A classification of one-dimensional local domains based on the invariant $(c-\delta)r-\delta$.

Anna Oneto[a] and Elsa Zatini[b*]

[a] *Ditpem, Università di Genova, P.le Kennedy, Pad. D - I 16129 Genova (Italy);*

[b] *Dima, Università di Genova, Via Dodecaneso 35 - 16146 Genova (Italy)*

Abstract. Let $(R, \mathfrak{m})$ be a one-dimensional, local, Noetherian domain and let $\overline{R}$ be the integral closure of R in its quotient field K. We assume that R is not regular, analitycally irreducible and residually rational. The usual valuation $v : K \longrightarrow \mathbb{Z} \cup \infty$ associated to $\overline{R}$ defines the numerical semigroup $v(R) = \{v(a),\ a \in R,\ a \neq 0\} \subseteq \mathbb{N}$. The aim of the paper is to study the non-negative invariant $b := (c-\delta)r - \delta$, where c, δ, r denote the conductor, the length of $\overline{R}/R$ and the Cohen Macaulay type of R, respectively. In particular, the classification of the semigroups $v(R)$ for rings having $b \leq 2(r-1)$ is realized. This method of classification might be successfully utilized with similar arguments but more boring computations in the cases $b \leq q(r-1)$, for reasonably low values of q. The main tools are type sequences and the invariant k which estimates the number of elements in $v(R)$ belonging to the interval $[c-e, c)$, e being the multiplicity of R.

Introduction. Let $(R, \mathfrak{m})$ be a one-dimensional, local, Noetherian domain and let $\overline{R}$ be the integral closure of R in its quotient field K. We assume that R is not regular and analitycally irreducible, i.e. $\overline{R}$ is a DVR with uniformizing parameter t and a finite R-module. We also suppose R to be residually rational, i.e. $R/\mathfrak{m} \simeq \overline{R}/t\overline{R}$. Called $v : K \longrightarrow \mathbb{Z} \cup \infty$ the usual valuation associated to $\overline{R}$, the image $v(R) = \{v(a),\ a \in R,\ a \neq 0\} \subseteq \mathbb{N}$ is a numerical semigroup. Starting from the following classical invariants:

c, the *conductor* of R, i.e. the minimal $j \in v(R)$ such that $j + \mathbb{N} \subset v(R)$,

$\delta := \ell_R(\overline{R}/R)$, the number of gaps of the semigroup $v(R)$ in $\mathbb{N}$,

$r := \ell_R((R : \mathfrak{m})/R)$, the *Cohen Macaulay type* of R,

the new invariant

$$b := (c-\delta)r - \delta$$

has been recently considered in the literature. The general problem of classifying rings according to the size of b has been examined by several authors. First, Brown and Herzog in [2] characterize all the one-dimensional reduced local rings having $b = 0$ or $b = 1$. Successively, in [3], [4], [6], Delfino, D'Anna and Micale

*Corresponding author.

E-mail addresses: oneto@diptem.unige.it (A.Oneto), zatini@dima.unige.it (E.Zatini).

consider the rings for which $b \leq r$. Partial answers in the case $b > r-1$ are given in [5].

In [10, Section 4] we obtain some improvements of the quoted results. This is done by using the expression of the invariant b in terms of the type sequence $[r_1, .., r_n]$ (defined in (1.1)), where $n := c - \delta$ and r_1 equals the Cohen-Macaulay type r of R, namely:

$$b = \sum_{i=1}^{n} (r - r_i).$$

So, employing the properties of the type sequence, we get as a straightforward consequence of the preceding formula the well known bounds

$$0 \leq b \leq (n-1)(r-1)$$

(for the positivity see [2], Theorem 1; for the upper bound see [3], Proposition 2.1). Also, we recover in an immediate way the two extremal cases:

$b = 0$, corresponding to the so called rings *of maximal length*, i.e. the rings having maximal type sequence $[r, r, ..., r]$;

$b = (n-1)(r-1)$, corresponding to the *almost Gorenstein* rings, i.e. the rings having minimal type sequence $[r, 1, ..., 1]$.

Actually, for any integer $q \in \mathbb{N}$ it is natural to ask if it is possible to characterize the rings verifying

$$(q-1)(r-1) \leq b \leq q(r-1).$$

In Section 3 we write explicitly all the possible values of $v(R)$ for $1 \leq q \leq 2$ (see Theorems (3.3), (3.4), (3.6)), but we outline that the method used here is absolutely general and analogous although more tedious computations might be repeated for greater values of q.

To achieve our results, we utilize heavily the number

$$k := \ell_R(R/(\mathfrak{C} + xR)),$$

where $\mathfrak{C} := t^c\overline{R}$ denotes the *conductor ideal* of R in $\overline{R}$ and x an element of R such that $v(x) = e(R)$, the *multiplicity*. In [5] it is established that $b = r-1 \Longrightarrow k = 1$ or 2 [5, Proposition 2.4], and that $b = r-1$ and $k = 2 \Longrightarrow r = e-2$ [5, Corollary 2.13]. In [6] the lower bound $rk - e + 1 \leq b$ is found. Improvements of these results and several other inequalities relating the invariants k, b, r are now realized by means of the type sequence of R (see (1.4) and (2.1)). For this purpose we introduce in Section 1 a decomposition of the semigroup $v(R)$ as a disjoint union of subsets:

$$v(R) = \{0, e, 2e, ..., pe, c, \rightarrow\} \cup H_1 \cup .... \cup H_{k-1},$$

where $H_i := \{y_i, y_i + e, ..., y_i + l_i e\}$, $i = 1, ..., k-1$, p, $l_i \in \mathbb{N}$, and $\{y_i\}_{i=1,..,k-1}$ have distinct residues (mod e) (see Setting 1.6). This allows us to obtain in Section 2 the useful formula (2.2.1):

$$b = X + Y + Z$$

where $X := (k-1)(r-1) \geq 0$,

$Y := k - (e - r) \geq 0$,

$Z := (r+1)(p + \sum_1^{k-1} l_i) + k + h - pe - 1 \geq 0$.

Obviously $X + Y = rk - e + 1$, and so the integer Z measures how far is b from the lower bound proved in [6].

The advantage of this formula is evident for low values of b. For instance, for rings having $b \in \{0, 1, 2\}$ we state in a quite simple way all the possible value sets (see Theorems (3.1), (3.8), (3.9)). Nevertheless, a such type of classification might be accomplished for greater values of b with similar arguments.

1 Preliminary results.

We begin by giving the setting of the paper.

Setting 1.1 Let $(R, \mathfrak{m})$ be a one-dimensional local Noetherian domain with residue field κ and quotient field K. We assume throughout that R is not regular with normalization $\overline{R} \subset K$ a DVR and a finite R-module, i.e., R is analytically irreducible. Let $t \in \overline{R}$ be a uniformizing parameter for $\overline{R}$, so that $t\overline{R}$ is the maximal ideal of $\overline{R}$. We also suppose that the field κ is isomorphic to the residue field $\overline{R}/t\overline{R}$, i.e., R is residually rational. We denote the usual valuation on K associated to $\overline{R}$ by v; that is, $v : K \longrightarrow \mathbb{Z} \cup \infty$, and $v(t) = 1$. By [9, Proposition 1] in this setting it is possible to compute for a pair of fractional nonzero ideals $I \supseteq J$ the length of the R-module I/J by means of valuations:

(1.1.1) $$\ell_R(I/J) = |v(I) \setminus v(J)|.$$

The set $v(R) := \{v(a) \mid a \in R,\ a \neq 0\} \subseteq \mathbb{N}$ is the *numerical semigroup* of R. Since the *conductor* $\mathfrak{C} := (R :_K \overline{R})$ is an ideal of both R and $\overline{R}$, there exists a positive integer c so that $\mathfrak{C} = t^c\overline{R}$, $\ell_R(\overline{R}/\mathfrak{C}) = c$ and $c \in v(R)$. Furthermore, denoting by $\delta := \ell_R(\overline{R}/R)$ the number of gaps of the semigroup $v(R)$ and $r := \ell_R((R : \mathfrak{m})/R)$ the *Cohen Macaulay type* of R, we define the invariant

$$b := (c - \delta)r - \delta.$$

We list the elements of $v(R)$ in order of size: $v(R) := \{s_i\}_{i \geq 0}$, where $s_0 = 0$ and $s_i < s_{i+1}$, for every $i \geq 0$. We put $e := s_1$ the *multiplicity* of R and $n = c - \delta$ the number such that $s_n = c$. For every $i \geq 0$, let R_i denote the ideal of elements whose values are bounded by s_i, that is,

$$R_i := \{a \in R \mid v(a) \geq s_i\}.$$

The ideals R_i give a strictly decreasing sequence

$$R = R_0 \supset R_1 = \mathfrak{m} \supset R_2 \supset \ldots \supset R_n = \mathfrak{C} \supset R_{n+1} \supset \ldots ,$$

which induces the chain of duals:

$$R \subset (R : R_1) \subset \ldots \subset (R : R_n) = \overline{R} \subset (R : R_{n+1}) = t^{-1}\overline{R} \subset \ldots.$$

Put $r_i := l_R((R : R_i)/(R : R_{i-1}))$, $i \geq 1$; the finite sequence of integers $[r_1, \ldots, r_n]$ is the *type sequence* of R.

