An application of type sequences to the blowing-up.

Anna Oneto and Elsa Zatini
(Beitrage zur Algebra und Geometrie, Vol 46, n.2, pp 471-489 (2005))

Abstract
Let I be an $\mathfrak{m}$-primary ideal of a one-dimensional, analytically irreducible and residually rational local Noetherian domain R. Given the blowing-up of R along I, we establish connections between the type-sequence of R and classical invariants like multiplicity, genus and reduction exponent of I.

1 Introduction

Let $(R, \mathfrak{m}, k)$ be a one-dimensional local Noetherian domain which is analytically irreducible and residually rational. In this paper we deal with the *blowing-up* $\Lambda := \Lambda(I) = \bigcup_{n\geq 0} I^n : I^n$ along a not principal $\mathfrak{m}$-primary ideal I of R.
The problem of finding relations involving the multiplicity $e := e(I)$, the genus $\rho := \rho(I) = l_R(\Lambda/R)$ and the reduction exponent $\nu := \nu(I)$, was first studied for $I = \mathfrak{m}$ by Northcott in the 1950s and later by Matlis (see [8]), Kirby (see [5]), Lipman (see [6]) and many others.

In this note we show that it is possible to describe the difference $2\rho - e\nu$ in terms of the *type sequence* $[r_1, ..., r_n]$ of R (r_1 is the *Cohen-Macaulay type*). Our main result is the formula of Theorem 4.7 in Section 4:

$$2\rho = e\nu + \sum_{i\notin\Gamma} (r_i - 1) - d(R : \Lambda) - l_R(\Lambda^{**}/\Lambda) - l_R(R : \Lambda/I^\nu).$$

Afterwards we use this statement to improve classical results concerning the equality

$$R : \Lambda = I^\nu$$

which has been studied by several authors under the hypothesis that R is Gorenstein. Starting from a theorem of Matlis valid for $\Lambda(\mathfrak{m})$ ([7], Theorem 13.4), Orecchia and Ramella ([14], Theorem 2.6) proved that if the associated graded ring $G(\mathfrak{m}) = \bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is Gorenstein, then $R : \Lambda = \mathfrak{m}^\nu$. Successively Ooishi, in the case of the blowing-up along an ideal I, proved that $2\rho \leq e\nu$ and that equality holds if and only if $R : \Lambda = I^\nu$ ([12], Theorem 3).

In Section 5, we consider the rings having *type sequence* $[r_1, 1, \dots, 1]$ which are called almost Gorenstein. For these rings we prove that the Ooishi's inequality $2\rho \leq e\nu$ becomes $2\rho \leq \nu e + r_1 - 1$ and that equality holds if and only if $R : \Lambda = I^\nu$ (Theorem 5.3).

In Section 6 we consider the case of the blowing-up along $\mathfrak{m}$. The study of the conductor $R : \Lambda$ provide some useful remarks when $e = \mu + 1$ (μ is the embedding dimension of R) and when the reduction exponent is 2 or 3.

2 Notations and Preliminaries.

Throughout this paper $(R, \mathfrak{m})$ denotes a one-dimensional local Noetherian domain with residue field k. For simplicity, we assume that k is an infinite field. Let $\overline{R}$ be the integral closure of R in its quotient field K; we suppose that $\overline{R}$ is a finite R-module and a DVR with a uniformizing parameter t, which means that R is analytically irreducible. We also suppose R to be residually rational, i.e., $k \simeq \overline{R}/t\overline{R}$. We denote the usual valuation associated to $\overline{R}$ by

$$v : K \longrightarrow \mathbb{Z} \cup \infty, \qquad v(t) = 1.$$

2.1

Under our hypotheses, for any fractional ideals $I \supseteq J$ the length of the R-module I/J can be computed by means of valuations (see [8], Proposition 1):

$$l_R(I/J) = \#(v(I) \setminus v(J)).$$

Given two fractional ideals I, J we define $I : J = \{x \in K \mid xJ \subseteq I\}$.

2.2

In the sequel we shall consider an $\mathfrak{m}$-primary ideal I of R which is not principal.

The *Hilbert function* and the *Hilbert-Poincaré series* of I are respectively

$$H_I(n) = l_R(I^n/I^{n+1}), \;\; n \geq 0, \qquad P_I(z) = \sum_{n \geq 0} H_I(n) z^n.$$

It is well-known that the power series $P_I(z)$ is rational:

$$P_I(z) = \frac{h_I(z)}{1-z}, \qquad \text{where} \qquad h_I(z) = h_0 + h_1 z + h_2 z^2 + ... + h_\nu z^\nu \in \mathbb{Z}[z],$$

$$h_0 = l_R(R/I), \quad h_i = l_R(I^i/I^{i+1}) - l_R(I^{i-1}/I^i), \text{ for all } i, \quad 1 \leq i \leq \nu.$$

The polynomial $h_I(z)$ is called the *h-polynomial* of I; moreover

$e(I) := h_I(1)$ is the *multiplicity* of I,

$\rho(I) := h'_I(1)$ is called *genus* of I, or *reduction number* of R if $I = \mathfrak{m}$.

We shall say that $h_I(z)$ is *symmetric* if $h_i = h_{\nu - i}$ for all i, $0 \leq i \leq \nu$.

The *blowing-up of* R *along* I is defined by

$$\Lambda := \Lambda(I) = \bigcup_{n \geq 0} I^n : I^n \qquad (cf.[6]).$$

Let $x \in I$ denote an element (called *a minimal reduction of* I) such that $I^{n+1} = xI^n$ for $n \gg 0$. Then (see [6], 1):

(1) $x^n \Lambda = I^n \Lambda, \quad \forall\, n \geq 0.$

(2) $e(I) = l_R(R/xR) = v(x) \geq H_I(n)$, for every $n \geq 0$.

(3) The least integer $\nu := \nu(I)$ such that $I^{n+1} = xI^n \quad \forall\, n \geq \nu$, is called the *reduction exponent* of I. It is known that $\nu(I) \leq e(I) - 1$ and that the following equalities hold:

$$\nu(I) \;=\; deg\; h_I(z) \;=\; min\{n \mid l_R(I^n/I^{n+1}) = e(I)\}$$
$$= min\{n \mid \Lambda = I^n : I^n\} \;=\; min\{n \mid I^n \Lambda = I^n\}.$$

(4) $\rho(I) = l_R(\Lambda/R)$. Hence

$$l_R(R/I^n) = e(I)n - \rho(I), \qquad \forall\, n \geq \nu.$$

(5) If $h_I(z)$ is symmetric, then $l_R(R/I^\nu) = \dfrac{e(I)\ \nu}{2}$.
This follows immediately from the fact that, if $h_I(z)$ is symmetric, then $2\rho(I) = e(I)\,\nu$ (see the proof of Lemma 3.3, [13]).

