - 2c \leq \sigma \leq 3\delta - 2c + p$.

iv) $c - p \leq \sum_{i=1}^{n} t_i \leq c$.

Proof. $i)$ follows from Prop. 3.4.

$ii)$ First inequality comes from $i)$, since $l_R(R/\theta_D) = l_R(R/\gamma) - l_R(\theta_D/\gamma)$; the second one holds since $\gamma \subseteq \theta_D$.

$iii)$ is obvious by $ii)$.

$iv)$ $l_R(R/\theta_D) + \delta = \sum_{i=1}^{n} t_i$, so the inequalities are immediate from $ii)$. □

Proposition 3.7 *Let p be the number defined in 3.5. Then:*

$$2(c-\delta) - p \ \leq\ \delta\ \leq\ (c-\delta)r_1 - p(r_1 - 1)\ \leq\ (c-\delta)r_1 - l_R(\theta_D/\gamma)(r_1 - 1).$$

Proof. Since $r_{i_1} = \ldots = r_{i_p} = 1$, and $r_i \leq r_1 \ \ \forall i$, using Prop. 3.1, *iii*) we get:

$$c-\delta+(c-\delta-p) \ \leq \ \delta = \sum_{1}^{c-\delta} r_i = c-\delta+\sum_{1}^{c-\delta}(r_i-1) \leq c-\delta+(c-\delta-p)(r_1-1).$$

To get the last inequality use Lemma 3.6, *i*). □

Corollary 3.8 *Let, as above,* $n = c-\delta$. *Then:*

i) $2\delta - c = \sum_{i=1}^{n}(r_i-1) \leq (c-\delta-p)(r_1-1) \leq l_R(R/\,\theta_D\,)(r_1-1)$.

ii) $2\delta - c \leq l_R(R/\,\theta_D\,)(t_1-2)$.

Proof. *i*) See the proof of Prop. 3.7, then use Lemma 3.6, *ii*).
ii) As in the proof of Prop. 3.7, using Prop. 3.1 and Prop. 3.2, we obtain: $2\delta - c + l_R(R/\,\theta_D\,) = \sum_{i=1}^{n}(t_i-1) \leq (c-\delta-p)(t_1-1) \leq l_R(R/\,\theta_D\,)(t_1-1)$. □

Corollary 3.9 *Either* $t_1 = 1$ *(i.e.* R *is Gorenstein) or* $t_1 \geq 3$.

From the first inequality of Prop. 3.7 we deduce the following

Corollary 3.10 $p \geq 2c - 3\delta$.

Of course, the above lower bound for p is significant in the case $2c - 3\delta > 0$. Using *iii*) of Lemma 3.6 we see that if $\sigma < 0$, then $2c-3\delta > 0$. Example 5 in 4.12 shows that the converse is false. The following bound for $l_R(R/\,\theta_D\,)$ is non trivial when $\sigma < 0$ (see Example 4 in 4.12).

Proposition 3.11 $l_R(R/\,\theta_D\,) \leq (2\delta - c)(r_1-1)$.

Proof. Let $\omega = (1, z_2, ..., z_{r_1})R$ and consider, as in [10], Satz 3), for every $i = 1, ..., r_1$ the R-module $\omega_i := (1, ..., z_i)R$. In particular ω_2 is two-generated, so by [3], Satz 2, $l_R(R/R:\omega_2) = l_R(\omega_2/R)$. It is clear that $\omega_{i+1}/\omega_i \simeq R/\mathfrak{b}_{i+1}$, where $\mathfrak{b}_{i+1} = Ann_R(\omega_{i+1}/\omega_i)$. By [10], Hilfssatz 4 and Satz 1 we obtain: $l_R(R:\omega_i/R:\omega_{i+1}) \leq l_R(R:\mathfrak{b}_{i+1}/R) \leq l_R(R/\mathfrak{b}_{i+1}) + 2\delta - c = l_R(\omega_{i+1}/\omega_i) + 2\delta - c$. Since $R = R:\omega_1 \supset R:\omega_2 \supset \ldots. \supset R:\omega_{r_1} = \theta_D$, we have $l_R(R/\,\theta_D\,) = l_R(R/R:\omega_2) + \sum_{i=2}^{r_1-1} l_R(R:\omega_i/R:\omega_{i+1}) \leq$
$l_R(\omega_2/R) + \sum_{i=2}^{r_1-1} l_R(\omega_{i+1}/\omega_i) + (2\delta-c)(r_1-2) = l_R(\,\omega\,/R) + (2\delta-c)(r_1-2)$. The thesis follows. □