In particular $r_1 = r$, the Cohen-Macaulay type of R. Moreover it is known that:

- $1 \leq r_i \leq r$ for every $i \geq 1$, and $r_i = 1$ for every $i > n$,
- $\delta = \sum_1^n r_i$,
- $2\delta - c = \sum_1^n (r_i - 1) = \sum_1^\infty (r_i - 1)$ (see, e.g. [10, Prop.2.7]).

Type sequence is a suitable tool to study the behavior of the invariant b.

Proposition 1.2 *We have:*

(1) $b = \sum_{i=1}^n (r - r_i)$.

(2) $0 \leq b \leq (n - 1)(r - 1)$.

Proof. For (1) see [10, Section 4].
(2). We have: $\sum_{i=1}^{n}(r-r_i)=\sum_{i=2}^{n}(r-r_i)\leq(n-1)(r-1)$, because $r_1=r$ and $r_i\geq 1$, for every $i\geq 1$. $\diamond$

Notation 1.3 Let R be as in (1.1). We set:

- $x\in\mathfrak{m}$ is an element such that $v(x)=e$; namely, $\ell_R(R/xR)=e$.
- For $a,b\in\mathbb{Z}$, $[a,b]=\{x\in\mathbb{Z}\mid a\leq x\leq b\}$.
- $i_0\in[1,n]$ is such that $s_{i_0-1}=\min\{y\in v(R)\mid y\geq c-e\}$.
  ($i_0=1\iff c=e$).
- $B:=[i_0,n]$ and $A:=[1,n]\setminus B$ $(|A|\leq n-1)$.
- $k:=\ell_R(R/(\mathfrak{C}+xR))$ $(1\leq k\leq e-1)$.

Theorem 1.4 *The following facts hold.*

(1) $k=|B|=\ell_R(\mathfrak{C}:_R\mathfrak{m}/\mathfrak{C})\geq e-r>0$.

(2) $k\leq\sum_{i\in B}r_i\leq e-1$. *If* $\sum_{i\in B}r_i=e-1$, *then* $s_{i_0-1}=c-e$.

Proof. (1) and the inequality $\sum_{i\in B}r_i\leq e-1$ of (2) are proved in [10, Lemma 4.2]. Since $r_i\geq 1$ for every i and $|B|=k$, the inequality $k\leq\sum_{i\in B}r_i$ is done.
Moreover, denoting by ω the canonical module of R (see [10] for the existence and the properties in our setting), we remark that
$\sum_{i\in B}r_i=\ell_R(\overline{R}/(R:R_{i_0-1}))=|v(\omega R_{i_0-1})_{<c}|$ and
$v(\omega R_{i_0-1})_{<c}\subseteq[c-e,c-2]$
(see the proof of the quoted lemma). Thus $\sum_{i\in B}r_i=e-1\Longrightarrow v(\omega R_{i_0-1})_{<c}=[c-e,c-2]$, and so s_{i_0-1}, the minimal element in $v(\omega R_{i_0-1})$, equals $c-e$. $\diamond$

The case $k=1$ is completely known and recalled below for the convenience of the reader.

Proposition 1.5 [10, Lemma 4.4] *The following facts are equivalent:*

(1) $k=1$.

(2) $v(R)=\{0,e,...,pe,c\rightarrow\}$.

(3) *The type sequence of* R *equals* $[e-1,....,e-1,r_n]$.

If R *satisfies these equivalent conditions, then:*

$$\delta=c-p-1,\ b=(p+1)e-c\leq r-1,\ r=e-1,\ r_n=e-1-b.$$

By virtue of (1.1.1) we have $k=|v(R)\setminus v(\mathfrak{C}+xR)|$. This fact allows to write $v(R)=v(\mathfrak{C}+xR)\cup\{0,y_1,...,y_{k-1}\}$, obtaining the description of $v(R)$ as a disjoint union of the sets H_i given in the next setting. The construction is significant for $k>1$.

Setting 1.6 Let $k > 1$. We set:

$v(R) = \{0, e, 2e, ..., pe, c, \rightarrow\} \cup H_1 \cup .... \cup H_{k-1}$, where

- p is the integer such that $c - e \leq pe < c$, in other words, $pe+2 \leq c \leq (p+1)e$. ($p \geq 0$ and $p = 0 \iff c = e$).
- $h := (p+1)e - c, \quad (0 \leq h \leq e-2)$.
- $H_i := \{y_i, y_i + e, ..., y_i + l_i e\}, \ i = 1, ..., k-1, \ l_i \in \mathbb{N}$.
- The integers $y_i \in \mathbb{N}$ are such that $e < y_1 < y_2 < ... < y_{k-1}, \quad y_i \notin e\mathbb{Z}$, $\overline{y_i} \neq \overline{y_j}$ $(mod\ e)$ for every $i, j \in \{1, .., k-1\}$.
- The integers $l_i, \ i = 1, ..., k-1$, are defined by the relations: $y_i + l_i e < c \leq y_i + (l_i + 1)e$.
- For $k = 2$ we shortly call $y := y_1, \quad l := l_1$.

Example 1.7 If $v(R) =< 10, 11, 26 >$, then:
$v(R) = \{0, 10, 20, 30, 40, 50 \rightarrow\} \cup H_1 \cup ... \cup H_7$ where $H_1 = \{11, 21, 31, 41\}$, $H_2 = \{22, 32, 42\}$, $H_3 = \{26, 36, 46\}$, $H_4 = \{33, 43\}$, $H_5 = \{37, 47\}$, $H_6 = \{44\}$, $H_7 = \{48\}$. According to notations previously introduced $y_1 = 11, y_2 = 22, y_3 = 26, y_4 = 33, y_5 = 37, y_6 = 44, y_7 = 48$ and $l_1 = 3, l_2 = l_3 = 2, l_4 = l_5 = 1, l_6 = l_7 = 0$. Moreover, $c = 50, p = 4, h = 0$.

Proposition 1.8 *Let $k > 1, p, h, \{l_i\}$ be the integers defined in* (1.3) *and* (1.6). *Then:*

(1) $r \in \{e-k, ..., e-1\}$.

(2) $0 \leq l_{k-1} \leq .... \leq l_2 \leq l_1 \leq p-1$.

(3) $c - \delta = p + k + \sum_1^{k-1} l_i$,
$\delta = (p+1)(e-1) - h - \sum_1^{k-1}(l_i + 1)$.

Proof. Assertion (1) follows immediately from (1.4.1).
(2). By definition of l_i and p, we have $(l_i + 1)e < y_i + l_i e < c \leq (p+1)e$; then $l_i + 1 \leq p$, for every $i = 1, ..., k-1$. Now note that
$y_i + l_i e < c \leq y_{i-1} + (l_{i-1} + 1)e \Longrightarrow y_i - y_{i-1} < (l_{i-1} + 1 - l_i)e \Longrightarrow l_i \leq l_{i-1}$.
(3). Using the integers defined in (1.6) $c - \delta$ and δ can be expressed as :

$$c - \delta = (p+1) + (l_1 + 1) + ... + (l_{k-1} + 1) = p + k + \sum_1^{k-1} l_i,$$

$$\delta = c - (c - \delta) = (p+1)e - h - (p + k + \sum_1^{k-1} l_i)$$

$$= (p+1)(e-1) - h - \sum_1^{k-1}(l_i + 1). \quad \diamond$$

It is natural to ask how the elements $y_1, ..., y_{k-1}$ introduced in (1.6) influence the Cohen Macaulay type of R. This will be analysed in the following (1.9), (1.11), (1.12).

Proposition 1.9 *Let* $k = \ell_R(R/(\mathfrak{C}+xR))$ *and let* $v(R)$ *be as in* (1.6)*. Further let* $x_1, ..., x_{k-1} \in \mathfrak{m}$ *be such that* $v(x_i) = y_i$*. The following facts are equivalent:*

(1) $r = e - 1$, *i.e.* R *is of maximal Cohen Macaulay type.*

(2) $v(R) \setminus v(xR : \mathfrak{m}) = \{0\}$.

(3) $y_1, ..., y_{k-1} \in v(xR : \mathfrak{m})$.