(6) The inclusion $R : \Lambda \supseteq I^\nu$ always holds and the equality $R : \Lambda = I^n$ implies that $n = \nu$ ([12], Proposition 1, [14], Lemma 1.5).

2.3

We shall consider also:

$$\begin{array}{rcl} v(R) & := & \{v(x),\ x \in R,\ x \neq 0\} \subseteq \mathbb{N}, \text{ the } \textit{numerical semigroup} \text{ of } R. \\ \gamma_R & := & R : \overline{R}, \text{ the } \textit{conductor ideal} \text{ of } R. \\ c & := & l_R(\overline{R}/\gamma_R), \text{ the } \textit{conductor} \text{ of } v(R), \text{ such that } \gamma_R = t^c\overline{R}. \\ \delta & := & l_R(\overline{R}/R), \text{ the } \textit{singularity degree of } R. \\ n & := & c - \delta = l_R(R/\gamma_R). \end{array}$$

2.4

In our hypotheses R has a *canonical module* ω, unique up to isomorphism.
We list below some well-known properties of ω, useful in the sequel (see [4]).
We always assume that $R \subseteq \omega \subset \overline{R}$.

(1) $\omega : \omega = R$ and $\omega : (\omega : I) = I$ for every fractional ideal I.

(2) If $I \supseteq J$, then $l_R(I/J) = l_R(\omega : J/\omega : I)$.

(3) $v(\omega) = \{j \in \mathbb{Z} \mid c-1-j \notin v(R)\}$, hence $c-1 \notin v(\omega)$ and $c+\mathbb{N} \subseteq v(\omega)$.

(4) R is Gorenstein *if and only if* $\omega = R$ *if and only if* $R : \omega = R$.
Otherwise $\gamma_R \subseteq R : \omega \subseteq \mathfrak{m}$.

(5) (see [9], Lemma 2.3). For every fractional ideal I,

$$s \in v(I\omega) \quad \textit{if and only if} \quad c - 1 - s \notin v(R : I).$$

2.5

We recall the notion of *type sequence* given for rings by Matsuoka in 1971, recently revisited in [2] and extended to modules in [10].
Let $n := c - \delta$, and let $s_0 = 0 < s_1 < \ldots < s_n = c$ be the first $n+1$ elements of $v(R)$. For each $i = 1, \ldots, n$, define the ideal $R_i := \{x \in R\ :\ v(x) \geq s_i\}$ and consider the chains:

$$R = R_0 \supset R_1 = \mathfrak{m} \supset R_2 \supset \ldots \supset R_n = \gamma_R$$
$$R = R : R_0 \subset R : \mathfrak{m} \subset R : R_2 \subset \ldots \subset R : R_n = \overline{R}$$

For every $i = 1, \ldots, n$, put $r_i := l_R(R : R_i/R : R_{i-1}) = l_R(\omega R_{i-1}/\omega R_i)$.
The *type sequence* of R, denoted by $t.s.(R)$, is the sequence $[r_1, \ldots, r_n]$.
We list some properties of type sequences useful in the sequel (see [2]):

(1) $r := r_1$ is the *Cohen-Macaulay type* of R.

(2) For every $i = 1, \dots, n$, we have $1 \leq r_i \leq r_1$.
(3) $\delta = \sum_1^n r_i$, and $2\delta - c = l_R(\omega/R) = \sum_1^n (r_i - 1)$.
(4) If $s_i \in v(R : \omega)$, then the correspondent r_{i+1} is 1 (see [9], Prop.3.4).

2.6
We recall that ring R is called *almost Gorenstein* if it satisfies the equivalent conditions
(1) $\mathfrak{m} = \mathfrak{m}\,\omega$.
(2) $r_1 - 1 = 2\delta - c$.
(3) $R : \omega \supseteq \mathfrak{m}$.
By the above property 2.5,(3), it is clear that R is *almost Gorenstein* if and only if $t.s.(R) = [r_1, 1, \dots, 1]$ and that *Gorenstein* means *almost Gorenstein* with $r_1 = 1$.

2.7
For any fractional ideal I of R we set $I^* := R : I$. Notice that:

$$I \subseteq I^{**} \subseteq I\omega.$$

In fact, $I^{**} = R : (R : I) \subseteq \omega : (R : I) = I\omega$.

2.8
We recall that the *integral closure* of an ideal I of R is $\overline{I} := I\overline{R} \cap R$ and that I is said to be *integrally closed* if $I = \overline{I}$.
In [11] Ooishi characterizes curve singularities which can be normalized by the first blowing-up along the ideal I in terms of integral closures:

$$(*) \qquad \Lambda = \overline{R} \quad \textit{if and only if} \quad I^n = \overline{I^n} \quad \text{for all } n \geq \nu.$$

We introduce a weaker notion of closure, namely the *canonical closure* of I as $\widetilde{I} := I\omega \cap R$. We'll see that this notion is particularly meaningful for almost Gorenstein rings. Recalling 2.7, we can easily see that $I \subseteq I^{**} \subseteq \widetilde{I} \subseteq \overline{I}$, so

$$I = \overline{I} \quad \textit{implies that} \quad I = I^{**} = \widetilde{I}.$$

For the canonical closure the analogue of statement $(*)$ is:

$$\Lambda = \omega\Lambda \quad \textit{if and only if} \quad I^n = \widetilde{I^n} \quad \text{for all } n \geq \nu.$$

This fact is shown in the next proposition.

Proposition 2.9
Let $\Lambda := \Lambda(I)$ be as above. We have the following groups of equivalent conditions:

(A) (A_1) $\omega \subseteq \Lambda$; (A_2) $\omega\Lambda = \Lambda$; (A_3) $\omega : \Lambda = R : \Lambda$;
(A_4) $I^n = \widetilde{I^n}$ $\forall\, n \geq \nu$; (A_5) $\omega I^n = I^n$ $\forall\, n \geq \nu$;
(A_6) *there exists $n > 0$ such that $\omega I^n = I^n$.*

(B) (B_1) $\Lambda = \Lambda^{**}$; (B_2) $\omega : \Lambda = \omega(R : \Lambda)$.

Moreover the following facts are equivalent

(1) *Conditions* (A) *hold.*

(2) *Conditions* (B) *hold and* $R : \Lambda \subseteq R : \omega$.

