Type-sequences of modules.

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Abstract. The main purpose of the paper is to find a suitable description of a class of modules, which we call *almost symmetric*, over a complete local k-algebra R of dimension one, k being an algebraically closed field of characteristic zero. One of the properties characterizing the modules M of this class, including the canonical module, is that the Cohen-Macaulay type $r_R(M)$ reaches the known bound $\delta + \delta(M) - c(M) + 1$. Another interesting property is obtained by extending to modules the notion of *type sequence*, given for rings in [1]. In fact, it is proved that the equality $r_R(M) = \delta + \delta(M) - c(M) + 1$ holds if and only if the type sequence of M is of the form $[r_R(M), 1, ..., 1]$. In the third section we investigate the meaning of the almost simmetry of modules in terms of properties of their value sets. In the last section we consider two particular cases: i) almost symmetric modules with $r_R(M) = 2$ (*almost canonical*), ii) modules over almost symmetric rings.

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Let $R = k[\![x_1, ..., x_n]\!]$ be a complete local k-algebra of dimension one with maximal ideal $\mathfrak{m}_R$, where k is an algebraically closed field of characteristic zero. We suppose R an integral domain. Throughout the paper we shall use the same notations and assumptions as in [9], which we list here for the convenience of the reader: $\overline{R} := k[\![t]\!]$ the integral closure of R in its quotient field $K := k\{\{t\}\}$; $\nu : k\{\{t\}\} \longrightarrow \mathbb{Z} \cup \infty$ the canonical valuation given by the degree in t; Γ the *value semigroup* of R; c the *conductor* and $\delta := dim_k(\overline{R}/R)$ the *singularity degree* of R; $r(R) := l_R(R : \mathfrak{m}_R)/R$ the *Cohen-Macaulay type* of R. Moreover calling $e := e(R)$ the multiplicity of R, we shall suppose that $t^e \in \mathfrak{m}_R$ (this is always possible via a suitable change of coordinates).
Analogously, for any fractional R-ideal $M \subset K$, $\Gamma(M) := \{\nu(m) \mid m \in M\}$ is the *value set* of M; $\Gamma(M)$ is a Γ-set, i.e. $\Gamma(M)+\Gamma \subset \Gamma(M)$; $c(M) :=$ the smallest integer such that $t^{c(M)}\overline{R} \subset M$ is the *conductor* of M, i.e. $c(M)+\mathbb{N} \subset \Gamma(M)$ and $c(M) - 1 \notin \Gamma(M)$; $\delta(M) := l_R(\overline{M}/M)$ where $\overline{M} := M \otimes_R \overline{R}/torsion$ the δ-*invariant* of M; $r_R(M) := l_R(M : \mathfrak{m}_R)/M$ the *Cohen-Macaulay type* of M. In virtue of [4], Lemma 1.1, we shall assume in the sequel, without loss of generality, that $R \subset M \subset \overline{R}$; this ensures that $\delta(M) = l_R(\overline{R}/M)$ and

$l_R(M/t^{c_k}\overline{R}) = c_k - \delta(M)$, where $c_k = c(M)$. We shall use indifferently both notations $c(M)$ or c_k for the conductor of an R-module M. The reason is the following. Set $c_0 = 0 < c_1 = 2 < ... < c_\delta = c$ the integer numbers such that $c_k - 1 \notin \Gamma$ for all $k = 0, ..., \delta$. They correspond naturally to a chain of overrings: $\overline{R} = R_0 \supset R_1 \supset ...R_k... \supset R_\delta = R$ defined by $R_k := R + t^{c_k}\overline{R}$. Then there exists $k \in \{0, ..., \delta\}$ such that $c(M) = c_k$. In view of what follows it is useful to note that M may be considered an R-module as well as an R_k-module.

- Two notions of type-sequence.

Given the value set of R $\Gamma = \{s_0 = 0, s_1, ..., s_{n-1}, s_n = c, \rightarrow\}$, where $n := c-\delta$, we consider for every $i = 0, ..., n$ the ideal $V_i := \{x \in R,\ \nu(x) \geq s_i\}$. Obviously $V_n = t^c\overline{R}$, $V_1 = \mathfrak{m}_R$, $V_0 = R$. Starting from the maximal sequence: $V_n \subset V_{n-1} \subset .... \subset V_1 \subset V_0$ we get the chain of R-ideals contained in K:

$$M = M : V_0 \subset M : V_1 \subset ... \subset M : V_n = t^{c_k - c}\overline{R}$$

Extending to the R-modules the terminology introduced in [1] for rings we put $t_i := l_R(M : V_i/M : V_{i-1})$, $i = 1, ..., n$, and we call *type-sequence* of M ($t.s.(M)$ for short) the sequence $[t_1, ..., t_n]$. Note that: $t_1 = l_R(M : \mathfrak{m}_R/M) = r_R(M)$ is the Cohen Macaulay type of M and

1.1 $c - c_k + \delta(M) = l_R(t^{c_k - c}\overline{R}/M) = r_R(M) + \sum_2^n t_i.$

Also in our case

1.2 $1 \leq t_i \leq t_1 \quad \forall\ i = 1, ..., n.$

Since the element $z := t^{c_k - 1 - s_{i-1}}$ is such that $z \in M : V_i$ and $z \notin M : V_{i-1}$, $t_i \geq 1$. To see the second inequality we recall the following result ([6], Satz 2): Let $(R, \mathfrak{m}_R)$ be a local one-dimensional Cohen-Macaulay ring and let $\mathfrak{a}, \mathfrak{b}, M$ be fractional ideals such that $\mathfrak{b} \subset \mathfrak{a}$; then : $l_R(M : \mathfrak{b}/M : \mathfrak{a}) \leq l_R(\mathfrak{a}/\mathfrak{b})\ r_R(M)$. Applying this with $\mathfrak{b} := V_i$ and $\mathfrak{a} := V_{i-1}$, since by definition $l_R(V_{i-1}/V_i) = 1$, we get $t_i := l_R(M : V_i/M : V_{i-1}) \leq r_R(M) = t_1$.

Of course, we can do the same regarding M as an R_k-module and we obtain the *k-type-sequence* of M ($k - t.s.(M)$ for short) $[l_1, ..., l_m]$, $m := c_k - k \leq n$, where $l_i := l_R(M : V_i^{(k)}/M : V_{i-1}^{(k)})$, $V_i^{(k)} := \{x \in R_k,\ \nu(x) \geq s_i\}$, $i = 1, ..., m$. The analogue of statement 1.1 is:

1.3 $\delta(M) = r_k(M) + \sum_2^m l_i$, where $l_1 = r_k(M)$ is the Cohen Macaulay type of M as R_k-module. As in the preceding case

$$1 \leq l_i \leq l_1 \quad \forall\ i = 1, ..., m.$$

Note that in general $\delta(M) \leq (c_k - k)r_k(M)$ and (case of "maximal" k-type-sequence)

1.4 $\delta(M) = (c_k - k)r_k(M) \iff k - t.s.(M) = [r_k, ..., r_k]$, $r_k := r_k(M)$

To go on in comparing the two notions of type-sequence we need to recall that for any fractional ideals N_1, N_2, $N_2 \subset N_1$, the length of the R-module N_1/N_2 can be computed by means of valuations (see [6]):

1.5 $\quad l_R(N_1/N_2) = \#\ \Gamma(N_1) \setminus \Gamma(N_2)$

1.6 Notations. For any numerical sets $H,\ K \subset \mathbb{Z}$ put $H - K := \{x \in \mathbb{Z} \mid x + K \subset H\}$ and $H^+ := H \cap \mathbb{N}$, $H^- := H \setminus H^+$.