Remark 3.12 The difference $a := (c-\delta)r_1 - \delta$ has been taken into account by several authors. In [10] it is proved that $a \geq 0$, when R is a one-dimensional local analytically unramified Cohen Macaulay ring. In [11] it had already been shown that $a \geq 0$, under more particular hypotheses. In [4] some general stucture theorems are presented for rings with $a = 0$ (the so called rings *of maximal length*) or $a = 1$ (the so called rings *of almost maximal length*).

Proposition 3.7 implies that $a \geq l_R(\,\theta_D\,/\gamma)(r_1-1)$. Hence:

$a < r_1 - 1 \implies \theta_D\, = \gamma$.
$a = r_1 - 1 \implies l_R(\,\theta_D\,/\gamma) \leq 1$.

The cases $a \leq r_1 - 1$ are studied in [6] and [7]. See also the following 5.2.

4. Relations between r_i's and t_i's.

Starting from the almost Gorenstein case, we are led to consider in a $t.s.$ $[r_1, \ldots, r_i, 1, 1, \ldots, 1]$ the index i of the last element r_i which is not 1. This number has a central role in Prop. 4.3 which involves R_i, θ_D and B. When $i = 1$, this proposition gives again the known characterizations of almost Gorenstein rings.

Lemma 4.1 *Let J be any proper ideal of R. If $\nu(R_i) \subseteq \nu(J)$, then $R_i \subseteq J$.*

Proof. In fact $\nu(R_i) \subseteq \nu(J) \implies \nu(R_i \cap J) = \nu(R_i) \implies R_i \cap J = R_i \implies$
$\implies R_i \subseteq J$. $\square$

Lemma 4.2 *The following facts hold:*

i) $r_{i+1} > 1 \implies c - 1 \in \nu(\omega^2 R_i)$.

ii) $c - 1 \in \nu(\omega^2 R_i) \iff R_i \not\subseteq \theta_D$.

iii) If $r_n > 1$, *then* $t_n \geq r_n + 1$.

Proof. $i)$ By Prop. 3.4, $r_{i+1} > 1 \implies s_i \notin \nu(\theta_D) \implies$
$c - 1 - s_i \in \nu(\omega^2) \setminus \nu(\omega) \implies c - 1 = s_i + (c - 1 - s_i) \in \nu(\omega^2 R_i)$.

$ii)$ By Lemma 2.3 $c-1 \in \nu(\omega^2 R_i) \iff 0 \notin \nu(R : \omega R_i)$. Suppose $c-1 \in \nu(\omega^2 R_i)$. If $R_i \subseteq \theta_D$, then $1 \in \theta_D : R_i = R : \omega R_i$, contradiction. Vice versa, if $R_i \not\subseteq \theta_D$, by Lemma 4.1 there exists an element $x \in R_i \setminus \theta_D$ such that $\nu(x) \notin \nu(\theta_D)$; then $u\, x\, \omega \not\subseteq R$ for all units $u \in \overline{R}$.
It follows that $0 \notin \nu(R : \omega R_i)$.

$iii)$ We have: $r_n = l_R(\omega R_{n-1}/\omega R_n) = l_R(\omega R_{n-1}/\gamma) \leq$
$l_R(\omega^2 R_{n-1}/\gamma) = l_R(\omega^2 R_{n-1}/\omega^2 R_n) = t_n$. Looking at valuations we see that the above inequality is strict since $c - 1 \in \nu(\omega^2 R_{n-1}) \setminus \nu(\omega R_{n-1})$, by $i)$. $\square$

In [2] it is proved that

$$R \text{ is almost Gorenstein} \iff \mathfrak{m} = \omega\, \mathfrak{m} \iff r_1 - 1 = 2\delta - c.$$

Hence: R almost Gorenstein, not Gorenstein $\iff \theta_D = \mathfrak{m}$. In other words:
$$t.s.(R) = [r_1, \ldots, 1] \text{ with } r_1 > 1 \iff R_1 \subseteq \theta_D \text{ and } R_0 \not\subseteq \theta_D.$$
Next proposition is a generalization of this fact.