(4) $x_1, ..., x_{k-1} \in (xR : \mathfrak{m})$.

(5) $x_i x_j \in x\mathfrak{m}$ *for every* $i, j = 1, ..., k - 1$.

(6) $\ell_R(\mathfrak{m}/\mathfrak{m}^2) = e$, *i.e.* R *is of maximal embedding dimension.*

Proof. Since $e - r = \ell_R(R/xR) - \ell_R((xR : \mathfrak{m})/xR) = \ell_R(R/(xR : \mathfrak{m}))$, the equality $e - r = 1$ means $|v(R) \setminus v(xR : \mathfrak{m})| = 1$, and so $1 \iff 2$ is proved. In the same way we obtain that

$(*)$ $r = e - 1 \iff (xR : \mathfrak{m}) = \mathfrak{m} \iff \mathfrak{m}^2 = x\mathfrak{m}$. Moreover,

$(**)$ $v(x^{-1}\mathfrak{m}) \subseteq \mathbb{N} \Longrightarrow x^{-1}\mathfrak{m}\mathfrak{C} \subseteq \mathfrak{C} \Longrightarrow \mathfrak{m}\mathfrak{C} = x\mathfrak{C} \Longrightarrow \mathfrak{C} \subseteq (xR : \mathfrak{m})$.

Considering the chain of ideals

$R \supset \mathfrak{m} \supseteq \mathfrak{C} + (x, x_1, ..., x_{k-1})R \supset \mathfrak{C} + (x, x_1, ..., x_{k-2})R \supset ... \supset \mathfrak{C} + xR$,

we see that $\ell_R(R/(\mathfrak{C} + xR)) = k \Longrightarrow \mathfrak{m} = \mathfrak{C} + (x, x_1, ..., x_{k-1})R$, hence

$(***)$ $x_i\mathfrak{m} = (xx_i)R + (x_ix_j)R + x_i\mathfrak{C}$ for every $j = 1, ..., k - 1$.

By $(*)$ we have immediately $1 \iff 6$ and $1 \Longrightarrow 5$.

$5 \Longrightarrow 4$. By the assumption $x_ix_j \in x\mathfrak{m}$, $\forall\, i, j = 1, ..., k - 1$ and by the obvious inclusion $x_i\mathfrak{C} \subseteq \mathfrak{m}\mathfrak{C} = x\mathfrak{C}$, from $(***)$ we get $x_i\mathfrak{m} \subseteq xR$, then $x_i \in (xR : \mathfrak{m})$.

The implication $4 \Longrightarrow 3$ is obvious.

Finally, $3 \Longrightarrow 2$ holds by $(**)$. ◇

Remark 1.10 It is clear from (1.9) (see the equivalence 1-5) that the condition $y_i + y_j - e \in v(R)$ for every $i, j = 1, ..., k - 1$, is necessary to have maximal Cohen Macaulay type. Unfortunately, it is not sufficient. For example, if $R = \kappa[[t^6, t^9 + t^{10}, t^{14}, t^{16}, t^{17}, t^{19}]]$, then $k = 2$, $y = 9$, $2y - e = 12 \in v(R)$, but $r = e - 2$. In this case (1.9.5) does not hold, because $(t^9 + t^{10})^2 \notin xR$.

Proposition 1.11 *Let* $k = \ell_R(R/(\mathfrak{C} + xR))$ *and let* $v(R)$ *be as in* (1.6)*.*

(1) $r = e - k$, $k \geq 2$, $\iff v(R) \setminus v(xR : \mathfrak{m}) = \{0, y_1, ..., y_{k-1}\}$.

(2) *If* $r < e - 1$, *then*
   (*a*) $2y_1 < c + e$,
   (*b*) $p \leq 2l_1 + 2$ *and* $p = 2l_1 + 2 \Longrightarrow h > 0$.

(3) *If* $r = e - k$, *then*
   (*a*) $y_1 + y_j < c + e$, *for every* $j = 1, ..., k - 1$.
   (*b*) $p \leq l_1 + l_{k-1} + 2$ *and* $p = l_1 + l_{k-1} + 2 \Longrightarrow h > 0$.

(4) *If* $p \geq 3$ *and* i *is such that* $l_i = 0$, *then* $2y_i > c + e$.

Proof. (1). By means of (∗∗) stated in the proof of (1.9), we have the inclusions $(\mathfrak{C} + xR) \subseteq (xR : \mathfrak{m}) \subseteq R$. Since $e - r = \ell_R(R/(xR : \mathfrak{m}))$ and $k = \ell_R(R/(\mathfrak{C} + xR))$, it follows that $e - r = k \iff (\mathfrak{C} + xR) = (xR : \mathfrak{m})$.
To see (2.a), suppose $2y_1 \geq c+e$, then $y_i + y_j \geq c+e$ for every $i, j = 1, ..., k-1$. Let $x_i \in \mathfrak{m}$ be elements such that $v(x_i) = y_i$ and let $s \in \mathfrak{m}$. If $s \in (\mathfrak{C}+xR)$, then $x_i s \in \mathfrak{m}(\mathfrak{C}+xR) \subseteq xR$. If $s \notin (\mathfrak{C}+xR)$, then $v(s) = y_j$, for some j, $1 \leq j \leq k-1$, hence $v(x_i s) = y_i + y_j \geq c+e \Longrightarrow x_i s \in x\mathfrak{C} \subset xR$. In both cases $x_i \in (xR : \mathfrak{m})$, and so $y_i \in v(xR : \mathfrak{m})$. Thus $v(R) \setminus v(xR : \mathfrak{m}) = \{0\}$ and $r = e - 1$ by (1.9), a contradiction.
To see (2.b), consider that by (1.3) and (1.6):

$y_1 \geq c - (l_1 + 1)e = (p - l_1)e - h.$

Combining this with the preceding (2.a), we obtain

$(2p - 2l_1)e - 2h \leq 2y_1 < c + e = (p + 2)e - h.$

Thus $(p - 2l_1 - 2)e < h$ and since $h \leq e - 2$, we see that $p \leq 2l_1 + 2$ and also that $p = 2l_1 + 2 \Longrightarrow h > 0$.
To prove (3.a), it suffices to show that $y_1+y_{k-1} < c+e$. Suppose $y_1+y_{k-1} \geq c+e$, then $y_i + y_{k-1} \geq c + e$ for all i. Let $x_{k-1} \in \mathfrak{m}$ be an element such that $v(x_{k-1}) = y_{k-1}$. As in (2.a), we get $x_{k-1} \in (xR : \mathfrak{m})$, and so $y_{k-1} \in v(xR : \mathfrak{m})$, a contradiction, since the assumption $e - r = k$ means $v(R) \setminus v(xR : \mathfrak{m}) = \{0, y_1, ..., y_{k-1}\}$ (see item 1).
We prove now (3b). As in (2.b),

$y_j \geq c - (l_j + 1)e = (p - l_j)e - h,$ for $j = 1, ..., k - 1$,

and by (3.a)

$(2p - l_j - l_1)e - 2h \leq y_1 + y_j < c + e = (p + 2)e - h.$ Hence

$(p - l_j - l_1 - 2)e < h \leq e - 2$, for every $j = 1, ..., k - 1$.

We conclude

$p \leq l_1 + l_j + 2 \leq l_1 + l_{k-1} + 2$ and also the last assertion.

For (4), note that $l_i = 0 \Longrightarrow y_i + e \geq c$, and that $p \geq 3 \Longrightarrow c > 3e$.
Thus: $2y_i \geq 2c - 2e = c + (c - 2e) > c + e$, as desired. ⋄

We may describe the particular case $k = 2$ in a more precise way.

Proposition 1.12 *Assume $k = 2$. With setting (1.6) we have:*

(1) $r = e - 1 \iff$ *one of the following conditions is satisfied:*

- (*a*) $2y \geq c + e$;
- (*b*) $2y = (2q + 1)e < c + e$, $q \geq 1$, $p \geq 2$ *and* $y \in v(xR : \mathfrak{m})$.

(2) $r = e - 2 \iff 2y < c + e$ *and if* $2y = (2q + 1)e$, *then* $y \notin v(xR : \mathfrak{m})$.