Proof
Let's begin to prove that the equalities $I^n = \widetilde{I^n} \quad \forall\, n \geq \nu$ imply that $\omega \subseteq \Lambda$.
Let k_1 be the minimal exponent such that $I^{k_1} \subseteq R : \omega$ (k_1 exists since $R : \omega \supseteq \gamma_R$). If $k_1 \geq \nu$, then $I^{k_1} = \widetilde{I^{k_1}} = \omega I^{k_1} \cap R = \omega I^{k_1}$ and this yields $\omega \subseteq I^{k_1} : I^{k_1} = I^{\nu} : I^{\nu} = \Lambda$. If $k_1 < \nu$, then $I^{\nu} \subseteq I^{k_!} \subseteq R : \omega$, hence $\omega I^{\nu} \subseteq R$. Thus, $I^{\nu} = \widetilde{I^{\nu}} = I^{\nu}\omega \cap R = I^{\nu}\omega$, which means $\omega \subseteq \Lambda$.
All the other implications in group (A) and also that ones in group (B) hold by the properties of the canonical module.
To prove *(A) implies (B)*, note that by 2.7 $\Lambda^{**} \subseteq \omega\Lambda = \Lambda$.
Moreover, if (A) holds, then $(R : \Lambda)\omega \subseteq (R : \Lambda)\Lambda \subseteq R$, hence $R : \Lambda \subseteq R : \omega$.
Under the further assumption $R : \Lambda \subseteq R : \omega$, we can prove *(B) implies (A)* because the fact $\omega : \Lambda = \omega(R : \Lambda) \subseteq \omega(R : \omega) \subseteq R$ leads to $\Lambda \supseteq \omega$.

Remark 2.10

(1) If R is almost Gorenstein, then $R : \Lambda \subseteq R : \omega$, hence conditions (A) and (B) above are equivalent.

(2) If I is a canonical ideal, i.e., $I \simeq \omega$, then conditions (A) and (B) hold, because Λ is reflexive and $R : \Lambda \subseteq R : \omega$ (see [9], Remark 2.5).

3 The first Formula.

In the following we use the notation introduced in Section 2.
$\Lambda := \Lambda(I) = \bigcup_{n>0} I^n : I^n$ is the *blowing-up of* R in an $\mathfrak{m}$-primary ideal I which is not principal and $e := e(I)$, $\nu := \nu(I)$, $\rho := \rho(I)$ are respectively the *multiplicity*, the *reduction exponent* and the *genus* of I.
Moreover we consider $\gamma_R := R : \overline{R}$, $\delta := l_R(\overline{R}/R)$, $c := l_R(\overline{R}/\gamma_R)$,
$\gamma_\Lambda := \Lambda : \overline{R}$, $\delta_\Lambda := l_R(\overline{R}/\Lambda)$, $c_\Lambda := l_R(\overline{R}/\gamma_\Lambda)$.
Finally, $x \in I$ denotes a *minimal reduction* of I.

3.1 We begin with a few remarks involving the conductor ideals respect to the canonical inclusions $R \subseteq \Lambda \subseteq \overline{R}$. We have the following diagram:

$$\begin{array}{ccc} \gamma_\Lambda & & \\ \cup\!| & & \\ \gamma_R & \subseteq & R : \Lambda \\ \cup\!| & & \cup\!| \\ (R : \Lambda)\ \gamma_\Lambda & & I^{\nu} \\ \cup\!| & & \cup\!| \\ x^{\nu}\ \gamma_\Lambda & = & I^{\nu} : \overline{R} \end{array}$$

Proposition 3.2

(1) $c - c_\Lambda \le e\nu$.

(2) $l_R(R/\gamma_R) - l_R(\Lambda/\gamma_\Lambda) = c - c_\Lambda - \rho = e\nu - \rho - l_R(\gamma_R/x^\nu\gamma_\Lambda) \le l_R(R/I^\nu)$.

(3) The following facts are equivalent:

(a) $c - c_\Lambda = e\nu$.

(b) $\gamma_R = x^\nu\gamma_\Lambda$.

(c) $\gamma_R \subseteq I^\nu$.

(d) $l_R(R/\gamma_R) - l_R(\Lambda/\gamma_\Lambda) = l_R(R/I^\nu)$.

Proof

(1) Considering the diagram in 3.1 we see that:
$c - c_\Lambda = l_R(\gamma_\Lambda/\gamma_R) = l_R(\gamma_\Lambda/x^\nu\gamma_\Lambda) - l_R(\gamma_R/x^\nu\gamma_\Lambda) = e\nu - l_R(\gamma_R/x^\nu\gamma_\Lambda)$.

(2) Since $\rho = \delta - \delta_\Lambda$, using part (1) of the proof we obtain:

$$\begin{aligned} l_R(R/\gamma_R) - l_R(\Lambda/\gamma_\Lambda) &= (c-\delta) - (c_\Lambda - \delta_\Lambda) = c - c_\Lambda - \rho \\ &= e\nu - \rho - l_R(\gamma_R/x^\nu\gamma_\Lambda) \le l_R(R/I^\nu). \end{aligned}$$

(3) Equivalences *(a) if and only if (b)* and *(b) if and only if (d)* are immediate by item (2).
To prove *(b) implies (c)*, we note that $\gamma_R = x^\nu\gamma_\Lambda = I^\nu : \overline{R} \subseteq I^\nu$.
Conversely, assumption (c) implies that $\gamma_R = \gamma_R : \overline{R} \subseteq I^\nu : \overline{R} \subseteq \gamma_R$, hence $\gamma_R = I^\nu : \overline{R} = x^\nu\gamma_\Lambda$.

Remark 3.3

(1) In view of item (2) of the above proposition we have the inequality

$$l_R(R/\gamma_R) - l_R(\Lambda/\gamma_\Lambda) \ge -\rho$$

and, in the case $I = \mathfrak{m}$, $l_R(R/\gamma_R) - l_R(\Lambda/\gamma_\Lambda) \ge e - \rho$.
In Example 7.1 we show that both these minimal values can be reached.

(2) Conditions (3) of 3.2 imply that $R : I^\nu \subseteq \overline{R}$, but if this inclusion holds we need not have the above equivalent conditions (see Example 7.2).

(3) Conditions (3) of 3.2 imply the Conductors Transitivity Formula:

$$\gamma_R = (R : \Lambda)\ \gamma_\Lambda.$$

Example 7.3 shows that the converse does not hold.

(4) Conditions (3) of 3.2 do not imply that $R : \Lambda = I^\nu$. This can be seen in Example 7.4; however next lemma shows that the converse is true.

Lemma 3.4
If $R : \Lambda = I^\nu$, *then we have*

(1) The equivalent conditions of Proposition 3.2,(3) hold.

(2) $\Lambda^{**} = \Lambda$.

Proof
(1) It is clear considering the diagram in 3.1. To prove part (2), observe that condition $R : \Lambda = I^\nu = \Lambda I^\nu$ implies $\Lambda^{**} = R : \Lambda I^\nu = I^\nu : I^\nu = \Lambda$.

From the above considerations we obtain a first formula connecting the invariants ρ, e, ν associated to the ideal I with the invariants c, δ of R by means of the length of the quotient $R : \Lambda / I^\nu$. This formula will be successively improved in Theorem 4.7 by using type sequences.