Next proposition shows that each invariant l_i represents the positive contribution of the corresponding t_i and it states an upper bound for the difference $t_i - l_i$.

Proposition 1.7

i) For every $i = 1, ..., m \quad l_i = \#\ (\Gamma(M : V_i) \setminus \Gamma(M : V_{i-1}))^+ \ \leq\ t_i$*;*

ii) if for some $i \in \{1, ..., m\}$ $t_i = 1$, *then the corresponding* $l_i = 1$*;*

iii) $\sum_1^m (t_i - l_i) \ \leq\ \delta - k \ \leq\ c - c_k$*;*

iv) $\sum_1^m (t_i - l_i) = \delta - k$ *if and only if* $t.s.(M) = [t_1, ..., t_m, 1, ..., 1]$.

Proof. Recall that $m = c_k - k$, $n = c - \delta$.
i) By formula 1.5 $t_i = \#\,\Gamma(M : V_i) \setminus \Gamma(M : V_{i-1})$ and since $V_i^{(k)} = V_i + t^{c_k}\overline{R}$, we have $M : V_i^{(k)} = (M : V_i) \cap (M : t^{c_k}\overline{R}) = (M : V_i) \cap \overline{R}$. Therefore $l_i = l_R((M : V_i) \cap \overline{R}\ /(M : V_{i-1}) \cap \overline{R}) = \#\ (\Gamma(M : V_i) \setminus \Gamma(M : V_{i-1}))^+$.
ii) By 1.3 $l_i \geq 1$ and by i) $l_i \leq t_i = 1$.
iii) Combining formula 1.1: $c - c_k + \delta(M) = \sum_1^n t_i$ with formula 1.3: $\delta(M) = \sum_1^m l_i$, we get $c - c_k + \sum_1^m l_i = \sum_1^n t_i$, hence $\sum_1^m (t_i - l_i) = c - c_k - (\sum_{m+1}^n t_i) \leq c - c_k - (n - m) = \delta - k$. The second inequality is obvious, since by definition $c_k - k \leq c - \delta$.
iv) follows from the preceding computation.

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The present paragraph is devoted to finding suitable characterizations of the modules having type-sequence of the form $[r, 1, ..., 1]$, or, equivalently, maximal Cohen-Macaulay type, which we call almost symmetric. Our study has been inspired by the papers of several authors, Barucci, D'Anna, Delfino, Dobbs, Fontana, Fröberg, who considered analogous properties in the case of rings (see [1], [2], [3]). Canonical modules play a crucial rôle in our context.

Note: all isomorphisms in this section are realized by units of $\overline{R}$, so that isomorphic modules have the same value set.

The *dualizing module* of R is: $\omega_R = \{\alpha \in k\{\{t\}\}\ dt \mid res(f\alpha) = 0\ \ \forall f \in R\}$. By means of the isomorphism $k\{\{t\}\}\ dt \simeq k\{\{t\}\}$ which maps $dt \longmapsto 1$ we shall identify ω_R with a fractional ideal. We fix as *canonical ideal* of R the ideal $\widetilde{\omega} := \epsilon t^c \omega_R$, where $\epsilon \in \overline{R}$ is a unit such that $R \subset \widetilde{\omega} \subset \overline{R}$; it follows that $c(\widetilde{\omega}) = c$ and $\Gamma(\widetilde{\omega}) = \{j \in \mathbb{Z} \mid c-1-j \notin \Gamma\}$. Moreover it is well known that $r_R(M) = 1 \iff M \simeq \widetilde{\omega}$, ([5], Korollar 6.12), hence "minimal" type-sequence, i.e. $t.s.(M) = [1, ..., 1]$, means $M \simeq \widetilde{\omega}$ and "minimal" k-type-sequence, i.e. $k - t.s.(M) = [1, ..., 1]$, means $M \simeq \widetilde{\omega}_k := \widetilde{\omega}_{R_k}$. Some properties of $\widetilde{\omega}$ are fundamental in our computations ([5], 2.): for any fractional ideals $\mathfrak{a}, \mathfrak{b}$, $\mathfrak{a} = \widetilde{\omega} : (\widetilde{\omega} : \mathfrak{a})$ and $l_R(\mathfrak{a}/\mathfrak{b}) = l_R(\widetilde{\omega}:\mathfrak{b}/\widetilde{\omega}:\mathfrak{a})$. Moreover: $\widetilde{\omega} : \widetilde{\omega} = R$.

Using the well known 'duality' $\omega_R : R_k = \omega_{R_k}$ ([5], Korollar 5.14), we obtain by a straightforward calculation:

2.1 $\widetilde{\omega}_k \simeq t^{c_k - c}\widetilde{\omega} \cap \overline{R} \simeq \widetilde{\omega} : t^{c-c_k} R_k$

Proof. Let $\widetilde{\omega} = \epsilon t^c \omega_R$, $\widetilde{\omega}_k = \epsilon' t^{c_k} \omega_{R_k}$. $\omega_{R_k} = \omega_R : R_k = \omega_R : (R + t^{c_k}\overline{R}) = \omega_R \cap t^{-c_k}\overline{R}$ yields $(\epsilon')^{-1} t^{-c_k} \widetilde{\omega}_k = \epsilon^{-1} t^{-c} \widetilde{\omega} \cap t^{-c_k}\overline{R}$, which is the first assertion because ϵ, ϵ' are units of $\overline{R}$. Moreover $\widetilde{\omega}_k = \epsilon' t^{c_k} \omega_{R_k} = \epsilon' t^{c_k} \omega_R : R_k = \epsilon' t^{c_k} \epsilon^{-1} t^{-c} \widetilde{\omega} : R_k = \epsilon' \epsilon^{-1} \widetilde{\omega} : t^{c-c_k} R_k$.

2.2 $\widetilde{\omega} : \gamma(M) = \epsilon t^c \omega_R : t^{c_k}\overline{R} = t^{c-c_k}\overline{R}$

2.3 $\mathfrak{m}_R(\widetilde{\omega} : M) = \widetilde{\omega} : (\widetilde{\omega} : (\mathfrak{m}_R(\widetilde{\omega} : M))) = \widetilde{\omega} : (M : \mathfrak{m}_R)$

We recall now the following generalization of [6] Satz 5 and a useful corollary:

2.4 ([9], Lemma 1.1) *Let N be a finitely generated torsion free R-module of rank 1. If $\Gamma(N) \subset \{j \in \mathbb{Z} \mid c-1-j \notin \Gamma\}$, then there exists a unit $u \in \overline{R}$ such that $uN \subset \widetilde{\omega}$. If $\Gamma(N) = \{j \in \mathbb{Z} \mid c-1-j \notin \Gamma\}$, then $uN = \widetilde{\omega}$.*

The above statement is slightly more general than the quoted result of [9]; we have only to prove that the assumption 'N containing $t^c\overline{R}$' is redundant.
If $c(N) \leq c$, then $N \supset t^c\overline{R}$. If $c(N) > c$, then consider the k-vector space $N/(t^c\overline{R} \cap N)$ instead of $N/t^c\overline{R}$ and repeat Step 1, Step 2, until to find the unit u such that $Res(unr/t^c) = 0 \ \forall\, r \in R, \ \forall\, n \in N, \ \nu(n) < c$. Since for any $n \in N$, $\nu(n) \geq c$, certainly $Res(unr/t^c) = 0 \ \forall\, r \in R$, we can conclude.