Proof. First recall that by (1.4.1) one has $r \geq e-2$. For implication $\Longrightarrow$ in (1), note that $y \in v(xR : \mathfrak{m})$, by (1.9), and so $2y - e \in v(\mathfrak{m})$. Then regarding the structure of $v(R)$, we have the claim. For the opposite implication, note that in case (*a*) for any $s \in \mathfrak{m}$ such that $v(s) = y$, $v(x^{-1}s^2) = 2y - e \geq c \Longrightarrow x^{-1}s^2 \in \mathfrak{C} \Longrightarrow s^2 \in x\mathfrak{m}$; now use again (1.9) to conclude.
(2) is immediate by (1). ⋄

2 Bounds for the invariant b.

Starting from the preliminary result (1.2) we go on in studying the integer b. First (see (2.1)) we find lower and upper bounds using the properties of the type sequence, then (see (2.2)) we express b in terms of the integers k, p, l_i, h occurring in the decomposition of $v(R)$ as in (1.6). This description becomes quite simple in the particular cases $k = 2, 3$ (see (2.3) and (2.4)). The last result of the present section (see(2.5)) furnishes informations according to the range $(q-1)(r-1) < b \leq q(r-1)$, that will be basic in the next section.

Proposition 2.1 *With Notation* 1.3, *the following facts hold.*

(1) $(e-r-1)(r-1) \leq rk - e + 1 \leq b - \sum_{i\in A}(r - r_i) \leq k(r-1)$.

(2) $b = (k-1)(r-1) + \sum_{i\in A}(r-r_i) \iff \sum_{i\in B} r_i = e-1$ *and* $k = e - r$.

(3) $b = k(r-1) + \sum_{i\in A}(r-r_i) \iff r_i = 1$ *for every* $i \in B$.

(4) *The following conditions are equivalent:*

  (*a*) $b = (e-r-1)(r-1)$.

  (*b*) $b = (k-1)(r-1)$.

  (*c*) $e - r = k$, $\sum_{i\in B} r_i = e-1$ *and* $r_i = r$ *for every* $i \in A$.

  If these conditions hold, then $s_{i_0-1} = c - e$.

(5) $b \geq (r-1)s$, *where* $s := |\{i \in [1,n] \mid r_i = 1\}|$.

Proof. Write the invariant $b = \sum_{i=1}^{n}(r-r_i)$ in the following form:

$$(*) \qquad b = \sum_{i\in B}(r-r_i) + \sum_{i\in A}(r-r_i) = rk - \sum_{i\in B} r_i + \sum_{i\in A}(r-r_i).$$

Using that $\sum_{i\in B} r_i \leq e-1$ (see 1.4.2), we obtain

$$(**) \qquad rk - (e-1) \leq b - \textstyle\sum_{i\in A}(r-r_i) \leq k(r-1).$$

Then, since $k \geq e - r$ by (1.4.1), the inequalities of (1) are clear.
(2). Supposing $b - \sum_{i\in A}(r-r_i) = (k-1)(r-1)$ we have by item 1 $(k-1)(r-1) \geq rk - e + 1$, hence $k \leq e - r$ and since always $k \geq e - r$, it follows that $k = e - r$. From $(*)$ $\sum_{i\in B} r_i = rk - (k-1)(r-1) = k + r - 1 = e - 1$. For the converse, it suffices to substitute $\sum_{i\in B} r_i = k + r - 1$ in $(*)$.
(3). Using $(*)$ we have $b - \sum_{i\in A}(r-r_i) = k(r-1) \iff \sum_{i\in B}(r-r_i) = k(r-1)$. Since $r - r_i \leq r - 1$ for every i and $k = |B|$, the last fact is equivalent to say that $r_i = 1$ for every $i \in B$.
(4), $a \Longrightarrow b$. By (1) we have immediately $\sum_{i\in A}(r-r_i) = 0$ and $(e-r-1)(r-1) = rk - e + 1 \Longrightarrow e - r = k$, as desired.
$b \Longrightarrow c$. By (1) we have $\sum_{i\in A}(r-r_i) \leq b - (rk - e + 1) = -k - r + e \leq 0$, then we can apply item 2 with $\sum_{i\in A}(r-r_i) = 0$.
$c \Longrightarrow a$. Substitute in $(*)$ the relations of (c).
The fact $s_{i_0-1} = c - e$ is immediate by (1.4.2).

By applying [10, Corollary 3.13.2], with $I = \mathfrak{C}$ we get (5). $\diamond$

Utilizing the description of the value set $v(R)$ introduced in (1.6), we obtain the next useful formula for the invariant b.

Theorem 2.2 *With Setting 1.6, assume $k > 1$. The following equalities hold:*

(1) $b = (r+1)\sum_1^{k-1}(l_i+1) - (p+1)(e-r-1) + h = X + Y + Z$

where $X := (k-1)(r-1) \geq 0,$

$Y := k - (e-r) \geq 0,$

$Z := (r+1)(p + \sum_1^{k-1} l_i) + k + h - pe - 1 \ \geq \sum_{i\in A}(r - r_i) \geq 0.$

(2) $c = (p + 1 + \sum_1^{k-1}(l_i+1))(r+1) - b.$

Proof. (1). To get the desired formula it suffices to substitute in the equality $b = (c-\delta)r - \delta$ the expressions of $c - \delta$ and δ given in (1.8.3). The positivity of Y is clear by (1.4.1). To prove the positivity of Z we use the second inequality of (2.1.1): $X + Y = kr - e + 1 \leq b - \sum_{i\in A}(r - r_i)$, and so we have the conclusion: $Z = b - (X+Y) \geq \sum_{i\in A}(r - r_i) \geq 0.$
(2). Since $b + c = (r+1)(c-\delta)$, (2) follows easily. $\diamond$

Lemma 2.3 Case $k = 2$. *With Setting 1.6, assume $k = 2$.*

(1) *If $r = e - 1$, then* $b = (l+1)e + h \leq (l+2)e - 2.$

Further: $b = (l+2)e - 2 \iff h = e - 2.$

(2) *If $r = e - 2$, then:*

$b = (l+1)(e-1) + h - p - 1,$

$c = (p + l + 2)(e-1) - b.$

Further we have:

(a) $l + 1 \leq p \leq 2l + 2$ *and* $p = 2l + 2 \implies h > 0.$

(b) $(l+1)(e-3) \leq b \leq (l+1)(e-2) + e - 3.$ *In particular*

$b = (l+1)(e-3) \iff p = 2l+2, h = 1$ *or* $p = 2l+1, h = 0.$

$b = (l+1)(e-2) + e - 3 \implies p = l+1, \ p > 1, \ h = e-2, \ y = e+1.$

Proof. For $k = 2$, we write $v(R) = \{0, e, 2e, ..., pe, c, \rightarrow\} \cup \{y, y+e, ..., y+le\}$, with $r \in \{e-2, e-1\}$, $c - \delta = p + 2 + l$, $c = (p+1)e - h$. (see 1.8), (1.6)). Then the expressions of b, in items 1,2, come from (1.8.4) with $k = 2$ and $e - r = 1, 2$, respectively. To complete the proof of item 1 recall that $h \leq e - 2$.
The bounds for p in item 2 come from (1.8.2) and (1.11.3) and the value of c comes from (2.2.2).
Rewriting b in the form

$b = (l+1)(e-2) + (l-p) + h,$

and recalling that $l - p \leq -1, \ h \leq e-2$, we obtain the upper bound for b.
Rewriting b in the form

$b = (l+1)(e-3) + (2l+2-p) + (h-1)$,
and using part a, we obtain the lower bound and also $b = (l+1)(e-3) \iff p = 2l+2, h = 1$ or $p = 2l+1, h = 0$.
Finally, note that $b = (l+1)(e-2) + e - 3 \Longrightarrow h = (p-l-1) + e - 2 \geq e - 2 \Longrightarrow p = l+1,\ h = e-2 \Longrightarrow c = pe + 2$ and since by definition of l $y + le < c$, it follows that $y < e+2$, hence $y = e+1$ and $p > 1$. $\diamond$

Lemma 2.4 Case $k = 3$. *With Setting 1.6, assume* $k = 3$.

(1) *If* $r = e-3$, *then* $b = (l_1 + l_2 + 2)(e-2) + h - 2(p+1)$. *Moreover,*
$p < l_1 + l_2 + 2 \Longrightarrow b \geq (l_1 + l_2 + 2)(e-4) + h,\ h \geq 0.$
$p = l_1 + l_2 + 2 \Longrightarrow b = (l_1 + l_2 + 2)(e-4) + h - 2,\ h > 0.$

(2) *If* $r = e-2$, *then* $b = (l_1 + l_2 + 2)(e-1) + h - p - 1$ *and* $p \leq 2l_1 + 2$. *Further,* $p = 2l_1 + 2 \Longrightarrow h > 0$.

(3) *If* $r = e-1$, *then* $b = (l_1 + l_2 + 2)e + h$.