Proposition 3.5

(1) $2\rho = e\nu + (2\delta - c) - l_R(R : \Lambda / I^\nu) - l_R(\omega\Lambda/\Lambda)$.

(2) The following facts are equivalent:

  (a) $2\rho = e\nu + (2\delta - c)$.

  (b) Λ *is Gorenstein and* $c - c_\Lambda = e\nu$.

  (c) $R : \Lambda = I^\nu$ *and* $\omega\Lambda = \Lambda$.

  (d) $R : \Lambda = I^\nu \subseteq R : \omega$.

Proof
From $2\rho = 2\delta - 2\delta_\Lambda = 2\delta - c - (2\delta_\Lambda - c_\Lambda) + c - c_\Lambda + e\nu - e\nu$, we get

$$(*) \qquad 2\rho = e\nu + (2\delta - c) - (2\delta_\Lambda - c_\Lambda) - (e\nu - c + c_\Lambda)$$

Hence the equivalence *(a) if and only if (b)* of (2) is clear.
Since $I^\nu \subseteq R : \Lambda \subseteq R$, we have

$$(**) \qquad l_R(R/R : \Lambda) = l_R(R/I^\nu) - l_R(R : \Lambda/I^\nu) = e\nu - \rho - l_R(R : \Lambda/I^\nu).$$

From the inclusions $R \subseteq \Lambda \subseteq \omega\Lambda$ and $R \subseteq \omega \subseteq \omega\Lambda$, we obtain that

$$l_R(R/R : \Lambda) = l_R(\omega\Lambda/\omega) = l_R(\omega\Lambda/\Lambda) + \rho - (2\delta - c).$$

Substituting this in the first member of $(**)$ we get the first formula and also the equivalence *(a) if and only if (c)*.
Finally, *(c) if and only if (d)* follows by using Proposition 2.9.

4 Formulas involving type sequences.

We keep the notation of the above section. We have seen in 3.5 that

$$2\rho \leq e\nu + (2\delta - c).$$

Using the notion of *type sequence* we insert a new term in this inequality (see Theorem 4.7):

$$2\rho \leq e\nu + \sum_{i \notin \Gamma} (r_i - 1) \leq e\nu + (2\delta - c).$$

We study also conditions to have equalities. To do this we introduce the positive invariant $d(R : \Lambda)$, which plays a crucial role in this context.

Definition 4.1
Let, as above, $s_0 = 0, s_1, \ldots, s_n = c$ be the first $n+1$ elements of $v(R)$, $n = c-\delta$. Let $t.s.(R) = [r_1, ..., r_n]$ be the type sequence of R. We call $d(R:\Lambda)$ the number

$$d(R:\Lambda) := l_R(\overline{R}/\Lambda^{**}) - \sum_{i\in\Gamma} r_i$$

where Γ denotes the numerical set $\Gamma := \{i \in \{1, .., n\} \mid s_{i-1} \in v(R:\Lambda)\}$.

Note that

$$\#\Gamma = l_R(R:\Lambda/\gamma) = l_R(\overline{R}/\omega\Lambda)$$

The following proposition ensures that $d(R:\Lambda) \geq 0$.

Proposition 4.2
We have

$$l_R(\overline{R}/\omega\Lambda) \leq \sum_{i\in\Gamma} r_i \leq l_R(\overline{R}/\Lambda^{**}).$$

Proof
The first inequality is obvious since $r_i \geq 1 \ \forall\, i$. For the second one we shall use property (5) of 2.4 with $I = R:\Lambda$:

$$s \in v(I\omega) \ \textit{ if and only if } \ c-1-s \notin v(\Lambda^{**}).$$

If $x_{i-1} \in I$ is such that $v(x_{i-1}) = s_{i-1}$, then by definition

$$r_i = l_R(\omega R_{i-1}/\omega R_i) = l_R(x_{i-1}\omega + \omega R_i/\omega R_i) = \#\{v(x_{i-1}\omega + \omega R_i) \setminus v(\omega R_i)\}.$$

Since $v(x_{i-1}\omega) \subseteq v(I\omega)$, the assignment $y \to c-1-y$ defines an injective map

$$\bigcup_{i\in\Gamma}\{v(x_{i-1}\omega + \omega R_i) \setminus v(\omega R_i)\} \longrightarrow \mathbb{N} \setminus v(\Lambda^{**}).$$

From the fact that the numerical sets

$$\{v(x_{i-1}\omega + \omega R_i) \setminus v(\omega R_i)\}, \quad i \in \{1, \ldots, n\},$$

are disjoint by construction we deduce that

$$\sum_{i\in\Gamma} r_i \leq l_R(\overline{R}/\Lambda^{**}).$$

The next proposition collects some useful properties of the invariant $d(R:\Lambda)$ and allows us to find sufficient conditions to have $d(R:\Lambda) = 0$.

Proposition 4.3
Let $i_o \in \mathbb{N}$ be such that $e(R:\Lambda) = s_{i_0}$. Then

(1) $d(R:\Lambda) = l_R(\omega\Lambda/\Lambda^{**}) - \sum_{i\in\Gamma}(r_i - 1)$.

(2) If $\omega \subseteq \Lambda^{**}$, i.e., $R:\Lambda \subseteq R:\omega$, then $d(R:\Lambda) = 0$.

(3) $d(R:\Lambda) = \sum_{i>i_0,\ i\notin\Gamma} r_i - l_R(\Lambda^{**}/R^*_{i_0})$.

(4) If $R:\Lambda$ is integrally closed, then $d(R:\Lambda) = 0$.

Proof

(1) $d(R:\Lambda) = l_R(\overline{R}/\Lambda^{**}) - \sum_{i\in\Gamma} r_i = l_R(\omega\Lambda/\Lambda^{**}) - (\sum_{i\in\Gamma} r_i - l_R(\overline{R}/\omega\Lambda))$.

(2) The inclusion $\omega \subseteq \Lambda^{**}$ implies that $\omega\Lambda = \Lambda^{**}$, hence the thesis by (1), recalling that $d(R:\Lambda) \geq 0$.

(3) After writing $l_R(\overline{R}/\Lambda^{**}) = l_R(\overline{R}/R^*_{i_0}) - l_R(\Lambda^{**}/R^*_{i_0})$, the thesis is clear since

$$l_R(\overline{R}/R^*_{i_0}) = \sum_{i>i_0} r_i.$$

(4) This results from the above item, because the fact that $R:\Lambda$ is integrally closed means that $R:\Lambda = R_{i_0}$.

The next theorem provides a link between the *type sequence* of R and the *genus* ρ of the ideal I.

Theorem 4.4

(1) $\rho = \sum_{i\notin\Gamma} r_i - l_R(\Lambda^{**}/\Lambda) - d(R:\Lambda) \leq r\ l_R(R/R:\Lambda)$.