2.5 ([9], Prop.1.2) *For any R-module $M \subset \overline{R}$ with conductor $c(M) = c_k$ there exist units $u, u' \in \overline{R}$ such that $uM \subset \widetilde{\omega}_k$ and $u' t^{c-c_k} M \subset \widetilde{\omega}$.*

In the rest of the paper we shall refer to these inclusions as *canonical immersions of M*. Unfortunately, as next example shows, it is not always possible to realize both immersions by means of the same unit.

Examples

- If $R := k[\![t^{15}, t^{21}, t^{25} + t^{28}, t^{32}]\!]$ and $R_k := k[\![t^{15}, t^{21}, t^{25} + t^{28}, t^{29}, \rightarrow]\!]$, $k = 25$, then we have $\widetilde{\omega} = < 1, t^{10}, t^{11}(1-t^3), t^{13}(1-t^6), t^{17}, t^{20}(1-t^3) > R$ and $\widetilde{\omega}_k = R_k + (1-t^3)^{-1} < t, t^2, t^4, t^5, t^6, t^8, t^9, t^{10}, t^{11}, t^{12}, t^{14}, t^{18} > R_k$. Let $M := \widetilde{\omega}_k$ and suppose that there exists a unique unit $u \in \overline{R}$ such that $uM \subset \widetilde{\omega}_k$ and $ut^{c-c_k}M \subset \widetilde{\omega}$. Then $u \in \widetilde{\omega}_k : \widetilde{\omega}_k = R_k$ and $t^{c-c_k}M \subset \widetilde{\omega}$.

This would imply in particular, since $c=70$ and $c_k=29$, that $t^{41}\in\widetilde{\omega}$. If so, then $t^{69}=(t^{25}+t^{28})t^{41}-t^{30}(t^{25}+t^{28})t^{11}(1-t^3)-t^{72}\in\widetilde{\omega}$, contradiction.

- We note also that in general it does not exist a unit $u\in\overline{R}$ such that $R\subset uM\subset\widetilde{\omega}$. Let $R:=k[\![t^5,t^8,t^{22}]\!]$ and $M:=(1+t^2)R+t^8R+t^{13}R$. In this case we have $c=20$ and $\widetilde{\omega}=<1,t^2>R$. Suppose there exists $u=1+b_1t+b_2t^2+b_3t^3+.....$, $b_i\in k$, such that $1=u\;m$, $m\in M$, i.e. $1=(1+b_1t+b_2t^2+b_3t^3+.....)\;(a_1+a_1t^2+a_2t^5+a_2t^7+a_3t^8+.....)$, $a_i\in k$. Then an easy calculation gives $u=1-t^2+t^4+......$ Since $ut^{17}=t^{17}-t^{19}+t^{21}+....\in uM$ and $t^{17}\in\widetilde{\omega}$, the inclusion $uM\subset\widetilde{\omega}$ would imply $c-1=19\in\Gamma(\widetilde{\omega})$, absurd.

As a first consequence we can get an elementary result on valuations:

Lemma 2.6 *For any fractional ideal* $N\subset K$

$$\Gamma(\widetilde{\omega}:N)=\Gamma(\widetilde{\omega})-\Gamma(N)$$

Proof. Since the inclusion $\subset$ holds in general, we have to prove $\supset$. Let j be such that $j+\Gamma(N)\subset\Gamma(\widetilde{\omega})\implies\Gamma(t^jN)\subset\Gamma(\widetilde{\omega})\implies$ by the quoted lemma 2.4 $ut^jN\subset\widetilde{\omega}$ for some unit $u\in\overline{R}\implies ut^j\in\widetilde{\omega}:N\implies j\in\Gamma(\widetilde{\omega}:N)$.

and, considering again the fixed immersions of M, other interesting relations:

Lemma 2.7

i) $\widetilde{\omega}:t^{c-c_k}M\;\simeq\;\widetilde{\omega}_k:M$;

ii) $l_R(\widetilde{\omega}/u't^{c-c_k}M)=\delta+\delta(M)-c(M)$;

iii) $l_R(\widetilde{\omega}_k/uM)=k+\delta(M)-c(M)$.

Proof. i) By 2.1 we have: $\widetilde{\omega}:t^{c-c_k}M\;=\;(\widetilde{\omega}:t^{c-c_k}R_k):M\;\simeq\;\widetilde{\omega}_k:M$.
ii) Using duality and 2.2 we get $\delta+\delta(M)-c(M)\;=\;l_R(\overline{R}/R)-l_R(M/\gamma(M))\;=\;l_R(\overline{R}/R)-l_R(\overline{R}/\widetilde{\omega}:t^{c-c_k}M)\;=\;l_R(\widetilde{\omega}:t^{c-c_k}M/u'R)\;=\;l_R(\widetilde{\omega}/u't^{c-c_k}M)$.
iii) Using also i) $k+\delta(M)-c(M)\;=\;l_R(\overline{R}/R_k)-l_R(\overline{R}/\widetilde{\omega}_k:M)\;=\;l_R(\widetilde{\omega}_k:M/uR_k)\;=\;l_R(\widetilde{\omega}_k/uM)$.

Remark 2.8 About assertion ii) we could be a little more precise.
From $l_R(\widetilde{\omega}:\mathfrak{m}_R/\widetilde{\omega})=1$ we deduce $\widetilde{\omega}:\mathfrak{m}_R=\widetilde{\omega}+t^{c-1}\overline{R}$, hence:
$r_R(M)=l_R(u't^{c-c_k}M:\mathfrak{m}_R/u't^{c-c_k}M)\;\leq\;l_R(\widetilde{\omega}:\mathfrak{m}_R/u't^{c-c_k}M)=$
$=l_R(\widetilde{\omega}/u't^{c-c_k}M)+1$. Thus ii) of 2.7 implies the well known result

2.9 $r_R(M)-1\leq\;\delta+\delta(M)-c(M)$

Notice that inequality 2.9 follows also immediately from 1.1; a generalization of this formula is given in [9], Prop.1.4.

At this point we are able to describe a family of modules, including the canonical module, in which the Cohen-Macaulay type achieves the maximal value. We need the following lemma:

Lemma 2.10 *For any fractional ideal* $N\subset\widetilde{\omega}$

$$r_R(N)-1=\;l_R(\widetilde{\omega}/N)\iff\mathfrak{m}_R\;=\;\mathfrak{m}_R\;(\widetilde{\omega}:\;N)$$

Proof. From $l_R(\widetilde{\omega}/N) = l_R(\widetilde{\omega} : N/R) = l_R(\widetilde{\omega} : N/\mathfrak{m}_R(\widetilde{\omega} : N)) + l_R(\mathfrak{m}_R(\widetilde{\omega} : N)/\mathfrak{m}_R) - 1 = r_R(N) + l_R(\mathfrak{m}_R(\widetilde{\omega} : N)/\mathfrak{m}_R) - 1$ we infer the thesis.