Proposition 4.3 *Let $1 \leq i \leq n$ and let $B = \omega^m$ be the blowing up of the canonical module of R. The following are equivalent:*

i) $R_i \subseteq \theta_D$ *and* $R_{i-1} \not\subseteq \theta_D$.

ii) $B \subseteq R : R_i$ *and* $B \not\subseteq R : R_{i-1}$.

iii) $t.s.(R) = [r_1, \ldots, r_i, 1, 1, \ldots, 1]$ *with* $r_i > 1$.

iv) $t.s.(\theta_D) = [t_1, \ldots, t_i, 1, 1, \ldots, 1]$ *with* $t_i > 1$.

Proof. $i) \iff ii)$ $R_i \subseteq \theta_D \iff \omega R_i = R_i \iff \omega^m R_i = R_i \iff B \subseteq R : R_i$.

$i) \implies iii)$ By hypothesis $s_j \in \nu(\theta_D)$ $\forall j \geq i \implies r_j = 1$ $\forall j > i$. We have to prove that $r_i > 1$. If $r_i = 1$, then by Prop. 3.2, $i)$, $\omega R_{i-1} = x_{i-1}R + \omega R_i \subseteq R \implies R_{i-1} \subseteq \theta_D$, absurd.

$iii) \implies iv)$ $r_i = l_R(\overline{R}/R : R_{i-1}) - l_R(\overline{R}/R : R_i) = l_R(\overline{R}/R : R_{i-1}) - (n-i)$ and analogously, by Prop. 3.2, $ii)$, $t_i = l_R(\overline{R}/\theta_D : R_{i-1}) - (n-i) \implies t_i \geq r_i > 1$.

$iv) \implies iii)$ If $i = n$, the implication is true by Prop. 3.2, $ii)$. Let $i \leq n-1$. Surely, by Prop. 3.2, $r_i > 1$ and by Lemma 4.2, $iii)$, $r_n = 1$. If $r_j > 1$ with $i < j < n$ and all the subsequents equal to 1, as above we would get $t_j \geq r_j > 1$, contradiction.

$iii) \implies i)$ $r_n = 1 \implies \omega R_{n-1} = x_{n-1}R + \gamma \subseteq R \implies R_{n-1} \subseteq \theta_D$. If also $r_{n-1} = 1$, then $\omega R_{n-2} = x_{n-2}R + \omega R_{n-1} \subseteq R$, then $R_{n-2} \subseteq \theta_D$ and so on. If $R_{i-1} \subseteq \theta_D$, then $r_i = 1$, and this concludes the proof. □

Proposition 4.4 *If $i \leq n-1$ is such that $r_i > 1$ and $r_j = 1$ for all j, $i+1 \leq j \leq n$, then*

$$t_i = r_i + 1.$$

Proof. By Prop. 4.3 we have $R_i \subseteq \theta_D$, hence $r_i = l_R(\omega R_{i-1}/R_i)$ and $t_i = l_R(\omega^2 R_{i-1}/R_i)$. Since, by Lemma 4.2, $i)$, $c-1 \in \nu(\omega^2 R_{i-1})$, our thesis will follow by proving that $\nu(\omega^2 R_{i-1}) = \nu(\omega R_{i-1}) \cup \{c-1\}$. Hence, let $m \in \nu(\omega^2 R_{i-1}) \setminus \nu(\omega R_{i-1})$: we claim that $m = c-1$.
By Lemma 2.3 $c-1-m \in \nu(R : R_{i-1})$. Let $m = \nu(x)$, $x \in \omega^2 R_{i-1}$ and $c-1-m = \nu(y)$, $y \in R : R_{i-1}$. If $\nu(y) > 0$, then $yR_{i-1} \subseteq R_i$, hence $c-1 = \nu(xy) \in \nu(\omega^2 R_i) = \nu(R_i)$, absurd. Hence $\nu(y) = 0$ and the thesis is achieved. □

Proposition 4.5 *The following are equivalent:*

i) $s_{n-1} \in \nu(\theta_D)$.

ii) $s_{n-1} = c-2$.

iii) $r_n = 1$.

Proof. Recall that $\omega R_n = \gamma$.