Proof. Recall that by (1.4) $e - r \leq 3$ and by (1.6)
$v(R) = \{0, e, 2e, ..., pe, c, \rightarrow\} \cup \{y_1, ..., y_1 + l_1 e\} \cup \{y_2, ..., y_2 + l_2 e\}$.
Formula (2.2.1) with $k = 3$ becomes
$b = (r+1)(l_1 + l_2 + 2) - (p+1)(e-r-1) + h$.
By substituing r with $e-1, e-2, e-3$, we get the desired expressions for b in items 3, 2, 1, respectively. To complete the proof of (1) and (2), apply (1.11.3) and (1.11.2), respectively. $\diamond$

Proposition 2.5 *Let* $q \in \mathbb{N}$ *be such that* $0 < b \leq q(r-1)$. *Then:*

(1) $r \geq e - q - 1$. *In particular,*

- *(a) If* $r = e - 1 - q$, *then* $b = q(r-1), q \leq e-3$, *and the equivalent conditions of* (2.1.4) *hold.*
- *(b) If* $r \geq e - q$, *then* $e - r \leq k \leq q$.

(2) *If* $(q-1)(r-1) < b \leq q(r-1)$, *we have:*

- *(c)* $k - 1 \leq q \leq n - 1$.
- *(d)* $(q-k-1)(r-1) < \sum_{i \in A}(r - r_i) \leq (q-k)(r-1) + e - 1 - k$.

Proof. (1). First we deduce the inequalities
$(\odot) \qquad (e-r-1)(r-1) \leq b \leq q(r-1)$,
by combining (2.1.1) with the assumption. Hence we get $r \geq e - q - 1$.
(a). If $r = e - 1 - q$, then relations $(\odot)$ give $b = (e-r-1)(r-1)$, and so the conditions of (2.1.4) hold.
(b). Assertion $(**)$ in the proof of (2.1) insures that $kr - (e-1) \leq b$. Hence assuming $r \geq e - q$, we have $rk \leq b + e - 1 \leq q(r-1) + e - 1 \leq (q+1)r - 1$; then $k \leq q$.
(2). Put $M := \sum_{i \in A}(r - r_i)$. We have to compare the two inequalities of (2.1.1)

$(k-1)(r-1)+M+k-(e-r) \leq b \leq k(r-1)+M$
with the assumption
$(q-1)(r-1) < b \leq q(r-1)$.
We obtain the following:
$(k-1)(r-1)+M+k-(e-r) \leq q(r-1)$, and also
$(q-1)(r-1) < k(r-1)+M$.
The first inequality gives $q \geq k-1$ and $M \leq (q-k)(r-1)+e-1-k$.
The second one says that $M > (q-k-1)(r-1)$. Moreover, combining the hypothesis with (2.1.2)
$(q-1)(r-1) < b \leq (n-1)(r-1)$,
and this implies $q \leq n-1$, as desired. ⋄

3 Classification.

Our aim is now to classify the value sets for one dimensional local domains having

$$0 \leq b \leq 2(r-1).$$

On this topic several results are present in the literature. For semigroup rings $R = \kappa[[t^\alpha; \alpha \in S]]$, $S \subset \mathbb{N}$ a numerical semigroup, Brown and Herzog in [2, Corollary after Theorem 4] illustrate the case $b = 1$. This result can be extended to rings R as in Setting 1.1 (see (3.1)). Successively D. Delfino in [5, Corollary 2.11 and Corollary 2.14] gives a characterization of rings satisfying the condition $b < r-1$ and exhibits all the possible value sets in the case $b \leq r$, under the additional assumption $r = e-1$. See also Proposition 2.7 from [3] for a further generalization. An exaustive description of the cases $0 < b < r-1$ can be found in [10, Th.4.6].

In this section we assume Setting 1.1 and Notation 1.3. Moreover, $t.s.(R)$ will denote the type sequence of R, defined in (1.1).

First we recall in (3.1) and (3.2) the quoted known results, which now become an easy consequence of our preceding statements.

Theorem 3.1 Case $b = 0$.
The following conditions are equivalent:

(1) $b = 0$.

(2) *Either R is Gorenstein, or* $v(R) = \{0, e, .., pe, (p+1)e \rightarrow\}$.

(3) $t.s.(R) = [r, ..., r]$.

Proof. By (2.1.1) $0 = b \geq (k-1)(r-1)$; hence either $r = 1$ or $k = 1$, and this last condition gives, by (1.5), $v(R) = \{0, e, .., pe, (p+1)e \rightarrow\}$, or equivalently, $t.s.(R) = [e-1, ..., e-1] = [r, .., r]$. Hence $1 \Longrightarrow 2 \Longleftrightarrow 3$ are clear. Of course, in the Gorenstein case we have $t.s.(R) = [1, .., 1]$. Implication $3 \Longrightarrow 1$ is immediate by (1.2.1). ⋄

Theorem 3.2 [10, Theorem 4.6.1] Case $0 < b < r-1$.
The following facts are equivalent:

(1) $0 < b < r-1$.

(2) $v(R) = \{0, e, .., pe, c \rightarrow\}$ *with* $pe + 2 < c \le (p+1)e$.

(3) $t.s.(R) = [e-1, e-1, ..., e-1, r_n]$, $r_n > 1$.

If these conditions hold, then:
$b < e-2$, $r = e-1$, $r_n = e-1-b$, $k = 1$, $c = (p+1)e - b$.

Theorem 3.3 Case $b = r - 1$.
If $b = r - 1 > 0$, *then either* $r = e-1$ *or* $r = e-2$.

1. Subcase $r = e-1$. *The following facts are equivalent:*

   (a) $b = r-1 > 0$ *and* $r = e-1$.
   (b) $v(R) = \{0, e, .., pe, pe+2 \rightarrow\}$, $e > 2$.
   (c) $t.s.(R) = [e-1, ..., e-1, 1]$, $e > 2$.
   (d) $b = r-1 > 0$ *and* $k = 1$.

2. Subcase $r = e-2$. *The following facts are equivalent:*

   (e) $b = r-1 > 0$ *and* $r = e-2$.
   (f) *either* $v(R) = \{0, e, 2e-1, 2e, 3e-1 \rightarrow\}$, $e > 3$,
   or $v(R) = \{0, e, y, 2e \rightarrow\}$, *with* $2y < 3e$, $e > 3$.
   (g) *either* $t.s.(R) = [e-2, e-2, 1, e-2]$, *with* $e > 3$,
   or $t.s.(R) = [e-2, r_2, r_3]$, *with* $r_2 + r_3 = e-1$, $e > 3$.
   (h) $b = r-1 > 0$ *and* $k = 2$.

Proof. Applying (2.5.1) with $q = 1$, we obtain that $r \ge e-2$. Further, if $b = r-1$, then $r = e-2 \iff k = 2$ by (2.5.1a) and (2.1.4); also, if $b = r-1$, then $r = e-1 \iff k = 1$ by (2.5.1b). This proves the first assertion and the equivalences $a \iff d$, $e \iff h$.

(1). First note that (a) implies $e > 2$; in fact, $e = 2$ would imply $r = 1, b = 0$.
$d \Longrightarrow b$. Since $k = 1$, the equivalent conditions of (1.5) hold, and
$v(R) = \{0, e, .., pe, c \rightarrow\}$, with $c = (p+1)e - b = pe + 2, e > 2$.
$b \Longrightarrow c$. If (b) holds, then by (1.5) $t.s.(R) = [e-1, ..., e-1, r_n]$, with $r - r_n = b = r-1$, hence $r_n = 1$, as in (c).
$c \Longrightarrow a$. By (1.2.1), (c) implies $r = e-1$ and $b = r - r_n = r-1$, as in (a).

(2). First note that (a) implies $e > 3$; in fact, $e = 3$ would imply $r = 1, b = 0$.
$h \Longrightarrow f$. Since $k = 2$ we use (2.3.2) recalling that $p \le 2l + 2$:
$e - 3 = b = (l+1)(e-1) + h - p - 1 \ge (l+1)(e-1) + h - 2l - 3$.
Hence we get $l(e-3) + h \le 1$ and the following possibilities occur by (2.3.2c):
$(l, p, h) = (0, 1, 0)$, or $(l, p, h) = (0, 2, 1)$, or $h = 0$, $e = 4$, $l = 1$.
- If $(l, p, h) = (0, 1, 0)$, then $c = 2e$, $v(R) = \{0, e, y, 2e \rightarrow\}$ with $2y < 3e$, $e > 3$.
- If $(l, p, h) = (0, 2, 1)$, then $v(R) = \{0, e, 2e, c \rightarrow\} \cup \{y\}$, with $c - \delta = 4$,