Theorem 2.11 *The following are equivalent:*

i) $r_R(M) - 1 = \delta + \delta(M) - c(M)$;

ii) $r_R(M) - 1 = l_R(\widetilde{\omega}/u' t^{c-c_k} M)$;

iii) $\mathfrak{m}_R \simeq \mathfrak{m}_R\, (\widetilde{\omega} :\ t^{c-c_k} M)$;

iv) $\widetilde{\omega} : \mathfrak{m}_R \simeq t^{c-c_k} M :\ \mathfrak{m}_R$;

v) $\mathfrak{m}_R\, \widetilde{\omega} \subset u' t^{c-c_k} M$;

vi) $t.s.(M) = [r_R(M), 1, ..., 1]$.

We call M *almost symmetric* if it fulfills the above equivalent conditions.
We call M *weakly almost symmetric* if it is almost symmetric as R_k-module.

Proof. i) $\Longleftrightarrow$ ii) is an immediate consequence of preceding 2.7 ii).
ii) $\Longleftrightarrow$ iii) by lemma 2.10 taking $N = u' t^{c-c_k} M$, because $r_R(N) = r_R(M)$.
iii) $\Longleftrightarrow$ iv) by duality.
iv) $\Longleftrightarrow$ v). As noted in remark 2.8 $\widetilde{\omega} : \mathfrak{m}_R = \widetilde{\omega} + t^{c-1}\overline{R}$, then condition iv) holds $\Longleftrightarrow$ $\widetilde{\omega} : \mathfrak{m}_R \subset u' t^{c-c_k} M :\ \mathfrak{m}_R \Longleftrightarrow \widetilde{\omega} \subset u' t^{c-c_k} M : \mathfrak{m}_R$ (because the inclusion $t^{c-1}\overline{R} \subset u' t^{c-c_k} M : \mathfrak{m}_R$ is always verified) $\Longleftrightarrow$ $\mathfrak{m}_R \widetilde{\omega} \subset u' t^{c-c_k} M$.
i) $\Longleftrightarrow$ vi). We know that $\sum_2^n t_i = c - c(M) + \delta(M) - r_R(M)$, $n = c - \delta$. Therefore, if $t_i = 1$, $i = 2, ..., n$, then we get i). Conversely, i) implies that $\sum_2^n t_i = c - \delta - 1$ and since $t_i \geq 1$, $i = 2, ..., n$, the thesis is proved.

Remark 2.12

- In virtue of 1.7 ii) we see immediately that:

M almost symmetric $\Longrightarrow$ M weakly almost symmetric.

- Conditions i),..,iv) can be viewed as the module theoretic analogue of the characterization of the almost Gorenstein rings given in prop. 20 of [2]. Notice that for $M = R$ relation iii) becomes $\mathfrak{m}_R = \mathfrak{m}_R\, \widetilde{\omega}$ and, more generally, for any overring $M \supset R$ having $c(M) = c$ it becomes $\mathfrak{m}_R = \mathfrak{m}_R\, \widetilde{\omega}_M$.
- Condition v) can be rewritten in the equivalent form: $u' M \supset \epsilon t^{c_k} \omega_{\mathfrak{m}_R : \mathfrak{m}_R}$.

In fact v) $\Longleftrightarrow$ $u' M \supset \epsilon t^{c_k} \mathfrak{m}_R\, \omega_R$ and, by duality, $\mathfrak{m}_R\, \omega_R = \omega_R : (\mathfrak{m}_R{:}\mathfrak{m}_R) = \omega_{\mathfrak{m}_R : \mathfrak{m}_R}$.

Corollary 2.13 *The following conditions are equivalent:*

i) $\mathfrak{m}_R : \mathfrak{m}_R$ *is almost symmetric as R-module;*

ii) R is almost symmetric and $r(R) = e - 1$;

iii) $\mathfrak{m}_R : \mathfrak{m}_R$ *is a Gorenstein ring.*

Proof. First observe that $r(R)=e-1 \iff l_R(t^e(\mathfrak{m}_R:\mathfrak{m}_R)/t^e\mathfrak{m}_R)=e$ $\iff t^e(\mathfrak{m}_R:\mathfrak{m}_R)=\mathfrak{m}_R$. Then apply 2.11 with $M=\mathfrak{m}_R:\mathfrak{m}_R$, $c_k=c-e$, $u'=1$. By v) of the quoted theorem M is almost symmetric means $\mathfrak{m}_R\,\widetilde{\omega}=t^e(\mathfrak{m}_R:\mathfrak{m}_R)=\mathfrak{m}_R$. i) $\iff$ ii) follows now immediately.
Moreover $\mathfrak{m}_R:\mathfrak{m}_R$ is a Gorenstein ring $\iff \mathfrak{m}_R:\mathfrak{m}_R \simeq \widetilde{\omega}_{\mathfrak{m}_R:\mathfrak{m}_R} \iff t^e(\mathfrak{m}_R:\mathfrak{m}_R) \simeq \widetilde{\omega}\,\mathfrak{m}_R \iff t^e(\mathfrak{m}_R:\mathfrak{m}_R) = \widetilde{\omega}\,\mathfrak{m}_R = \mathfrak{m}_R$. Then iii) $\iff$ ii).
A proof of ii) $\iff$ iii) is also in [2] , Prop.25.

Let $\mathcal{M}(R)$ be the reduced moduli variety for finitely generated torsion free R-modules of rank 1 constructed by G.M. Greuel and G. Pfister in [4]. It is well known that the number $\delta(M{:}M)$ represents the orbit dimension of M in $\mathcal{M}(R)$ (see also [9]).

Proposition 2.14 *Let M be almost symmetric, then:*

i) $M{:}M \subset \mathfrak{m}_R:\mathfrak{m}_R$

ii) $\delta - r(R) \le \delta(M{:}M) \le k$

iii) $c-c_k \le e$

iv) $r_R(M) \le r(R)+1$.

Moreover the following are equivalent:

v) $r_R(M) = r(R)+1$

vi) $\mathfrak{m}_R\,\widetilde{\omega} = u't^{c-c_k}M$

vii) $\delta(M:M) = \delta - r(R)$.

Proof. We have obviously that: $u't^{c-c_k}M : u't^{c-c_k}M \subset \widetilde{\omega} : u't^{c-c_k}M$, hence $\mathfrak{m}_R \subset \mathfrak{m}_R(M{:}M) \subset \mathfrak{m}_R(\widetilde{\omega}:u't^{c-c_k}M)=\mathfrak{m}_R$. This implies i).
Claim ii) is immediate since $R_k \subset M{:}M \subset \mathfrak{m}_R{:}\mathfrak{m}_R$.
Claim iii) holds because condition v) of 2.11 implies the inclusion $\Gamma(\mathfrak{m}_R\widetilde{\omega}) \subset \Gamma(t^{c-c_k}M)$, hence $e \in c-c_k+\Gamma(M)$.
From $r(R)=l_R(\widetilde{\omega}/\mathfrak{m}_R\widetilde{\omega})=l_R(\widetilde{\omega}/u't^{c-c_k}M)+l_R(u't^{c-c_k}M/\mathfrak{m}_R\widetilde{\omega})=r_R(M)-1+d$ where $d:=l_R(u't^{c-c_k}M/\mathfrak{m}_R\widetilde{\omega})$ we can deduce $r_R(M)\le r(R)+1$ and v) $\iff$ vi).
Combining the last inequality with $c_k-\delta(M) \le \delta(M{:}M) \le k$ (see [9], Prop. 2.1) and $c_k-\delta(M)=\delta-r_R(M)+1$ (see 2.11, i)), we obtain: $\delta-r(R) \le \delta-r_R(M)+1 \le \delta(M:M) \le k$. So v) $\iff$ vii).