$i) \implies ii)$. If $c-2 \notin \nu(R)$, then $1 \in \nu(\omega)$. But this would imply that s_{n-1} and $s_{n-1}+1 \in \nu(\omega R_{n-1}) \setminus \nu(\gamma) \implies r_n > 1 \implies s_{n-1} \notin \nu(\theta_D)$, absurd.

$ii) \implies iii)$ Obviously $\nu(\omega R_{n-1}) \setminus \nu(\gamma) = \{s_{n-1}\}$. □

Corollary 4.6 $B = \overline{R} \iff r_n > 1$.

Proof. $B = \overline{R} \iff 1 \in \nu(\omega) \iff c-2 \notin \nu(R)$. □

Corollary 4.7 *If $\theta_D = R_i$ for some i, then the equivalent conditions of Proposition 2.7 hold.*

Proof. $B \subseteq R : R_i$ by Prop. 4.3 $\implies R : B \supseteq R_i = \theta_D \implies R : B = \theta_D$, since the other inclusion is always true. $\square$

In the particular case $\theta_D = R_n$ we obtain:

Proposition 4.8 *Set, as above, $n_i := c(R : R_i) - \delta(R : R_i)$ and $m_i := c(\theta_D : R_i) - l_R(\overline{R} / \theta_D : R_i)$. The following facts are equivalent:*

i) $\theta_D = \gamma$.

ii) $\omega^2 = \overline{R}$.

iii) $t_i = s_i - s_{i-1}$ *for each* $i = 1, ..., n$.

iv) $m_i = 0$ *for each* $i = 0, ..., n$.

v) $\theta_D : R_i = t^{c-s_i}\,\overline{R}$ *for each* $i = 0, ..., n$.

vi) $\omega^{**} = \overline{R}$.

If the above conditions hold, then

a) $t_1 = e$.

b) $\forall\, i > 1, \quad r_i > t_i \iff n_i > n_{i-1}$.

Proof. $i) \iff ii)$ See Prop. 2.6, $ii)$.

$ii) \implies iii)$ In fact $t_i = l_R(\omega^2 R_i / \omega^2 R_{i-1}) = l_R(R_i \overline{R} / R_{i-1} \overline{R}) = s_i - s_{i-1}$.

$iii) \implies iv)$ We have seen in Prop. 3.1 that $t_i = s_i - s_{i-1} + m_i - m_{i-1}$. Hypothesis $iii)$ implies that $m_1 = m_2 = ... = m_n = c(\overline{R}) - \delta(\overline{R}) = 0$.

$iv) \implies v)$ $m_i = 0 \implies \nu(\theta_D : R_i) = [c - s_i, +\infty)$. Since the inclusion $t^{c-s_i}\,\overline{R} \subseteq \theta_D : R_i$ holds for every $i = 0, ..., n$, the equality of the value sets implies the other inclusion.

$v) \implies i)$ Take in v) $i = 0$.

$vi) \implies ii)$ and $i) \implies vi)$ are immediate by Prop. 2.6.

$a)$ $t_1 = s_1 - s_0 = e$.

$b)$ Using Prop. 3.1 iii), it is immediate. $\square$

Our conjecture $t_1 \geq r_1$ is true for rings having maximal C.M. type, namely $r_1 = e - 1$. In this case we get a more precise result.

Proposition 4.9 *Let $e \geq 3$. If for some $1 \leq i \leq n$ $r_i = e - 1$, then $t_i = e$. Moreover, for the same i we have: $s_{i-1} = (i-1)e, \quad s_i = ie$.*

Proof. Since $t^e R_{i-1} \subseteq R_i \subset R_{i-1}$, we have the chain $t^e\,\omega\, R_{i-1} \subseteq \omega\, R_i \subseteq \omega\, R_{i-1}$.
Hypothesis $r_i = e - 1$ implies that $l_R(\omega\, R_i / t^e\,\omega\, R_{i-1}) = 1$ and since $c - 1 + e \in \nu(\omega R_i) \setminus \nu(t^e \omega R_{i-1})$, it follows that

$$(*) \qquad \omega R_i = t^e \omega R_{i-1} + zR \quad \text{with } \nu(z) = c - 1 + e.$$

Analogously, considering the chain $t^e\,\omega^2 R_{i-1} \subseteq \omega^2 R_i \subseteq \omega^2 R_{i-1}$, we see that the thesis $t_i = e$ is equivalent to $t^e \omega^2 R_{i-1} = \omega^2 R_i$. It will be sufficient to prove this last equality. From $(*)$ we have $\omega^2 R_i = t^e \omega^2 R_{i-1} + z\omega$. Now,

$z \in \gamma \subseteq R_i$ for every $i \implies z\omega \subseteq \omega R_i \implies \omega^2 R_i = t^e \omega^2 R_{i-1} + zR$. By Lemma 4.2 $r_i > 1 \implies c-1 \in \nu(\omega^2 R_{i-1})$, then $\nu(z) \in \nu(t^e \omega^2 R_{i-1})$: we obtain that $t^e \omega^2 R_{i-1} = \omega^2 R_i$, as claimed.