$c=(p+1)e-h=3e-1,\ 2y<c+e=4e-1 \Longrightarrow y\leq 2e-1,$
$c-e\in v(R) \Longrightarrow y=2e-1$. Hence $v(R)=\{0,e,2e-1,2e,3e-1\rightarrow\},\ e>3$.
- If $h=0,\ e=4,\ l=1$, then $e-3=b=(l+1)(e-1)+h-p-1 \Longrightarrow$
$p=4=2l+2 \Longrightarrow h>0$, which is absurd. Hence $h \Longrightarrow f$ is proved.
$f \Longrightarrow g$. Denoting $R_0=\kappa[[t^d,\ d\in v(R)]]$ the monomial ring such that $v(R_0)=v(R)=\{0,e,2e-1,2e,3e-1\rightarrow\}$, we have $r(R_0)=e-2$. Since $r(R)\leq r(R_0)$ and $r(R)\geq e-2$ by (1.4.1), we conclude that $r(R)=e-2$. The other invariants are easily derived from $v(R)$: $c-\delta=4,\ \delta=3e-5,\ b=(c-\delta)r-\delta=e-3$. By substituting in (2.1.1), we obtain $\sum_{h\in A}(r-r_h)=0$, hence $r_2=e-2$ and $r_3+r_4=e-1$, as desired.
The same reasoning holds for $v(R)=\{0,e,y,2e\rightarrow\}$.
To see $g \Longrightarrow e$, it suffices to recall that $b=\sum_{h=1}^{n}(r-r_h)$, see (1.2.1). ⋄

Theorem 3.4 Case $r-1<b<2(r-1)$.
We have $r-1<b<2(r-1)$ *if and only if* $v(R)$ *is one of the following:*

(1) $v(R)=\{0,e,....,pe,c\rightarrow\}\cup\{y\}$, *with* $y\notin e\mathbb{Z}$,
and either $2y\geq c+e,\ pe+5\leq c\leq min\{y+e,(p+1)e\},\ e\geq 5$,
or $e=2e',\ y=3e',\ p=2,\ 4e'+5\leq c\leq 5e',\ e\geq 10,\ y\in v(xR:\mathfrak{m})$.

(2) $v(R)=\{0,e,2e,c\rightarrow\}\cup\{y\}$, *with* $y\notin e\mathbb{Z},\ 2y<c+e$ *and:*
if $2y\neq 3e$, *then* $2e+3\leq c\leq 3e-2,\ e\geq 5$;
if $2y=3e$, *then* $e=2e',\ 4e'+3\leq c\leq 5e',\ e\geq 6,\ y\notin v(xR:\mathfrak{m})$.

(3) $v(R)=\{0,e,y,c\rightarrow\}$,
with $y\notin e\mathbb{Z},\ e\geq 5,\ 2y<c+e,\ e+4\leq c\leq 2e-1$.

In each case $k=2$; *in case* (1), $r=e-1$ *and* $b\geq r+1$; *in cases* (2) *and* (3), $r=e-2$.

Proof. Assume $r-1<b<2(r-1)$.
Step 1. Claim: $k=2$ and $e-2\leq r\leq e-1,\ r>2$.
We have $r>2$, since $r=2\Longrightarrow 1<b<2$, which is absurd. Further (2.1.1) gives $(k-1)(r-1)\leq b$, and so $k\leq 2$. But $k=1$ would imply $b\leq r-1$ by (1.5), then $k=2$. We conclude using (1.4.1).
Now utilizing the notation in (1.6) we write:

$$(*)\begin{cases} v(R)=\{0,e,2e,...,pe,c,\rightarrow\}\cup\{y,y+e,...,y+le\},\ p\geq 1,\ y>e,\ y\notin e\mathbb{Z},\\ y+le<c=(p+1)e-h\leq y+(l+1)e,\quad l+1\leq p.\end{cases}$$

Step 2. Claim: $l=0$ and $e\geq 5$. Further, if $r=e-2$, then $p\leq 2$.
- If $r=e-1$, then, by (2.3.1) we know that $b=(l+1)e+h,\ l,h\geq 0$. Hence $b<2(r-1)=2e-4 \Longrightarrow (l-1)e+h<-4 \Longrightarrow l=0,\ h<e-4$; further we get : $c=(p+1)e-h\geq pe+5,\ e\geq 5$ and $b=h+e\geq e=r+1$.
- If $r=e-2$, we have $(l+1)(e-3)\leq b$ and $l+1\leq p\leq 2l+2$ by (2.3.2). Then $b<2(r-1)=2(e-3)\Longrightarrow l=0$ and $p\leq 2$; also, the assumption $e-3<b<2e-6$ implies $e\geq 5$.
Step 3. When $r=e-1$, recalling the relations proved in Step 2, we obtain $v(R)=\{0,e,....,pe,c\rightarrow\}\cup\{y\}$, with $e\geq 5,\ pe+5\leq c$, as in item 1. Recall that by definition of p and l we have $c\leq(p+1)e$ and $c\leq y+e$. Moreover, by

(1.12.1) one of the following conditions is satisfied:
either (a) $2y \geq c+e$, or (b) $2y = (2q+1)e < c+e$, $p \geq 2$ and $y \in v(xR : \mathfrak{m})$. Further, as noted in (1.11.4), $p \geq 3$, $l = 0 \Longrightarrow 2y > c+e$, hence in case (b) we have $p = 2$ and consequently $(2q+1)e < c+e \leq 4e \Longrightarrow q = 1$. This proves (1).

Step 4. When $r = e-2$, we have by Step 2 that $l = 0$ and $p \leq 2$.
In the case $p = 2$ we get item 2. In fact from (2.3.2) we obtain $c = 4e-4-b$ and the bounds for c follow at once. The last assertion in item 2 comes from (1.12). Analogously, in the case $p = 1$ we get item 3. Notice that when $p = 1$ we cannot have $2y = 3e < c+e$, because $c+e \leq 3e-1$.

To complete the proof, let $v(R)$ be as in items $(1),(2),(3)$; we claim that $r-1 < b < 2(r-1)$. In every case $k = 2$; in case (1) $r = e-1$ and in cases (2), (3) $r = e-2$ by (1.12). The rest is a direct computation based on relation (2.2.2): $c = (p+2)(r+1) - b$. $\diamond$

Example 3.5 We supply an example for each case of the above proposition.

• Case (1) with $2y \geq c+e$.
Let $R = \kappa[[t^5, t^{10}, t^{12}, t^{15}, \rightarrow]]$. Then: $y = 12, p = 2, c = 15, r = 4, b = 5$.

• Case (1) with $2y = 3e$.
Let $R = \kappa[[t^{10}, t^{15}, t^{20}, t^{25}, \rightarrow]]$. Then: $y = 15, p = 2, c = 25, r = 9, b = 15$.

• Case (2).
Let $R = \kappa[[t^{10}, t^{15}+t^{16}, t^{20}, t^{25}, \rightarrow]]$. As above, $y = 15, p = 2, c = 25$, but $r = 8$ by 1.9 since $(t^{15}+t^{16})^2 \notin x\mathfrak{m}$. Then $b = 11$.

• Case (3).
Let $R = \kappa[[t^5, t^6, t^9, \rightarrow]]$. Here $y = 6, p = 1, c = 9, r = 3, b = 3$.

Theorem 3.6 Case $b = 2(r-1)$.
$b = 2(r-1) > 0$ if and only if $v(R)$ is one of the following:

1. *(a)* $v(R) = \{0, e, e+2, e+4 \rightarrow\}$, $e \geq 4$.
   (b) $v(R) = \{0, e, 2e, 2e+4 \rightarrow\} \cup \{y\}$, $e \geq 4$, $y \in v(xR : \mathfrak{m})$.
   (c) $v(R) = \{0, e, 2e, ..., pe, pe+4 \rightarrow\} \cup \{y\}$, $e \geq 4$, $y \geq (p-1)e+4$, $p \geq 3$.
2. *(a)* $v(R) = \{0, e, e+1, e+3 \rightarrow\}$, $e \geq 4$.
   (b) $v(R) = \{0, e, y, 2e, 2e+2 \rightarrow\}$, $e \geq 5$, $2e+4 \leq 2y < 3e+2$, $2y \neq 3e$.
   (c) $v(R) = \{0, e, 2e, 3e-1, 3e, 4e-1, 4e, 5e-1 \rightarrow\}$, $e \geq 4$.
   (d) $v(R) = \{0, e, 2e, y, 3e, y+e, 4e \rightarrow\}$, $e \geq 4$, $2y < 5e$.
3. *(a)* $v(R) = \{0, e, y_1, y_2, 2e \rightarrow\}$, $e \geq 5$, $y_1 + y_2 < 3e$.
   (b) $v(R) = \{0, e, 2e-2, 2e-1, 2e, 3e-2 \rightarrow\}$, $e \geq 5$.

Further:
in case 1, $r = e-1$ *and* $\ell_R(R/(\mathfrak{C}+xR)) = 2$;
in case 2, $r = e-2$ *and* $\ell_R(R/(\mathfrak{C}+xR)) = 2$;
in case 3, $r = e-3$ *and* $\ell_R(R/(\mathfrak{C}+xR)) = 3$.