Proposition 2.15

i) If S is any overring such that $R\subset S\subset \mathfrak{m}_R:\mathfrak{m}_R$, then $\widetilde{\omega}_S$ is an almost symmetric R-module;

ii) in particular $\widetilde{\omega}_{\mathfrak{m}_R:\mathfrak{m}_R}$ is R-almost symmetric of C.M. type $r(R)+1$;

iii) $\widetilde{\omega}_k$ is almost symmetric $\iff c-c_k \le e$.

Proof. i) Let $c(S) = c_k$ and let ϵ be the unit such that $\epsilon t^c \omega = \widetilde{\omega}$. Multiplying by ϵt^c the chain $\omega_R : (\mathfrak{m}_R{:}\mathfrak{m}_R) \subset \omega_R : S \subset \omega_R$ and using duality we get: $\mathfrak{m}_R \widetilde{\omega} \subset \epsilon t^c \omega_S \subset \widetilde{\omega}$. Since there exists a unit $\tau \in \overline{R}$ such that $\omega_S = \tau t^{-c_k} \widetilde{\omega}_S$, we can write: $\mathfrak{m}_R \widetilde{\omega} \subset \epsilon \tau t^{c-c_k} \widetilde{\omega}_S \subset \widetilde{\omega}$. So $\widetilde{\omega}_S$ verifies condition v) of 2.11.
ii) Apply i) and proposition 2.14. It is easy to see that $\widetilde{\omega}_{\mathfrak{m}_R:\mathfrak{m}_R}$ verifies condition vii).
iii) Implication $\Longleftarrow$ is iii) of 2.14. Implication $\Longrightarrow$ follows from i), because $R_k \subset \mathfrak{m}_R{:}\mathfrak{m}_R \Longleftrightarrow c - c_k \le e$.

3

Our purpose is now to investigate the meaning of the almost simmetry of modules in terms of properties of their value sets.

First of all by analogy with the notion of $t.s.(M)$ studied in the first section, we introduce the concept of type-sequence for the Γ-set $\Gamma(M)$. This is a natural generalization of the definition of type-sequence given in [1] for numerical semigroups.

Given the value set of R $\Gamma = \{s_0 = 0, s_1, ..., s_{n-1}, s_n = c, \to\}$, where $n := c - \delta$, we consider for every $i = 0, ..., n$ the ideal $S_i := \{s \in \Gamma,\ s \ge s_i\}$. Obviously $S_n = [c, \to]$, $S_1 = \Gamma(\mathfrak{m}_R)$, $S_0 = \Gamma$ and, in general, $S_i = \Gamma(V_i)$. Starting from the maximal sequence: $S_n \subset S_{n-1} \subset .... \subset S_0$ we get the chain:

$$\Gamma(M) = \Gamma(M) - S_0 \subset \Gamma(M) - S_1 \subset ... \subset \Gamma(M) - S_n = [c_k - c, \to]$$

and we put $\tau_i := \#\ (\Gamma(M) - S_i) \setminus (\Gamma(M) - S_{i-1})$, $i = 1, ..., n$. We shall call *type-sequence* of $\Gamma(M)$ ($t.s.(\Gamma_M)$ for short) the sequence $[\tau_1, ..., \tau_n]$. Note that $(\Gamma(M) - S_1) \setminus (\Gamma(M) - S_0) = (\Gamma(M) - \Gamma(\mathfrak{m}_R)) \setminus \Gamma(M)$ is exactly the set denoted in [9] by $A_\Gamma(M) \cup \{c_k - 1\}$, so $\tau_1 - 1 = \alpha_R(M) := \#\ A_\Gamma(M)$ is the invariant introduced in [9].
We can naturally repeat the same process regarding this time $\Gamma(M)$ as a Γ_k-set, $\Gamma_k := \Gamma(R_k)$, obtaining the *k-type-sequence* of $\Gamma(M)$ $(k - t.s.(\Gamma_M)$ for short) $[\lambda_1, ..., \lambda_m]$, $m := c_k - k \le n$, where $\lambda_i := \#\ (\Gamma(M) - S_i^{(k)}) \setminus (\Gamma(M) - S_{i-1}^{(k)})$ and $S_i^{(k)} := \{s \in \Gamma_k,\ s \ge s_i\}$, $i = 1, ..., m$.
Since again $(\Gamma(M) - S_1^{(k)}) \setminus (\Gamma(M) - S_0^{(k)}) = (\Gamma(M) - \Gamma(\mathfrak{m}_k)) \setminus \Gamma(M)$ is the set denoted in [9] by $A_{\Gamma(R_k)}(M) \cup \{c_k - 1\}$, it follows that $\lambda_1 - 1 = \alpha_k(M) := \#\ A_{\Gamma(R_k)}(M)$ is the invariant introduced in [9].
The analogue of statements 1.1, 1.2, 1.3, is:

3.1 $\quad c - c_k + \delta(M) = \#\ [c_k - c, \to] \setminus \Gamma(M) = \sum_1^n \tau_i, \qquad \tau_1 = \alpha_R(M) + 1$

$\quad 1 \le \tau_i \le \tau_1 \quad \forall\ i = 1, ..., n.$

Proof. The first row is the definition of type-sequence. To prove $1 \le \tau_i$ it suffices to observe that $c_k - 1 - (s_i - 1) \in \Gamma(M) - S_i$, $\notin \Gamma(M) - S_{i-1}$. To prove $\tau_i \le \tau_1$, we may consider the monomial module $M_0 = \sum_\gamma t^\gamma k$, $\gamma \in \Gamma(M)$, over the monomial ring $R_0 = \sum_\gamma t^\gamma k$, $\gamma \in \Gamma$. Since $\Gamma(M_0 : V_i(M_0)) =$

$\Gamma(M_0) - \Gamma(V_i(M_0)) = \Gamma(M) - S_i,\ \ t_i(M_0) = \tau_i\ \ \forall\ i = 1, ..., n,$ we conclude by 1.2 that $\tau_i = t_i(M_0)\ \le\ t_1(M_0) = \tau_1$.

3.2 $\quad \delta(M) = \sum_1^m \lambda_i, \qquad \lambda_1 = \alpha_k(M) + 1$

$$1 \le \lambda_i \le \lambda_1 \ \ \forall\ i = 1, ..., m$$

Proof. This is 3.1 when M is regarded as an R_k-module.