To prove the other equalities, note that by definition $s_i \leq s_{i-1} + e$. As already remarked $r_i = e-1$ implies that $\nu(\omega R_i) = \nu(t^e \omega R_{i-1}) \cup \{c-1+e\}$. Hence $s_i \in \nu(t^e \omega R_{i-1})$, but $s_i \geq s_{i-1} + e \implies s_i = s_{i-1} + e = ie$. □

For rings of C.M. type 2, we have a complete description of the type sequences of R and θ_D. In this case the arrow $\implies$ of Prop. 3.4 becomes $\iff$.

Proposition 4.10 *Suppose* $r_1 = 2$. *Then:*

$$s_i \in \nu(\theta_D) \implies r_{i+1} = t_{i+1} = 1$$

$$s_i \notin \nu(\theta_D) \implies r_{i+1} = 2,\ t_{i+1} = 3.$$

Proof. We have from Corollary 3.8, i) and Prop. 3.11 that $l_R(R/\theta_D) = 2\delta - c$ hence $l_R(\theta_D/\gamma) = 2c - 3\delta$. The elements of the type sequence $[r_1, ..., r_n]$, $n = c - \delta$, of R are 1 or 2, suppose p times 1 and $n-p$ times 2. Then $\delta = \sum_{i=1}^n r_i = p + 2(n-p) \implies p = 2c - 3\delta$. Hence $p = l_R(\theta_D/\gamma)$ and $r_{i+1} = 1 \iff s_i \in \theta_D$ (see Prop. 3.4). By hypothesis ω is two-generated, say $\omega = (1, z)$, then $1, z, z^2$ constitue a system of generators for ω^2 ; hence $t_1 \leq 3$, and Corollary 3.9 implies that $t_1 = 3$. Consider now the type sequence of θ_D, by Prop. 3.2, $r_i = 1 \implies t_i = 1$. Suppose that for some i either $t_i = 2$ or $r_i = 2$ and $t_i = 1$. Then $\delta + l_R(R/\theta_D) = \sum_{i=1}^n t_i < l_R(\theta_D/\gamma) + 3 l_R(R/\theta_D) \implies$ $\delta < c - \delta + 2\delta - c$, absurd. The thesis follows. □

Another case in which our conjecture $t_1 \geq r_1$ is true comes directly from Corollary 3.8:

Proposition 4.11 *If* $l_R(R/\theta_D)(r_1 - 2) \leq 2\delta - c$, *then* $r_1 \leq t_1$.

Proof. If $r_1 > t_1$, from Corollary 3.8, *ii*), we get $2\delta - c \leq l_R(R/\theta_D)(t_1 - 2) < l_R(R/\theta_D)(r_1 - 2)$. □

Example 4.12 Suppose $R = \mathbb{C}[[t^h]]$, $h \in \nu(R)$, is a semigroup ring. The first three examples show that the converses of Prop. 3.2, ii), Prop. 3.4 and Prop. 4.9 are false.

1. Let $\nu(R) = \{0, 10, 11, 17, 20 \rightarrow\}$, then $\theta_D = \gamma$, $\delta = 16$, $c - \delta = 4 < 12 = 2\delta - c$, $t.s.(R) = [7, 2, 5, 2]$, $t.s.(\theta_D) = [10, 1, 6, 3]$.
In this case $t_2 = 1$ and $r_2 > 1$.

2. Let $\nu(R) = \{0, 5, 6, 10 \rightarrow\}$, then $\theta_D = \gamma$, $\delta = 7$, $c - \delta = 3 < 4 = 2\delta - c$, $t.s.(R) = [3, 1, 3]$, $t.s.(\theta_D) = [5, 1, 4]$. In this case $t_2 = r_2 = 1$. But $s_1 = 5 \notin \nu(\theta_D)$.