Proof. Let, as above, $k = \ell_R(R/(\mathfrak{C}+xR))$. First we assume $b = 2(r-1) > 0$ and we observe that by (2.1.1) $(k-1)(r-1) \leq b = 2(r-1)$, then $k \leq 3$. Since $k = 1$ implies $b \leq r-1$ by (1.5), one of the following cases occurs:

$$\left[\begin{array}{llll} or & k=2 & and & r=e-1 \\ or & k=2 & and & r=e-2 \\ or & k=3 & and & r=e-3. \end{array}\right.$$

In case $k = 2$ by Setting 1.6 we have:

$$(*)\begin{cases} v(R)=\{0,e,2e,...,pe,c,\rightarrow\} \cup \{y, y+e, ..., y+le\},\ p \geq 1,\ y > e,\ y \notin e\mathbb{Z}, \\ y + le < c = (p+1)e - h \leq y + (l+1)e, \quad l+1 \leq p. \end{cases}$$

Step 1. Assuming $r = e-1$ and $k = 2$, we prove that $v(R)$ has the form described in item 1. By (2.3.1) and the assumption we have the equalities $b = (l+1)e + h = 2e-4$; hence $(l-1)e + h = -4 \Longrightarrow l = 0,\ h = e-4,\ e \geq 4,\ c = (p+1)e - h = pe+4$. Now, $l = 0 \Longrightarrow y \geq c - e = (p-1)e + 4$, and so

$$v(R)=\{0,e,...,pe,pe+4\rightarrow\} \cup \{y\},\ \text{with}\ (p-1)e+4 \leq y \leq pe+2,\quad e \geq 4.$$

For $p = 1$ we get (1.a). In fact, by (1.12.1) $2y \geq c + e = 2e + 4 \Longrightarrow y \geq e + 2 \Longrightarrow y = e+2$. For $p = 2$ we get (1.b). For $p \geq 3$ we get (1.c).

Step 2. Assuming $r = e-2$ and $k = 2$, we prove that $v(R)$ satisfies item 2. First, by (2.3.2) we have that $l+1 \leq p \leq 2l+2$ and also that

$$(**)\quad (l+1)(e-3) \leq (l+1)(e-1) + h - p - 1 = b.$$

Then $b = 2(e-3) > 0$ implies $(l+1)(e-3) \leq 2(e-3)$, i.e. $l \leq 1$.

Case $l = 0$, and consequently $1 \leq p \leq 2$.

($\cdot$) If $l = 0,\ p = 1$, then by $(**)$, $h = e-3$, thus $c = e+3$, and (2.a) holds.

($\cdot$) If $l = 0,\ p = 2$, then $h = e-2, c = 2e+2$, hence (2.b) holds.

Case $l = 1$. Now, relation $(**)$ combined with the assumption $b = 2e-6$ implies $h - p - 1 = -4,\ 2 \leq p \leq 4$ and two possibilities occur:

($\cdot$) $p = 4,\ h = 1,\ c = 5e-1$. The relation $c \leq y + (l+1)e$ gives $y \geq 3e-1$, the relation $2y < c + e$ gives $y \leq 3e-1$. Hence (2.c) holds.

($\cdot$) $p = 3,\ h = 0,\ c = 4e$; hence (2.d) holds.

Step 3. Assuming $r = e-3$ and $k = 3$, we prove that $v(R)$ has the form described in item 3. First, by Setting 1.6 and by (2.4.1) we have:

$$(\overline{*})\begin{cases} v(R)=\{0,e,...,pe,c\rightarrow\} \cup \{y_1, y_1+e, ..., y_1+l_1e\} \cup \{y_2, y_2+e, ..., y_2+l_2e\} \\ p \geq 1,\ y_2 > y_1 > e,\ y_i \notin e\mathbb{Z}, \\ y_i + l_ie < c = (p+1)e - h \leq y_i + (l_i+1)e, \quad l_i + 1 \leq p, \\ b = (l_1 + l_2 + 2)(e-2) + h - 2(p+1). \end{cases}$$

By (1.11.3), since $r = e-k$, then $p \leq l_1 + l_2 + 2$.

($\cdot$) If $p < l_1 + l_2 + 2$, then substituting $b = 2(e-4) > 0$ in $(\overline{*})$ we get $(l_1+l_2)(e-4)+h \leq 0,\ h \geq 0$. Hence $h = l_1 = l_2 = 0,\ p = 1,\ c = 2e,\ y_1+y_2 < c+e$ by (1.11.3), and so we have (3.a).

($\cdot$) If $p = l_1+l_2+2$, then analogously we get $(l_1+l_2)(e-4)+h-2 = 0$, with $0 < h \leq 2$. The case $h = 1$ is impossible. In fact, $h = 1 \Longrightarrow l_1+l_2 = 1$ (in particular, by (1.8.2), $l_2 \leq l_1$, hence $l_2 = 0,\ l_1 = 1$), $e = 5,\ p = 3,\ c = (p+1)e-h = 19$. The relation of (1.6) $c \leq y_i + (l_i+1)e$ gives $y_1 \geq 19-10 = 9,\ y_2 \geq 19-5 = 14$, but $y_1 + y_2 < c + e = 24$ by (1.11.3); the only possibility would be $y_1 = 9,\ y_2 = 14$. Absurd that $\overline{y_1} = \overline{y_2}\ (mod\ 5)$. Hence $h = 2,\ l_1 = l_2 = 0,\ p = 2,\ c = 3e-2$ and

$$v(R)=\{0,e,2e,3e-2,\rightarrow\} \cup \{y_1, y_2\}.$$

Since $l_1 = 0$, the bound $c \leq y_1 + e$ gives $y_1 \geq 2e-2$. Recalling that by (1.11.3)

$y_1 + y_2 < c + e$, we conclude $y_1 = 2e - 2, y_2 = 2e - 1$, as in $(3.b)$.

Viceversa, we assume in the following $v(R)$ having the form described in items 1,2,3, and we prove that $b = 2(r - 1) > 0$.
For a $v(R)$ as in item 1 we see that $r = e - 1$ using (1.12). In fact, in case $(1.a)$ we have $y = e + 2,\ 2y = c + e$ and in case $(1.c)$:

$2y \geq 2(p - 1)e + 8 > c + e = (p + 1)e + 4.$

In conclusion in each case of item 1 we have $\ell_R(R/(\mathfrak{C}+xR)) = 2,\ r = e-1,\ l = 0$. Using (2.3.1) $b = e + h = 2e - 4 = 2(r - 1)$, as desired.
In case $(2.a)$, $y = e + 1 \notin v(xR : \mathfrak{m})$, then $r = e - 2$ by (1.9). In case $(2.b)$ by hypothesis $2y < c + e$ and $2y \neq 3e$, then $r = e - 2$ by (1.12). In case $(2.c)$ we get by a direct calculation $v(xR_0 : \mathfrak{m}) \setminus v(R_0) = \{4e + 1, ..., 5e - 2\}$, then $r = r(R_0) = e-2$. In case $(2.d)$ $2y < c+e$ and $2y \notin e\mathbb{Z}$, then $r = e-2$ by (1.12). In conclusion, in each case of item 2 one has: $\ell_R(R/(\mathfrak{C} + xR)) = 2,\ r = e - 2$, and so by (2.3.2) $b = (l + 1)e - 1) + h - p - 1$. Putting in this formula

$(\cdot)$ $l = 0,\ p = 1,\ h = e - 3$, in case $(2.a)$,
$(\cdot)$ $l = 0,\ p = 2,\ h = e - 2$, in case $(2.b)$,
$(\cdot)$ $l = 1,\ p = 4,\ h = 1$, in case $(2.c)$,
$(\cdot)$ $l = 1, p = 3,\ h = 0$, in case $(2.d)$,

we get $b = 2e - 6 = 2(r - 1)$, as desired.
In both cases of item 3 we have $r = e-3$. In fact, $y_1+y_2-e \notin v(R) \Longrightarrow y_1, y_2 \notin v(xR : \mathfrak{m}) \Longrightarrow e-r = 3$ by (1.11.1). Hence $\ell_R(R/(\mathfrak{C}+xR)) = 3,\ r = e-3,\ l_1 = l_2 = 0$, and by (2.4.3) $b = 2(e - 2) + h - 2(p + 1)$. Putting in this formula

$(\cdot)$ $h = 0,\ p = 1$ in case $(3.a)$,
$(\cdot)$ $h = p = 2$ in case $(3.b)$,

we get $b = 2e - 8 = 2(r - 1)$, as desired. $\diamond$

With similar arguments one can evaluate the semigroups $v(R)$ of rings having $b > 2(r - 1)$. For instance, if $2(r - 1) < b \leq 3(r - 1)$ there are few possible cases and the classification is tedious but easy. Now, for each $q \geq 3$ we construct a family of rings of multiplicity e and Cohen Macaulay type $r = e - 1$ having $b = q(r - 1)$ or $(q - 1)(r - 1) < b < q(r - 1)$.