The analogue of 1.7 is:

Proposition 3.3

i) *For every* $i = 1, ..., m \quad \lambda_i = \#\ ((\Gamma(M) - S_i) \setminus (\Gamma(M) - S_{i-1}))^+\ \le\ \tau_i$*;*

ii) *if for some* $i \in \{1, ..., m\}\ \ \tau_i = 1,$ *then the corresponding* $\lambda_i = 1$*;*

iii) $\sum_1^m (\tau_i - \lambda_i)\ \le\ \delta - k\ \le\ c - c_k$*;*

iv) $\sum_1^m (\tau_i - \lambda_i) = \delta - k$ *if and only if* $t.s.(M) = [\tau_1, ..., \tau_m, 1, ..., 1]$.

Proof. i) Since $S_i^{(k)} = S_i \cup [c_k, \rightarrow],\ \Gamma(M) - S_i^{(k)} = (\Gamma(M) - S_i) \cap \Gamma(M) - [c_k, \rightarrow])$ $= (\Gamma(M) - S_i) \cap \mathbb{N}$. Therefore $\lambda_i = \#\ (\Gamma(M) - S_i^{(k)}) \setminus (\Gamma(M) - S_{i-1}^{(k)}) =$ $\#\ (\Gamma(M) - S_i)^+ \setminus (\Gamma(M) - S_{i-1})^+ = \#\ ((\Gamma(M) - S_i) \setminus (\Gamma(M) - S_{i-1}))^+$.
For the rest of the proof see 1.7 replacing t_i with τ_i and l_i with λ_i.

3.4 $\quad$ ([9], remark after 1.3) In particular we have:

$$\begin{array}{ccc} \lambda_1 = \alpha_k(M) + 1 & \le & \alpha_R(M) + 1 = \tau_1 \\ | \vee & & | \vee \\ l_1 = r_k(M) & \le & r_R(M) = t_1 \end{array}$$

According to the terminology used in [2] for semigroups, given the Γ-set $\Gamma(M)$ we call $B_1 := \{c_k - 1 - x,\ x \in \Gamma\}$ the set of *holes of the first type* and $B_2 := \{x \in \mathbb{Z},\ x \notin \Gamma(M)$ and $c_k - 1 - x \notin \Gamma\}$ the set of *holes of the second type.* $B_1 \cap B_2 = \emptyset$ by definition and $B(M) := B_1 \cup B_2$ is the whole set of holes of $\Gamma(M)$.

Proposition 3.5

i) $B_2 = \Gamma(t^{c_k - c}\widetilde{\omega}) \setminus \Gamma(M)$*;*

ii) $B_2^+ = \{x \notin \Gamma(M) \mid c_k - 1 - x \notin \Gamma_k\} = \Gamma(\widetilde{\omega}_k) \setminus \Gamma(M)$*;*

iii) $\#\ B_2 = \delta - (c(M) - \delta(M))$*;*

iv) $\#\ B_2^+ = k - (c(M) - \delta(M))$.

Proof. i) and ii) follow directly from definitions.
iii) By i) and 1.5 $\#\ B_2 = l_R(\widetilde{\omega}/u' t^{c - c_k} M) = \delta + \delta(M) - c(M)$, where the last equality is ii) of 2.7.
iv) Analogously $\#\ B_2^+ = l_R(\widetilde{\omega}_k / uM) = k + \delta(M) - c(M)$, where the last equality is iii) of 2.7.

3.6 $A_\Gamma(M) \subset B_2$.

Proof. Let $j \in A_\Gamma(M)$; we have to prove that $j - c_k + c \in \Gamma(\widetilde{\omega})$, i.e., $c-1-(j-c_k+c) \notin \Gamma$. On the contrary, by definition of $A_\Gamma(M)$, $j+(c_k-1-j) = c_k - 1 \in \Gamma(M)$, the desired contradiction.

Theorem 2.11 in terms of valuations is:

Theorem 3.7 *The following are equivalent:*

i) $A_\Gamma(M) = B_2$;

ii) $\alpha_R(M) = \delta + \delta(M) - c(M)$;

iii) $\Gamma(\mathfrak{m}_R) = \Gamma(\mathfrak{m}_R) + \Gamma(\widetilde{\omega} : t^{c-c_k}M)$;

iv) $t.s.(\Gamma_M) = [\alpha_R(M)+1, 1, ..., 1]$.

We call $\Gamma(M)$ *almost symmetric* if it fulfills the above equivalent conditions. We call $\Gamma(M)$ *weakly almost symmetric* if it is almost symmetric as R_k-module.

Proof. i) $\Longleftrightarrow$ ii) follows from 3.5 and 3.6.
i) $\Longrightarrow$ iii): $B_2 \subset A_\Gamma(M) \Longrightarrow \Gamma(t^{c_k-c}\widetilde{\omega}) + \Gamma(\mathfrak{m}_R) \subset \Gamma(M) \Longrightarrow \Gamma(\widetilde{\omega}) + \Gamma(\mathfrak{m}_R) \subset \Gamma(t^{c-c_k}M) \Longrightarrow \Gamma(\widetilde{\omega}) + \Gamma(\mathfrak{m}_R) + \Gamma(\widetilde{\omega} : t^{c-c_k}M) \subset \Gamma(t^{c-c_k}M) + \Gamma(\widetilde{\omega} : t^{c-c_k}M) \subset \Gamma(\widetilde{\omega}) \Longrightarrow \Gamma(\mathfrak{m}_R) + \Gamma(\widetilde{\omega} : t^{c-c_k}M) \subset \Gamma(\widetilde{\omega}) - \Gamma(\widetilde{\omega}) = \Gamma$. From this, since $\Gamma(\widetilde{\omega} : t^{c-c_k}M) \subset \mathbb{N}$ by i) of lemma 2.7, we can deduce that $\Gamma(\mathfrak{m}_R) + \Gamma(\widetilde{\omega} : t^{c-c_k}M) \subset \Gamma(\mathfrak{m}_R)$.
iii) $\Longrightarrow$ i): Let $j \in B_2 \Longrightarrow j \notin \Gamma(M)$ and $j = c_k - c + x$, $x \in \Gamma(\widetilde{\omega}) \Longrightarrow j \neq c_k - 1$. Claim: $j + \Gamma(\mathfrak{m}_R) \subset \Gamma(M)$. Now $x + \Gamma(\mathfrak{m}_R) \subset \Gamma(\widetilde{\omega}) \Longrightarrow$ using the hypothesis, $x + \Gamma(\mathfrak{m}_R) + \Gamma(\widetilde{\omega} : t^{c-c_k}M) \subset \Gamma(\widetilde{\omega})$, i.e.,
$j + \Gamma(\mathfrak{m}_R) + \Gamma(\widetilde{\omega} : M) \subset \Gamma(\widetilde{\omega}) \Longrightarrow j + \Gamma(\mathfrak{m}_R) \subset \Gamma(M)$, by 2.6.
ii) $\Longleftrightarrow$ iv): By 3.1 $c - c_k + \delta(M) = (\alpha_R(M)+1) + \sum_2^n \tau_i$. Hence the hypothesis $\tau_i = 1 \ \forall\ i = 2, ..., n$ implies ii). Conversely, if ii) holds, then $\sum_2^n \tau_i = c - \delta - 1$ and since $\tau_i \geq 1$, $i = 2, ..., n$, claim iv) is proved.

- As in the case of modules in virtue of 3.3 ii) we see immediately that:

$\Gamma(M)$ almost symmetric $\Longrightarrow$ $\Gamma(M)$ weakly almost symmetric.