3. Let $\nu(R) = \{0, 10, 11, 12, 14, 17, 20 \rightarrow\}$. Then: $c = 20$, $\delta = 14$, $r_1 = 5$, $\omega = \langle 0, 1, 3, 4, 6 \rangle$, $\omega^2 = \overline{R}$, hence $\theta_D = \gamma$. $t.s(R) = [5, 1, 1, 3, 2, 2]$, $t.s.(\theta_D) = [10, 1, 1, 2, 3, 3]$. In this case $t_1 = 10$, but $r_1 = 5 < e - 1$, moreover $r_4 > t_4 = 2$.

4. Let $\nu(R) = \langle 13, 121, 133, 163, 164, 166, 168, 170, 171\rangle$. We have $\delta = 181$, $c = 322$, $r_1 = 4$, $\theta_D = \langle 121, 166, 168, 198, 216, 223, 234, 241, 248, 266\rangle$.
   Hence $l_R(R/\theta_D) = 43$ and $\sigma = -3$.
   Here bound in Prop. 3.11 is better than bound in Lemma 3.6, $ii)$. In fact:
   $2\delta - c = 40 < l_R(R/\theta_D) = 43 < (2\delta - c)(r_1 - 1) = 120 < c - \delta = 141.$
   The type sequences $t.s.(R)$ and $t.s.(\theta_D)$ are respectively:
   [4 4 4 4 4 3 2 2 2 2 2 1 2 2 1 2 1 1 1 1 1 2 1 1 1 1 2 2 1 1 2 1 1 1 1 2 2 1 1 2 1 1
   1 1 2 2 1 1 2 1 1 1 1 2 2 1 1 2 1 1 1 1 2 1 1 1 2 1 1 1 1 1 2 1 ... 1]
   [10 10 10 10 8 6 3 3 3 3 3 1 3 2 1 3 1 1 1 1 1 2 1 1 1 1 3 2 1 1 2 1 1 1 1 3 2 1 1 2 1 1
   1 1 3 2 1 1 2 1 1 1 1 3 2 1 1 2 1 1 1 1 3 1 1 1 2 1 1 1 1 3 1 ... 1]

5. Let $\nu(R) = \{7, 8, 9, 10, 12 \rightarrow\}$. We have $\delta = 7$, $r_1 = 3$, $c = 12$. and R is almost Gorenstein, so $\theta_D = \mathfrak{m}$, hence $\sigma = 1$, but $3\delta - 2c < 0$.

5. Minimality and maximality.

In the comparison between the type sequences of the ring and of the Dedekind different, properties like minimality and maximality are completely equivalent.

• **Minimal type sequences.** In [2] one can find the properties of *almost Gorenstein* rings. Analogous properties for fractional ideals are considered in [13]: a fractional ideal I is called of *minimal type sequence* (*m.t.s.* for short) if and only if $t.s.(I) = [r(I), 1, ..., 1]$, where $r(I)$ is the Cohen Macaulay type of I as an R-module. Since it is well known that $r(I) = 1 \iff I \simeq \omega$, it follows in particular that $t_1 = 1 \iff R$ is Gorenstein.

Next proposition deals with the m.t.s. property in the not Gorenstein case.

Proposition 5.1 *Let R be not Gorenstein. The following are equivalent:*

$i)$ *R is almost Gorenstein.*

$ii)$ *θ_D is m.t.s.*

$iii)$ $\omega^{**} = R : \mathfrak{m}$.

$iv)$ $B = R : \mathfrak{m}$.

In this case $t_1 = r_1 + 1$.

Proof. $i) \iff ii)$ is equivalence $iii) \iff iv)$ of Prop. 4.3 for $i = 1$.

$i) \Longrightarrow iii)$ is immediate, since when R is almost Gorenstein, we have $\theta_D = \mathfrak{m} = \mathfrak{m}\,\omega$ and by Prop. 2.6 $\omega^{**} = \omega^2 = R : \mathfrak{m}$. Last equality is proved in [2], Prop. 28.

$iii) \Longrightarrow iv)$ ω^{**} is a ring $\Longrightarrow \omega^{**} = \omega^2 = B$ by Prop.2.7.