Example 3.7 Let $q \geq 3$. Following notations of Setting 1.6 we consider

$v(R) = \{0, e, 2e, ..., pe, c \rightarrow\} \cup \{y, y + e, ..., y + le\}$,

with $e > p,\ p = 2q,\ l = q - 2$. In this case $k = 2$. Using (1.12) we see that $r = e-1$, because $y+(q-1)e \geq c > 2qe \Longrightarrow y > (q+1)e \Longrightarrow 2y > 2(q+1)e \geq c + e$. Then by (2.3.1) $b = (q - 1)e + h$, with $0 \leq h \leq e - 2$. Now, with an additional hypothesis on the conductor, we are in goal. In fact:

1) Assuming $c = pe + p$, we have $h = (p + 1)e - c = -p + e = -2q + e$, then $b = (q - 1)e + (-2q + e) = q(e - 2) = q(r - 1)$.

2) Assuming $c > pe+p$, i.e. $e-h > 2q$, we have $(q-1)(e-2) < (q-1)e \leq b = (q-1)e+h = q(e-2)+2q-e+h < q(e-2)$, hence $(q-1)(r-1) < b < q(r-1)$.

As a further application of the previous results we describe exhaustively the cases $b = 1$ and $b = 2$ (see next (3.8), (3.9); for $b = 1$ see also [2], Section 4).

With regard to the formula

$$b = \sum_{i=1}^{n}(r - r_i)$$

it becomes natural to consider the invariant $b$as a measure of how far is the type sequence $[r_1, ..., r_n]$ from the maximal one $[r, ..., r]$. For instance, for $b = 1$ one expects a type sequence of the form $[r, ..., r-1, ..., r]$, for $b = 2$ $[r, ..., r-1, ..., r-1, ..., r]$ or $[r, ..., r-2, ..., r]$, and so on. Surprisingly, after finding by a direct computation all the possible value sets and the corresponding type sequences, we discover that very few choices are possible. For $b = 1$ (resp. $b = 2$) either $e \leq 4$ (resp. $e \leq 5$) or $t.s.(R) = [e-1, ..., e-1, e-1-b]$.

Corollary 3.8 Case $b = 1$. *Here t.s. stands for* $t.s.(R)$.
$b = 1$ *if and only if* $v(R)$ *is one of the following:*
$v(R) = \{0, 4, 7, 8, 11 \rightarrow\}$, *with t.s.* $[2, 2, 1, 2]$;
$v(R) = \{0, 4, 5, 8, \rightarrow\}$, *with t.s.* $[2, 1, 2]$;
$v(R) = \{0, e, ..., pe, (p+1)e-1, \rightarrow\}$, $e \geq 3$, *with t.s.* $[e-1, ..., e-1, e-2]$.

Proof. First recall that $b > 0 \Longrightarrow r > 1$ by (1.2.1). Let, as in (2.2.1), $b = X + Y + Z$, where $X := (k-1)(r-1) \geq 0$, $Y := k - (e-r) \geq 0$, and $Z := (r+1)(p + \sum_1^{k-1} l_i) + k + h - pe - 1 \geq 0$.
Assuming $b = 1$, we have to consider the choices:

	X	Y	Z
$a)$	1	0	0
$b)$	0	1	0
$c)$	0	0	1

In $a)$ $k = r = 2$, $2 - (e-2) = 0 \Longrightarrow e = 4$. By (3.3.2) with $e = 4$ we find:
$v(R) = \{0, 4, 7, 8, 11 \rightarrow\}$,
$v(R) = \{0, 4, 5, 8, \rightarrow\}$.
In $b)$ $k = 1$, $1 - (e-r) = 1$, which is absurd.
In $c)$ $k = 1$, $1 - (e-r) = 0 \Longrightarrow r = e-1$, $e \geq 3$, $Z = ep + 1 + h - pe - 1 = 1 \Longrightarrow h = 1 \Longrightarrow c = (p+1)e - 1$. By (1.5) we find:
$v(R) = \{0, e, ..., pe, (p+1)e-1, \rightarrow\}$, $e \geq 3$. ◇

Corollary 3.9 Case $b = 2$. *As above, t.s. stands for* $t.s.(R)$.
$b = 2$ *if and only if* $v(R)$ *is one of the following:*
$v(R) = \{0, 4, 5, 7, \rightarrow\}$, *with t.s.* $[2, 1, 1]$;
$v(R) = \{0, 4, 8, 11, 12, 15, 16, 19, \rightarrow\}$, *with t.s.* $[2, 2, 2, 1, 2, 1, 2]$;
$v(R) = \{0, 4, 8, 9, 12, 13, 16, \rightarrow\}$, *with t.s.* $[2, 2, 1, 2, 1, 2]$;
$v(R) = \{0, 5, 9, 10, 14, \rightarrow\}$, *with t.s.* $[3, 3, 1, 3]$;
$v(R) = \{0, 5, 6, 10, \rightarrow\}$, *with t.s.* $[3, 1, 3]$;
$v(R) = \{0, 5, 7, 10, \rightarrow\}$, *with t.s.* $[3, 2, 2]$;
$v(R) = \{0, 5, 6, 7, 10, \rightarrow\}$, *with t.s.* $[2, 1, 1, 2]$;
$v(R) = \{0, 5, 6, 8, 10, \rightarrow\}$, *with t.s.* $[2, 2, 1, 1]$;
$v(R) = \{0, 5, 8, 9, 10, 13, \rightarrow\}$, *with t.s.* $[2, 2, 1, 1, 2]$;
$v(R) = \{0, e, ..., pe, (p+1)e-2, \rightarrow\}$, $e \geq 4$, *with t.s.* $[e-1, ..., e-1, e-3]$.

Proof. As in the preceding proof, assuming $b = 2$, we have to consider the following choices:

	X	Y	Z
$a)$	0	1	1
$b)$	1	0	1
$c)$	1	1	0
$d)$	2	0	0
$e)$	0	2	0
$f)$	0	0	2

First recall that $k = 1 \Longrightarrow r = e - 1$ by (1.5), and so $X = 0$ (with $r > 0$) $\Longrightarrow k - (e - r) = Y = 0$ and cases $a), e)$ are impossible.
In $b)$ $X = 1 \Longrightarrow k = r = 2,\ 2 - (e - 2) = Y = 0 \Longrightarrow e = 4$, hence $b = 2(r - 1)$ and we can apply (3.6.2) with $e = 4$. We find:
$v(R) = \{0, 4, 5, 7, \rightarrow\}$,
$v(R) = \{0, 4, 8, 11, 12, 15, 16, 19, \rightarrow\}$,
$v(R) = \{0, 4, 8, 9, 12, 13, 16, \rightarrow\}$.
In $c)$ $X = 1 \Longrightarrow k = r = 2,\ 2 - (e - 2) = Y = 1 \Longrightarrow e = 3,\ Z = 3(p + l) + 2 + h - 3p - 1 = 0 \Longrightarrow 3l + h + 1 = 0$, which is absurd.
In $d)$ the condition $X = (k - 1)(r - 1) = 2$ implies two possibilities:
$d_1)$ $k = 2,\ r = 3,\ 2 - (e - 3) = 0 \Longrightarrow e = 5$. We are in case $b = r - 1,\ r = e - 2$. By (3.3.2) with $e = 5$ we find:
$v(R) = \{0, 5, 9, 10, 14, \rightarrow\}$,
$v(R) = \{0, 5, 6, 10, \rightarrow\}$,
$v(R) = \{0, 5, 7, 10, \rightarrow\}$.
$d_2)$ $k = 3,\ r = 2,\ e = 5$. We are in case $b = 2(r - 1),\ r = e - 3$, and so by (3.6.3) with $e = 5$ we find:
$v(R) = \{0, 5, 6, 7, 10, \rightarrow\}$,
$v(R) = \{0, 5, 6, 8, 10, \rightarrow\}$,
$v(R) = \{0, 5, 8, 9, 10, 13, \rightarrow\}$.
In $f)$ $k = 1,\ r = e-1,\ Z = ep+1+h-pe-1 = 2 \Longrightarrow h = 2 \Longrightarrow c = (p+1)e-2$. By (1.5) we find:
$v(R) = \{0, e, ..., pe, (p + 1)e - 2, \rightarrow\},\ e \geq 4.$ ⋄

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