Theorem 3.8

i) M is almost symmetric if and only if $\Gamma(M)$ is almost symmetric and $r_R(M) - 1 = \alpha(M)$;

ii) M is weakly almost symmetric if and only if $\Gamma(M)$ is weakly almost symmetric and $r_k(M) - 1 = \alpha_k(M)$.

Proof. The implications $\Longleftarrow$ follow directly from definition.
(i), $\Longrightarrow$: By 3.6 and iii) of 3.5 inequalities hold: $r_R(M) - 1 \leq \alpha(M) \leq \#\, B_2 = \delta - (c(M) - \delta(M))$, which become equalities in the almost symmetric case.
(ii), $\Longrightarrow$: Similarly inequalities : $r_k(M) - 1 \leq \alpha_k(M) \leq \#\, B_2^+ = k - (c(M) - \delta(M))$ become equalities in the weakly almost symmetric case.

Corollary 3.9

i) If M is almost symmetric, then:
$$r_R(M) - r_k(M) = \alpha(M) - \alpha_k(M) = \delta - k.$$

ii) M is almost symmetric if and only if
M is weakly almost symmetric and $r_R(M) - r_k(M) = \delta - k$.

iii) If M is almost symmetric, then:
$$\delta - k \ \leq\ r_R(M) - 1 \ \leq\ e - 1.$$

Proof. i) We have observed that M almost symmetric $\Longrightarrow$ M weakly almost symmetric, so by definition $r_R(M) - r_k(M) = \delta - k$. Moreover by 3.8 $r_R(M) - 1 = \alpha(M)$ and $r_k(M) - 1 = \alpha_k(M)$.
ii) follows from i) and proposition 1.7, iv).
iii) is an immediate consequence of i).

4

In this section we want to characterize a subclass of almost symmetric R-modules, which we call almost canonical, having CM-type two. The name is motivated by the fact that they can be easily constructed by deleting one element in a minimal system of generators of the canonical module. In the last part we investigate CM-type and reflexiveness of modules over almost symmetric rings.

Theorem 4.1 *The following are equivalent:*

i) $\delta + \delta(M) - c(M) = 1$;

ii) M is almost symmetric and $r_R(M) = 2$;

iii) $t.s.(M) = [2, 1, ..., 1]$;

iv) $\Gamma(M)$ is almost symmetric and $\alpha_R(M) = 1$;

v) $t.s.(\Gamma_M) = [2, 1, ..., 1]$.

We call M *almost canonical* if it fulfills the above equivalent conditions.

Proof. ii) $\Longrightarrow$ i) by definition of almost symmetric.
i) $\Longrightarrow$ ii): In general $r_R(M) - 1 \leq \delta + \delta(M) - c(M)$, then $r_R(M) \leq 2$. But $r_R(M) = 1$ means $M \simeq \widetilde{\omega}$ ([5], Korollar 6.12) and in this case $\delta + \delta(M) - c(M) = 0$ (see prop.1.5 of [9]). Therefore $r_R(M) = 2$ and M is almost symmetric.
ii) $\Longleftrightarrow$ iii) by theorem 2.11.
ii) $\Longleftrightarrow$ iv) by theorem 3.8, since $r_R(M) \leq \alpha(M) + 1$.
iv) $\Longleftrightarrow$ v) by theorem 3.7.

In the case $M = R$ such a ring is called Kunz in [1]. Next structure theorem will be useful to justify the name "almost canonical":

Theorem 4.2 *M is almost canonical if and only if*
either $k = \delta - 1$ *and* $M \simeq \widetilde{\omega}_k$
or $c(M) = c$ *and* $l_R(\widetilde{\omega}/uM) = 1$.
In this last case either $M : M = R$ *or* $M \simeq \widetilde{\omega}_{M:M}$.

Proof. Suppose M is almost canonical, then $r_k(M) \le r_R(M) = 2$. $r_k(M) = 1$ means $M \simeq \widetilde{\omega}_k$ and $\delta - k = r_R(M) - r_k(M) = 1$ by 3.9. If $r_k(M) = 2$, then $\delta - k = 0$, i.e., $c(M) = c$ and by iv) of 2.11 $l_R(\widetilde{\omega}/uM) = 1$. On the other hand, if $k = \delta - 1$ and $M \simeq \widetilde{\omega}_k$, then by [9] $c(M) = \delta(M) + k = \delta(M) + \delta - 1$ i.e. M is almost canonical; if $c(M) = c$ and $l_R(\widetilde{\omega}/uM) = 1$, then by 2.7 $\delta + \delta(M) - c(M) = l_R(\widetilde{\omega}/uM) = 1$.
It remains to prove that if M is almost canonical and $c(M) = c$, i.e., $k = \delta$, then either $M : M = R$ or $M \simeq \widetilde{\omega}_{M:M}$.
By [9], Prop. 2.1, $c(M) - \delta(M) + \alpha_{\Gamma(M:M)}(M) \le \delta(M : M) \le \delta$. Hence, in our case, $\delta - 1 + \alpha_{\Gamma(M:M)}(M) \le \delta(M : M) \le \delta \implies$ either $\delta(M : M) = \delta$ or $\delta(M : M) = \delta - 1$. In the first case $M : M = R$. Suppose now $\delta(M : M) = \delta - 1$. Since $c(\widetilde{\omega}_{M:M}) = \delta(M : M) + \delta(\widetilde{\omega}_{M:M})$ and $c(\widetilde{\omega}_{M:M}) = c(M)$, using the hypothesis $c(M) = \delta(M) + \delta - 1$, we conclude $\delta(M) = \delta(\widetilde{\omega}_{M:M})$ i.e. $M \simeq \widetilde{\omega}_{M:M}$.

Remark 4.3 Let M be any overring of R with the same conductor $c(M) = c$. Then: $l_R(M/R) = 1 \implies \widetilde{\omega}_M$ is almost canonical as R-module.
Proof. By duality $1 = l_R(M/R) = l_R(\widetilde{\omega}/\widetilde{\omega}_M) \implies \delta(\widetilde{\omega}_M) = \delta(\widetilde{\omega}) + 1 = c - \delta + 1 = c(\widetilde{\omega}_M) - \delta + 1$.

Examples

- Let $R := k[\![t^5, t^{12}, t^{13}, t^{14}, t^{16}]\!]$ and $M := R + t^2R + t^3R + t^9R$. Computing $\Gamma(R) = \{0, 5, 10, 12, \rightarrow\}$, $\Gamma(M) = \{0, 2, 3, 5, 7, 8, 9, 10, 12 \rightarrow\}$, we see that M is almost canonical and $M = \widetilde{\omega}_{M:M}$.
- Let R as above and $M := R + tR + t^2R + t^3R$. Then $\Gamma(M) = \{0, 1, 2, 3, 5, 6, 7, 8, 10, \rightarrow\}$, $k = 8$, $r_R(M) = 2$, $r_k(M) = 1$ because of $A_\Gamma(M) = \{-2\}$ and $\delta + \delta(M) - c(M) = 9 + 2 - 10 = 1$, so M is almost canonical and $M \simeq \widetilde{\omega}_k$.
- Let R as above and $M := R + t^2R + t^3R + t^7R + t^9R + t^{12}R$. Here $\Gamma(M) = \{0, 2, 3, 5, 7, \rightarrow\}$, $A_\Gamma(M) = \{-5, -3, -2, 4\}$, hence $r_R(M) = 5$, $r_k(M) = 2$. In this case since $k + \delta(M) - c(M) = 5 + 3 - 7 = 1$, M is almost canonical as R_k-module, i.e., $k - t.s.(M) = [2, 1]$; but since $\delta + \delta(M) - c(M) = 9 + 3 - 7 \ne 4$, M is not almost canonical as R-module. According to theorem 4.2 we have: $M : M = R_k$.