(2) Let $i_o \in \mathbb{N}$ *be such that* $e(R:\Lambda) = s_{i_0}$. *Then*

$$\rho = \sum_{i\leq i_0} r_i - l_R(\Lambda^{**}/\Lambda) + l_R(\Lambda^{**}/R^*_{i_0}).$$

Proof

(1) From the inclusions $R \subseteq \Lambda \subseteq \Lambda^{**} \subseteq \overline{R}$ we obtain

$$\begin{aligned}\rho = l_R(\Lambda/R) &= \delta - l_R(\Lambda^{**}/\Lambda) - l_R(\overline{R}/\Lambda^{**})\\ &= \delta - l_R(\Lambda^{**}/\Lambda) - d(R:\Lambda) - \sum_{i\in\Gamma} r_i.\end{aligned}$$

Thus the first equality is clear since $\delta - \sum_{i\in\Gamma} r_i = \sum_{i\notin\Gamma} r_i$.

The inequality follows immediately, recalling that $r_i \leq r \;\; \forall\, i$ and that

$$l_R(R/R:\Lambda) = \#\big(\{1,\dots,n\}\setminus\Gamma\big).$$

(2) By substituting formula (3) of 4.3 in formula (1) above, we obtain

$$\begin{aligned}\rho &= \sum_{i\notin\Gamma} r_i - l_R(\Lambda^{**}/\Lambda) - \sum_{i>i_0, i\notin\Gamma} r_i + l_R(\Lambda^{**}/R^*_{i_0}) =\\ &= \sum_{i\leq i_0} r_i - l_R(\Lambda^{**}/\Lambda) + l_R(\Lambda^{**}/R^*_{i_0}).\end{aligned}$$

Remark 4.5
In the case $\Lambda = \overline{R}$ the inequality $\rho \leq r\ l_R(R/R:\Lambda)$ of Theorem 4.4 gives the well-known relation $\delta \leq r\ (c-\delta)$ ([8], Theorem 2).
The maximal value $\rho = r\ l_R(R/R:\Lambda)$ is achieved if and only if $r_i = r$ for all $i \notin \Gamma$, $\Lambda = \Lambda^{**}$ and $d(R:\Lambda) = 0$; this happens for instance if $I = \mathfrak{m}$ and $e = \mu$ (see 6.2), or if R is *Gorenstein*.

Corollary 4.6

(1) $e\nu + rl_R(R:\Lambda/I^\nu) \leq (r+1)\ l_R(R/I^\nu)$.

(2) If the h-polynomial is symmetric, then $l_R(R:\Lambda/I^\nu) \leq \dfrac{r-1}{r} \cdot \dfrac{e\nu}{2}$

Proof

(1) From the first item of the theorem we have:
$\rho = e\nu - l_R(R/I^\nu) \leq rl_R(R/R:\Lambda) = rl_R(R/I^\nu) - rl_R(R:\Lambda/I^\nu)$.
The thesis follows.

(2) By property (5) of 2.2 it suffices to substitute $l_R(R/I^\nu) = \dfrac{e\nu}{2}$ in (1).

Theorem 4.7

(1) $2\rho = e\nu + \sum_{i\notin\Gamma} (r_i - 1) - d(R:\Lambda) - l_R(\Lambda^{**}/\Lambda) - l_R(R:\Lambda/I^\nu)$.

(2) The following facts are equivalent:

(a) $2\rho = e\nu + \sum_{i\notin\Gamma} (r_i - 1)$.

(b) $R:\Lambda = I^\nu$ *and* $d(R:\Lambda) = 0$.

Proof
(1). We can rewrite formula (1) of Proposition 3.5 as:

$$2\rho = e\nu + \sum_{i\notin\Gamma} (r_i - 1) + \sum_{i\in\Gamma} (r_i - 1) - l_R(R:\Lambda/I^\nu) - l_R(\Lambda^{**}/\Lambda) - l_R(\omega\Lambda/\Lambda^{**}).$$

So using item (1) of Proposition 4.3, we obtain part (1).
(2) follows from part (1) by virtue of Lemma 3.4 recalling that $d(R:\Lambda) \geq 0$.

We remark that the equality $R:\Lambda = I^\nu$ does not ensure that $d(R:\Lambda) = 0$ (see Example 7.5).

5 Almost Gorenstein Rings.

In this section we deal with *almost Gorenstein* rings. The notations will be the same as in the preceding sections.

Under the hypothesis R *almost Gorenstein*, the formulas in 3.5, 4.4 and 4.7 involving the genus $\rho(I)$ are considerably simplified and allow us to extend some well-known results concerning the equality $R:\Lambda = I^\nu$. Recently Barucci and

Fröberg stated the equivalence *(a) if and only if (c)* of next Theorem 5.3 in the case R *almost Gorenstein* and $\Lambda = \Lambda(\mathfrak{m})$ (see [3], Proposition 26).

First, inspired by the famous result of Bass "*A one-dimensional Noetherian local domain R is Gorenstein if and only if each nonzero fractional ideal of R is reflexive*" (see [1], Theorem 6.3), we notice that:

Proposition 5.1
R is almost Gorenstein if and only if $\omega J = J^{**}$ for every not principal fractional ideal J.

Proof
Suppose R almost Gorenstein. By 2.7 it suffices to prove that $\omega J \subseteq J^{**}$. Since $R : J = \mathfrak{m} : J$, we have $(R : J)J\omega \subseteq \mathfrak{m}\omega = \mathfrak{m}$, hence $\omega J \subseteq J^{**}$.
The opposite implication follows immediately by taking $J = \mathfrak{m}$.

Corollary 5.2
If R is an almost Gorenstein ring, then

(1) $\Lambda^{**} = \omega\Lambda$ *and* $d(R : \Lambda) = 0$.

(2) $\rho = r - 1 + l_R(R/R : \Lambda) - l_R(\Lambda^{**}/\Lambda)$.

Proof

(1) The second equality follows from Proposition 4.3, (1).

(2) Apply Formula (1) of Theorem 4.4, observing that in the almost Gorenstein case
$$\sum_{i\notin\Gamma} r_i = r - 1 + l_R(R/R : \Lambda).$$

Under the assumption R *almost Gorenstein*, since
$$\omega\Lambda = \Lambda^{**}, \quad \sum_{i\notin\Gamma}(r_i - 1) = r - 1 = 2\delta - c \quad \text{and} \quad d(R : \Lambda) = 0$$
both Proposition 3.5 and Theorem 4.7 give the next theorem.

Theorem 5.3
Assume that R is an almost Gorenstein ring and let $\Lambda = \Lambda(I)$. Then:

(1) $2\rho = e\nu + r - 1 - l_R(R : \Lambda/I^\nu) - l_R(\Lambda^{**}/\Lambda)$.