$i) \iff iv)$ has been proved by D'Anna in [5], Prop.3.4. □

• **Maximal type sequences.** Recalling that in general $t.s.(R) = [r_1, ..., r_n]$, with $r_1 \leq e - 1$ and $r_i \leq r_1$, of course "maximal" type sequence means $t.s.(R) = [e-1, ..., e-1]$. In [7] and [6] the authors characterize all the rings whose type sequence is closer to the maximal one in the following sense:

$t.s.(R) = [e-1, ..., e-1, e-1-a]$. For simplicity, we call *a-maximal* a type sequence of this form.

Proposition 5.2 *(See [6] and [7]). Let* $a \in \mathbb{N}$ *be such that* $a \leq r_1 - 1$*. The following facts are equivalent:*

i) $(c-\delta)r_1(R) - \delta = a$ *and* $r_1 = e-1$.

ii) $\nu(R) = \{0, e, 2e, ..., (n-1)e, ne-a, \rightarrow\}$.

iii) $t.s.(R) = [e-1, ..., e-1, e-1-a]$.

Moreover, if $a \leq r_1 - 2$*, then condition* $r_1 = e-1$ *in i) is superflous.*

We want to show now that the a-maximality of $t.s.(R)$ is equivalent to the a-maximality of $t.s.(\theta_D)$, i.e. $t.s.(\theta_D) = [e, ..., e, e-a]$, (see Prop. 5.4). To do this we need some more or less well known results, that we list below for our convenience.
In the following $\langle l_1, ..., l_i \rangle$ denotes the $\nu(R)$-set generated by $l_1, ..., l_i$ and, for any numerical set $H \subset \mathbb{Z}$, $H + l := \{h + l,\ h \in H\}$.

Lemma 5.3 *Let* $0 \leq a \leq e-2$ *and let* $\nu(R) = \{0, e, 2e, ..., (n-1)e, ne-a, \rightarrow\}$. *In this case* $c = ne - a$, $n = c - \delta$.

i) Canonical ideals:
For $a = 0$ *then* $\nu(\omega) = \langle 0, 1, 2, ..., e-2 \rangle$. *Call it* $\nu(\omega_0)$.
For any $a \geq 1$, *change the last* a *generators by addying* 1 *to each one, i.e.* $\nu(\omega_a) = \langle 0, 1, ..., e-a-2, e-a, ..., e-1 \rangle$.
In particular, $\nu(\omega_{e-2}) = \langle 0, 2, 3, ..., e-1 \rangle$.

ii) Type sequence of R :
$t.s.(R) = [e-1, ..., e-1, e-1-a]$.

iii) Omega square:
for $a = 0, ..., e-3$ $\quad \omega^2 = \overline{R}$,
for $a = e-2$ $\quad \nu(\omega^2) = \{0, 2, \rightarrow\}$.

iv) Type sequence of θ_D :
for $a = 0, ..., e-3$ $\quad t.s.(\theta_D) = [e, e, ..., e, e-a]$,
for $a = e-2$ $\quad t.s.(\theta_D) = [e, e, ..., e,\ 1]$.

v) Dedekind different:
for $a = 0, ..., e-3$ $\quad \theta_D = \gamma$,
for $a = e-2$ $\quad \theta_D = zR + \gamma$ *with* $\nu(z) = (n-1)e$.

Proof. *i)* Just remember that $\nu(\omega) = \{j \in \mathbb{Z} \mid c-1-j \notin \nu(R)\}$.

ii) For every $a = 0, ..., e-2$ and for every $i = 0, ..., n-1$, we have $\nu(\omega R_i) = \nu(\omega) + ie$. Then for every $i = 0, ..., n-2$,
$$\nu(\omega R_i) \setminus \nu(\omega R_{i+1}) = \{0, 1, ..., e-a-2, e-a, ..., e-1\} + ie.$$
So we obtain that $r_{i+1} = l_R(\omega R_i / \omega R_{i+1}) = e-1$.