Proposition 4.4 *Suppose R almost symmetric and M such that $c(M) = c$. Then:*

i) *M is almost symmetric;*

ii) $r(R) = l_R(M/R) + r_R(M)$.

Proof. As observed in the second remark of 2.12 the hypothesis R almost symmetric means $\mathfrak{m}_R = \mathfrak{m}_R\, \widetilde{\omega}$.

i) Claim: $\mathfrak{m}_R = \mathfrak{m}_R(\widetilde{\omega} : M)$. Let $uM \subset \widetilde{\omega}$ be the canonical immersion. Then $u\mathfrak{m}_R \subset \mathfrak{m}_R(\widetilde{\omega} : M) \subset \mathfrak{m}_R\ \widetilde{\omega} = \mathfrak{m}_R$. Since u is a unit, the preceding inclusions are equalities.
ii) $r(R) = l_R(\widetilde{\omega}/\mathfrak{m}_R\ \widetilde{\omega}) = l_R(\widetilde{\omega}/\mathfrak{m}_R) = l_R(\widetilde{\omega}/\widetilde{\omega} : M) + l_R(\widetilde{\omega} : M/u\mathfrak{m}_R) = l_R(M/R) + l_R(\widetilde{\omega} : M/\mathfrak{m}_R\ (\widetilde{\omega} : M)) = l_R(M/R) + r_R(M)$; we have here used i) and 1.5 to say that $l_R(\widetilde{\omega} : M/u\mathfrak{m}_R) = l_R(\widetilde{\omega} : M/\mathfrak{m}_R(\widetilde{\omega} : M))$.

Next counterexample shows that the implication "M almost symmetric as R_k-module $\implies$ R_k almost symmetric" fails, even if R is almost symmetric. Let $R := k[\![t^4, t^9, t^{14}, t^{19}]\!]$ and $M := R + t^5R + t^{10}R + t^{15}R$. Then $\Gamma(R) = \{0, 4, 8, 9, 12, 13, 14, 16, \rightarrow\}$ and $\Gamma(M) = \{0, 4, 5, 8, 9, 10, 12, \rightarrow\}$. Since $\delta = 9$, $c = 16$, $r(R) = 3$, R is an almost symmetric ring. We can easily verify that $\delta(M) = 6$, $c(M) = 12$, $r_R(M) = 4$, $r_k(M) = 3$, hence M is almost symmetric as R_k-module, but R_k is not almost symmetric and formula ii) of Theorem 2.11 does not hold. In fact $k = 8$, $\delta(R_k) = 8$, $c(R_k) = 12$, $r(R_k) = 3$.

Corollary 4.5 *Suppose R almost symmetric and $r(R) = 2$. There are exactly two isomorphism classes of R-modules having $c(M) = c$: either $M \simeq R$ or $M \simeq \widetilde{\omega}$.*

Proof. The conclusion follows immediately from assertion ii) of the above proposition: $l_R(M/R) + r_R(M) = 2$.

Remark 4.6 The only R-module M "reflexive" over R, i.e., $M = M^{**} := R : (R : M)$, such that $R \subset M \subset \overline{R}$ and $c(M) = c$ is $M = R$.
Proof. Suppose $M = M^{**}$ and $M \neq R$. Then, since $R : M \subset \mathfrak{m}_R$ and $(R : \mathfrak{m}_R)(R : M) \subset R$, it follows that $R : \mathfrak{m}_R \subset M^{**} = M$. Thus $c(M) - 1 = c - 1 \in \Gamma(R : \mathfrak{m}_R) \subset \Gamma(M)$, which is a contradiction.

Proposition 22 of [2] may be slightly generalized considering any R-module M instead of a strict overring of R in this way:

Proposition 4.7 *The inequality (*) $l_R(M/R) \leq l_R(R/R : M) + l_R(\widetilde{\omega}/R)$ is valid in general. If we consider the conditions:*

i) $\widetilde{\omega} \subset M : M$, *i.e.,* $\widetilde{\omega}\ M = M$

ii) $R : M = (R : M)\ \widetilde{\omega} = \widetilde{\omega} : M$

iii) $=$ *holds in (*)*

iv) $M = M^{**}$

v) $(R : M)\ \widetilde{\omega} = \widetilde{\omega} : M$

then $(i) \iff (ii) \iff (iii) \implies (iv) \iff (v)$. *Assuming R almost symmetric and $M \neq R$, all conditions are equivalent.*

Proof. To prove (*) it suffices to note that: $l_R(M/R) = l_R(\widetilde{\omega}/\widetilde{\omega} : M) \leq l_R(\widetilde{\omega}/R : M) = l_R(\widetilde{\omega}\ M/R) = l_R(R/R : M) + l_R(\widetilde{\omega}/R)$. Then the equivalences $(i) \iff (ii) \iff (iii)$ are clear. Moreover by duality: $l_R(M^{**}/M) = l_R(\widetilde{\omega} : M/(R : M)\ \widetilde{\omega})$, so $(iv) \iff (v)$. Finally $(ii) \implies (v)$ is trivial. Suppose now R almost symmetric and $M \neq R$. Since $R : M = \mathfrak{m}_R : M$ and $\mathfrak{m}_R = \mathfrak{m}_R\ \widetilde{\omega}$, we get $(R : M)\ M\ \widetilde{\omega} \subset \mathfrak{m}_R$, hence $(R : M)\ \widetilde{\omega} = R : M$. So $(v) \implies (ii)$.

Corollary 4.8 *If R is almost symmetric, then*
M is reflexive if and only if $M : M$ is reflexive.

Proof. Put $B := M : M$ and apply $(iv) \iff (i)$ of the preceding proposition. M reflexive $\iff \widetilde{\omega} \subset B \iff \widetilde{\omega} \subset B : B \iff B$ reflexive.

Remark 4.9

- Next counterexample shows that the hypothesis 'R almost symmetric' is needed to state the equivalence of conditions 4.7. Let $R := k[\![t^3, t^7, t^8]\!]$, $M := R + t^4R + t^5R$. Here $M = M^{**}$, but $\widetilde{\omega}\ M \neq M$ because $t \in \widetilde{\omega}\ M$.
- Relations between reflexiveness and canonical ideals are present in the literature. In the case M is an overring of R the equivalence $(iv) \iff (v)$ becomes the well known assertion:

$$\gamma_{M/R}\omega_R = \omega_M \iff M \text{ is reflexive}$$

$\gamma_{M/R} = R : M$ being the usual conductor. This idea has been extended to the not birational case by means of the notion of complementary module (see, for instance, [8], theorem 2.3 with the additional hypothesis '$\gamma_{M/R}\omega_R$ Cohen-Macaulay' and [7], Proposition 4.34).

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