(2) The following conditions are equivalent:

(a) $2\rho = e\nu + r - 1$.

(b) Λ *is Gorenstein and* $c - c_\Lambda = e\nu$.

(c) $R : \Lambda = I^\nu$.

(d) $\omega : \Lambda = I^\nu$.

In this case the equivalent conditions (A) of Proposition 2.9 hold.

Proof
We have only to prove *(c) if and only if (d).*
(c) implies (d). By Lemma 3.4 we have $\Lambda = \Lambda^{**} = \omega\Lambda$. Hence $\omega : \Lambda = R : \Lambda$ by duality.
To prove *(d) implies (c)*, we notice that $I^\nu \subseteq R : \Lambda \subseteq \omega : \Lambda$.

Corollary 5.4
If R is an almost Gorenstein ring and the h-polynomial is symmetric, then
$$l_R(R : \Lambda/I^\nu) \leq r - 1 \quad \textit{and the equality holds if and only if} \quad \Lambda = \Lambda^{**}.$$

Proof
The symmetry of the h-polynomial gives $2\rho = e\nu$ (see 2.2(5)), hence it suffices to substitute this in formula (1) of the theorem.

We note that the condition $l_R(R : \Lambda/I^\nu) = r - 1$ does not imply that the h-polynomial is symmetric: see for instance Example 7.7, where R is *almost Gorenstein* with $r(R) > 1$ and Example 7.6, where R is Gorenstein. Example 7.6 shows also that the hypotheses R Gorenstein and $2\rho = e\nu$ do not give the symmetry of the h-polynomial.

The following statement of Ooishi (see [12], Corollary 6) can be obtained as a direct consequence of our preceding results.

Corollary 5.5
If R is Gorenstein and the h-polynomial is symmetric, then the equivalent conditions (2) of Theorem 5.3 hold.

Another immediate consequence of Theorem 5.3 is the natural generalization of Theorem 10 of [12] to the almost Gorenstein case.

Corollary 5.6
Suppose R almost Gorenstein.
The equality $\gamma = I^\nu$ holds if and only if $\Lambda = \overline{R}$ and $2\delta = e\nu + r - 1$.

Formula (1) of Theorem 5.3 is very useful in applications, expecially when $\Lambda = \Lambda^{**}$. In the next theorem we prove that in the almost Gorenstein case the blowing up along a reflexive ideal I is reflexive; this is not always true (see Example 7.9). Nevertheless, in Example 7.4 we have R almost Gorenstein, Λ reflexive, but I not reflexive.

First we recall the following property (see [10], Corollary 3.15).

5.7
Let R be almost Gorenstein and let J be a fractional ideal not isomorphic to R, then
$$J \textit{ is reflexive} \quad \textit{if and only if} \quad J : J \supseteq R : \mathfrak{m}.$$

Theorem 5.8
Suppose R almost Gorenstein and let $\Lambda = \Lambda(I)$. Then

(1) *The equivalent conditions of the groups* (A), (B) *of Proposition 2.9 are equivalent to the following ones:*

(C) (C$_1$) $\Lambda \supseteq R:\mathfrak{m}$. (C$_2$) I^ν *is reflexive.*
(C$_3$) I^n *is reflexive* $\forall\ n \geq \nu$.
(C$_4$) I^n *is reflexive for some* $n \geq \nu$.

(2) If I is reflexive, then the equivalent conditions (A), (B), (C) *hold, in particular* Λ *is reflexive.*

Proof
(1) The equivalence of conditions (C) is immediately achieved by using 5.7.
In order to prove the equivalence (A) *if and only if* (C), we note that $\omega I^n = (I^n)^{**}$ by Proposition 5.1, hence

$$\omega I^n = I^n \quad if\ and\ only\ if \quad I^n = (I^n)^{**}.$$

(2) By applying as before Proposition 5.1 we deduce that $I = I^{**} = \omega I$; but this is equivalent to $\omega \subseteq I : I \subseteq \Lambda$.

6 Blowing up along the maximal ideal.

Our purpose is now to consider the special case $I = \mathfrak{m}$. We denote by $\Lambda := \Lambda(\mathfrak{m})$ *the blowing-up* of R along the maximal ideal, e the *multiplicity*, $\mu := l_R(\mathfrak{m}/\mathfrak{m}^2)$ the *embedding dimension*, r the *Cohen-Macaulay type* of R; $x \in \mathfrak{m}$ is a *minimal reduction* of $\mathfrak{m}$.

When $e = \mu$, namely $\mathfrak{m}$ is *stable*, we can prove that the Gorensteiness of the blowing up Λ is equivalent to the almost-Gorensteiness of the ring R.

When $e = \mu + 1$, we get an explicit formula for the length of the module $R : \Lambda/\mathfrak{m}^\nu$. It turns out that this length is zero if and only if R is Gorenstein and $\nu = 2$.

In the cases $\nu = 2$ and $\nu = 3$ we state formulas involving the conductor $R : \Lambda$ which extend some results of Ooishi valid for Gorenstein rings (see [12]).

We begin with two simple remarks, useful in the sequel.

Remark 6.1

(1) $l_R(R/R:\Lambda) = l_R(x(R:\mathfrak{m})/R:\Lambda) + (e - r)$.

(2) If R is almost Gorenstein, then $\Lambda = \Lambda^{**}$.

Proof

(1) We know that $x\Lambda = \mathfrak{m}\Lambda$ by property (1) of 2.2. Therefore the inclusion $\mathfrak{m} \subseteq x\Lambda$ implies that $R:\Lambda \subseteq x(R:\mathfrak{m}) \subseteq R \subseteq R:\mathfrak{m}$. From this chain we get the thesis.

(2) This is true by Theorem 5.8, since $I = \mathfrak{m}$ is reflexive.

6.2 CASE $e = \mu$.

We recall that $\mathfrak{m}$ is said to be *stable* if $\Lambda = \mathfrak{m} : \mathfrak{m}$. We have the following well known equivalent conditions for the *stability* of $\mathfrak{m}$ (see [7], Theorem 12.15):

(1) $\mathfrak{m}$ *is stable*
(2) $e = \mu$
(3) $\rho = e - 1$
(4) $r = e - 1.$

Proposition 6.3
If $\mathfrak{m}$ *is stable, then the following facts are equivalent:*

(1) R *is almost Gorenstein*

(2) Λ *is Gorenstein.*

Proof
By hypothesis $R : \Lambda = \mathfrak{m}$ and $\nu = 1$. Hence if R is almost Gorenstein, then Λ is Gorenstein by Theorem 5.3. Vice versa, the hypothesis $\Lambda = \mathfrak{m} : \mathfrak{m}$ implies that $c_\Lambda = c - e$. Thus if Λ is Gorenstein, then condition (2),(b) of Proposition 3.5 is satisfied and R is almost Gorenstein because

$$2\delta - c = 2\rho - e = e - 2 = r - 1.$$

6.4 CASE $e = \mu + 1$.
If $e = \mu + 1$, the structure of R is quite well understood, see e.g. [15]. From the form of the h-polynomial $h(z) = 1 + (\mu - 1)z + z^\nu$, one can infer that $\rho = \mu - 1 + \nu$. Moreover there are two possibilities depending on the Cohen-Macaulay type r:
(A): If $r < e - 2$, then $\nu = 2$;
(B): If $r = e - 2$, then $\mathfrak{m}^2 = x\mathfrak{m} + (w^2)R$, with $w \in \mathfrak{m} \setminus \mathfrak{m}^2$ and $\mathfrak{m}^3 \subset x\mathfrak{m}$ (see [15], Prop. 5.1).