Let now $i=n-1$. By definition $r_n=\#[\nu(\omega\, R_{n-1})\backslash\nu(\gamma)]$. Since $\nu(\omega\, R_{n-1})=\nu(\omega\,)+(n-1)e=\langle(n-1)e,(n-1)e+1,....,ne-a-2,ne-a,....,ne-1\rangle$, we see that only the first $e-a-1$ elements are smaller than $c=ne-a$ and we conclude that $r_n=e-a-1$.

iii) For $a=0,....,e-3$ we see that $1\in\nu(\omega\,)$, then $\omega^2=\overline{R}$.
For $a=e-2$, by item *i)* $\omega=\langle 0,2,3,....,e-1\rangle$, then $\omega^2=\{0,2,\rightarrow\}$.

iv) For $a=0,....,e-3$ and for $i=0,....,n-2$, using *iii)* we get

$$t_{i+1}=l_R(R_i\,\overline{R}\,/R_{i+1}\,\overline{R}\,)=e.$$

For $a=e-2$ and for $i=0,....,n-2$, we have $\nu(\omega^2\,R_i)\setminus\nu(\omega^2\,R_{i+1})=\{0,2,....,e-1,e+1\}+ie$ and we get again $t_{i+1}=e$.
It remains to compute the last component $t_n=\#[\nu(\omega^2\,R_{n-1})\setminus\nu(\gamma)]$. For $a=0,....,e-3$, $\nu(\omega^2\,R_{n-1})=\nu(R_{n-1}\,\overline{R}\,)=\{(n-1)e,\rightarrow\}$; in this set the elements $<c$ are $e-a$, so $t_n=e-a$. For $a=e-2$, we have by *i)* $r_n=1$, then by Prop. 3.2 also $t_n=1$.

v) The thesis follows from *iii)*, by applying Lemma 2.3. $\square$

Proposition 5.4 *Let* $e\geq 3$.

i) For $0\leq a<e-2$,
$t.s.(R)=[e-1,....,e-1,e-1-a]\iff t.s.(\,\theta_D\,)=[e,e,....,e,e-a]$.

ii) $t.s.(R)=[e-1,....,e-1,\ 1]\iff t.s.(\,\theta_D\,)=[e,e,....,e,\ 1]$.

Proof. Both implications $\Longrightarrow$ follow from Prop.5.2 and Lemma 5.3.

i) $\Longleftarrow$ Suppose $0\leq a<e-2$ and $t.s.(\,\theta_D\,)=[e,e,....,e,e-a]$. By Lemma 4.2 $r_n=\delta-\sum_{i=1}^{n-1}r_i<e-a$ and by hypothesis $\delta+l_R(R/\,\theta_D\,)=ne-a$. Then $ne-a-l_R(R/\,\theta_D\,)-\sum_{i=1}^{n-1}r_i<e-a\Longrightarrow\sum_{i=1}^{n-1}r_i>(n-1)e-l_R(R/\,\theta_D\,)=(n-1)(e-1)+(n-l_R(R/\,\theta_D\,))-1$, i.e. $\sum_{i=1}^{n-1}r_i\geq(n-1)(e-1)+(n-l_R(R/\,\theta_D\,))$.

On the other hand $\sum_{i=1}^{n-1}r_i\leq(n-1)r_1\leq(n-1)(e-1)$. The only possibility is $\sum_{i=1}^{n-1}r_i=(n-1)(e-1)$ and $l_R(R/\,\theta_D\,)=n$, i.e. $\theta_D=t^c\,\overline{R}$. Hence $r_i=e-1$ for $i=1,...,n-1$ and $r_n=ne-a-n-(n-1)(e-1)=e-a-1$.

ii) $\Longleftarrow$ Suppose $t.s.(\,\theta_D\,)=[e,e,....,e,1]$. By Lemma 4.2 $r_n=1$. As in the above item we find $\sum_{i=1}^{n-1}r_i=(n-1)(e-1)+n-l_R(R/\,\theta_D\,)-1$. Hence $n-l_R(R/\,\theta_D\,)-1\leq 0$, i.e. either $n-l_R(R/\,\theta_D\,)=0$ or $n-l_R(R/\,\theta_D\,)=1$.
In the first case $\theta_D=\gamma$, moreover $\delta=\sum_{i=1}^{n-1}r_i+1=(n-1)(e-1)\Longrightarrow$

$\delta=ne-n-e+1=ne-c+\delta-e+1\Longrightarrow c-1=ne-e$, which is a contradiction.
The other possibility leads to $l_R(\,\theta_D\,/\gamma)=1$ and $\sum_{i=1}^{n-1}r_i=(n-1)(e-1)$, hence $r_i=e-1$ for every $i=0,...,n-1$. $\square$

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