We begin with a technical lemma.

Lemma 6.5 *Assume that* $r = e - 2$. *Then there exists an element* $w \in \mathfrak{m}$ *with* $v(w) - e \notin v(\mathfrak{m} : \mathfrak{m})$ *such that:*

(1) $\mathfrak{m} = x(\mathfrak{m} : \mathfrak{m}) + wR$ *and* $w\mathfrak{m} \subset x(\mathfrak{m} : \mathfrak{m})$.

(2) $\mathfrak{m}^j = x\mathfrak{m}^{j-1} + w^jR =$
$$= x^{j-1}\mathfrak{m} + x^{j-2}w^2R + ... + xw^{j-1}R + w^jR, \quad \forall\ j = 2, ..., \nu.$$

(3) $\mathfrak{m}^3 \subseteq x\mathfrak{m}$.

(4) For every element $s \in \mathfrak{m} : \mathfrak{m}$ *such that* $v(s) > 0$ *we have*
$$sw^j \in x^{j-1}\mathfrak{m}, \quad \forall\ j = 2, ..., \nu.$$

Proof

(1) The assumption $r = e - 2$ means that $l_R(\mathfrak{m}/x(\mathfrak{m} : \mathfrak{m})) = 1$, hence by 2.1 there exists an element $w \in \mathfrak{m}$ such that $v(w) - e \notin v(\mathfrak{m} : \mathfrak{m})$ and $\mathfrak{m} = x(\mathfrak{m} : \mathfrak{m}) + wR$. To prove the inclusion $w\mathfrak{m} \subseteq x(\mathfrak{m} : \mathfrak{m})$ it suffices to consider the chain $x(\mathfrak{m} : \mathfrak{m}) \subseteq x(\mathfrak{m} : \mathfrak{m}) + w\mathfrak{m} \subset \mathfrak{m}$.

(2) We prove our claim by induction on j. Suppose $j = 2$. From (1) we have that $\mathfrak{m}^2 \subseteq x\mathfrak{m} + w\mathfrak{m} = x\mathfrak{m} + w(x(\mathfrak{m} : \mathfrak{m}) + wR) \subseteq x\mathfrak{m} + w^2R \subseteq \mathfrak{m}^2$. Suppose now the assertion true for j. Claim:

$$\begin{aligned}\mathfrak{m}^{j+1} &= x\mathfrak{m}^j + w^{j+1}R \\ &= x^j\mathfrak{m} + x^{j-1}w^2R + ... + xw^jR + w^{j+1}R.\end{aligned}$$

By using repeatedly the inductive hypothesis we get

$$\begin{aligned}\mathfrak{m}^{j+1} &= x\mathfrak{m}^j + w^j\mathfrak{m} \subseteq x\mathfrak{m}^j + w\mathfrak{m}^j = x\mathfrak{m}^j + w(x\mathfrak{m}^{j-1} + w^jR) \\ &\subseteq x\mathfrak{m}^j + w^{j+1}R \subseteq \mathfrak{m}^{j+1}.\end{aligned}$$

We are left to prove the second equality of the claim. We have:

$$\begin{aligned}\mathfrak{m}^{j+1} &= x^{j-1}\mathfrak{m}^2 + x^{j-2}w^2\mathfrak{m} + ... + xw^{j-1}\mathfrak{m} + w^j\mathfrak{m} \\ &\subseteq x^{j-1}\mathfrak{m}^2 + x^{j-2}w\mathfrak{m}^2 + ... + xw^{j-2}\mathfrak{m}^2 + w^{j-1}\mathfrak{m}^2 \\ &= x^{j-1}(x\mathfrak{m} + w^2R) + x^{j-2}w(x\mathfrak{m} + w^2R) + ... + xw^{j-2}(x\mathfrak{m} + w^2R) \\ &\quad + w^{j-1}(x\mathfrak{m} + w^2R) \\ &\subseteq x^j\mathfrak{m} + x^{j-1}w^2R + ... + x^2w^{j-1}R + xw^jR + w^{j+1}R \subseteq \mathfrak{m}^{j+1}.\end{aligned}$$

For the last but one inclusion we have used the fact that

$$\begin{aligned}&x^{j-1}w\mathfrak{m} + ... + x^2w^{j-2}\mathfrak{m} + xw^{j-1}\mathfrak{m} \\ &= x^{j-1}w(x(\mathfrak{m} : \mathfrak{m}) + wR) + ... + xw^{j-1}(x(\mathfrak{m} : \mathfrak{m}) + wR) \\ &\subseteq x^j\mathfrak{m} + x^{j-1}w^2R + ... + x^2w^{j-1}R + xw^jR.\end{aligned}$$

(3) As seen in the proof of item (2), $\mathfrak{m}^2 = x\mathfrak{m} + w\mathfrak{m}$. Hence $\mathfrak{m}^3 = x\mathfrak{m}^2 + w\mathfrak{m}^2 \subseteq x\mathfrak{m}$, because $w\mathfrak{m}^2 \subseteq x\mathfrak{m}$ by item (1).

(4) Let $s \in \mathfrak{m} : \mathfrak{m}$ be such that $v(s) > 0$. We proceed by induction on j. Suppose $j = 2$. By item (1) there exist $y \in \mathfrak{m} : \mathfrak{m}$ and $a \in R$ such that $sw = xy + aw$. If $v(a) = 0$, then $v(s - a) = 0$, contradicting the fact that $v(w) - e \notin v(\mathfrak{m} : \mathfrak{m})$. Hence $a \in \mathfrak{m}$. Thus $sw^2 = xyw + aw^2 \in x\mathfrak{m}$, because $\mathfrak{m}^3 \subseteq x\mathfrak{m}$. Assume now the inductive hypothesis $\dfrac{sw^j}{x^{j-1}} \in m$, then $\dfrac{sw^j}{x^{j-1}} = xz + bw$, with $z \in \mathfrak{m} : \mathfrak{m}$, $b \in R$, i.e., $(\dfrac{sw^{j-1}}{x^{j-1}} - b)w = xz$. Since the element $\dfrac{sw^{j-1}}{x^{j-1}}$ has a positive valuation, by the same reasoning as above we conclude that $b \in \mathfrak{m}$. Therefore $\dfrac{sw^{j+1}}{x^{j-1}} = xzw + bw^2 \in x\mathfrak{m}$, which is our thesis.

Proposition 6.6

(1) If $r = e - 2$, *then* $e = \mu + 1$.

(2) If $e = \mu + 1$, *then* $l_R(x(R : \mathfrak{m})/R : \Lambda) = 1$.

Proof

(1) By Lemma 6.5,(2), $\mathfrak{m}^2 = x\mathfrak{m} + w^2R$. Hence

$$l_R(\mathfrak{m}/\mathfrak{m}^2) = l_R(\mathfrak{m}/x\mathfrak{m}) - l_R(\mathfrak{m}^2/x\mathfrak{m}) = e - 1.$$

(2) We shall prove that the R-module

$$R:\mathfrak{m}/x^{\nu-1}(R:\mathfrak{m}^\nu)\simeq x(R:\mathfrak{m})/R:\Lambda$$

is monogenous generated by $\overline{1}$. We divide the proof in two parts, following cases (A), $r<e-2$ and (B), $r=e-2$ above.
Case (A). $\nu=2$. Since $\mathfrak{m}^2=x\mathfrak{m}+(a)R,\ a\notin x\mathfrak{m}$, we have that $x(R:\mathfrak{m}^2)=(R:\mathfrak{m})\cap\left(\frac{x}{a}\right)R$. If $y\in R:\mathfrak{m}$, then $ya\in\mathfrak{m}^2$ and we can write $ya=xr+as$, with $r\in\mathfrak{m},\ s\in R$, namely $y=\frac{x}{a}r+s$, so $\overline{y}=\overline{1}s$.
Case (B). We want to prove that if $s\in\mathfrak{m}:\mathfrak{m}$ has a positive valuation, then $s\in x^{\nu-1}(R:\mathfrak{m}^\nu)$. By item (2) and (4) of Lemma 6.5 we have

$$\frac{s\mathfrak{m}^\nu}{x^{\nu-1}}\in s\mathfrak{m}+s\frac{w^2}{x}R+....+s\frac{w^{\nu-1}}{x^{\nu-2}}R+s\frac{w^\nu}{x^{\nu-1}}R\subseteq\mathfrak{m}.$$

Theorem 6.7
Let $e=\mu+1$. Then

$$l_R(R:\Lambda/\mathfrak{m}^\nu)=r-1+(e-1)(\nu-2).$$

Proof
We have to compute the difference $l_R(R/\mathfrak{m}^\nu)-l_R(R/R:\Lambda)$. As recalled in 6.4 $e\nu-l_R(R/\mathfrak{m}^\nu)=\rho=e-2+\nu$. Combining the above results 6.1 and 6.6 we obtain $l_R(R/R:\Lambda)=e-r+1$. The conclusion follows.

Corollary 6.8
Let $e=\mu+1$. Then

(1) $R:\Lambda=\mathfrak{m}^\nu$ if and only if R is Gorenstein and $\nu=2$.

(2) $\sum_{i\notin\Gamma,i\neq 1}(r_i-1)=d(R:\Lambda)+l_R(\Lambda^{**}/\Lambda)+(\nu-2)$.

(3) R is almost Gorenstein if and only if $\nu=2$ and $\omega\Lambda=\Lambda$.

Proof

(1) It follows directly from Theorem 6.7.

(2) As recalled in 6.4 $\rho=\mu-1+\nu$, then the formula of Theorem 4.7 gives:
$$l_R(R:\Lambda/\mathfrak{m}^\nu)=(\mu+1)\nu-2(\mu-1+\nu)+\sum_{i\notin\Gamma}(r_i-1)-d(R:\Lambda)-l_R(\Lambda^{**}/\Lambda).$$
By comparing with Theorem 6.7 the thesis follows.

(3) This is clear after observing that item (2) combined with equality (1) of Proposition 4.3 becomes:
$$(2\delta-c)-(r-1)=l_R(\omega\Lambda/\Lambda)+(\nu-2).$$

Corollary 6.9
$r=e-2$ and $R:\Lambda=\mathfrak{m}^\nu$ if and only if R is Gorenstein with $e=3$.

6.10 CASE $\nu = 2$.

We recall that in this case the invariants ρ, e, μ are related by the equality:

$$\rho = 2e - \mu - 1.$$

Proposition 6.11

Assume $\nu = 2$.

(1) $2e + rl_R(R : \Lambda/\mathfrak{m}^2) \le (r+1)(\mu+1)$.

(2) *If R is almost Gorenstein, then* $e - (\mu+1) = \dfrac{r-1}{2} - \dfrac{1}{2} l_R(R : \Lambda/\mathfrak{m}^2)$.

In particular:

if R is Gorenstein, then $e = \mu + 1$ and $R : \Lambda = \mathfrak{m}^2$;

if R is a Kunz ring (namely almost Gorenstein of type 2), then

$$e = \mu + 1 \text{ and } l_R(R : \Lambda/\mathfrak{m}^2) = 1.$$

Proof

(1) The inequality follows directly from Corollary 4.6.

(2) This is Theorem 5.3 with $\nu = 2$, $\rho = 2e - \mu - 1$ and $\Lambda = \Lambda^{**}$.

We deduce from Proposition 6.11 that if R is almost Gorenstein, then:

$$(*) \qquad R : \Lambda = \mathfrak{m}^2 \quad \textit{if and only if} \quad e - (\mu+1) = \frac{r-1}{2} \quad \textit{and} \quad \nu = 2.$$

This equivalence was already known for Gorenstein rings: assertion $(*)$ in the case $r = 1$ is exactly Corollary 7 of [12].

We remark that there exist almost Gorenstein rings satisfying the condition

$$e - (\mu+1) = \frac{r-1}{2}$$

with $\nu \neq 2$ (see Example 7.3). The next corollary shows that this cannot happen when R is Gorenstein.

Corollary 6.12

Let R be a Gorenstein ring. Then the following conditions are equivalent:

(1) $e = \mu + 1$.

(2) $\nu = 2$.

(3) $R : \Lambda = \mathfrak{m}^2$.

Proof

If $e = \mu + 1$, we have that $\nu = 2$ by Corollary 6.8. To conclude the proof it suffices to apply Proposition 6.11.

In the case R Gorenstein, Corollary 6.8 combined with Corollary 6.12 gives Proposition 12 of [12]:

$$\gamma_R = \mathfrak{m}^2 \quad \textit{if and only if} \quad \Lambda = \overline{R} \quad \textit{and} \quad e = \mu + 1.$$

The following proposition states a more general relation between e and $\mu + 1$.