Notes on the geometry of Lagrangian torus fibrations

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1 Introduction

These notes have developed from the text of a talk[1] where Duistermaat's global theory of action-angle variables [6] was expounded. An obvious motivation for this theory is to understand the global structure of integrable systems, but one should also mention a recent renewal of interest due to the relevance of this theory to the approaches to string-theoretic mirror symmetry for Calabi-Yau varieties à la Strominger-Yau-Zaslow [12], especially in connection with the attempt to construct a mathematical theory of mirror symmetry [9, 10, 1, 4]. In turn, the possibility of associating a torus-fibred mirror complex manifolds to a Lagrangian torus fibration prospects intriguing scenarios for possible dualities of integrable systems.

We shall assume a basic knowledge of the local integrability result usually referred to as the Liouville-Arnold theorem [2, 5]. The main mathematical tools we shall need are fibre bundle theory and sheaf cohomology (at a very basic level). Useful references on these topics are [11, 7, 8, 3].

Let us summarize the main idea of Duistermaat's construction. Assume one is given a symplectic manifold M which is fibred on a base manifold B in such a way that the fibres are compact Lagrangian submanifolds. These data define intrinsically a "relative lattice" on B — more precisely, a covering $\Lambda \to B$ which is contained in the cotangent bundle T^*B as a Lagrangian submanifold and intersects every cotangent space in a lattice. Then one compares the fibration $M \to B$ with the torus bundle T^*B/Λ, which carries a natural symplectic structure such that the toric fibres and the zero section are Lagrangian. Whether the two fibrations are isomorphic just as smooth fibrations, or as symplectic manifolds fibred in Lagrangian submanifolds, or are topologically trivial, depends on the value and properties of an invariant of the fibration $M \to B$ (its Chern class) and on

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the monodromy of the fibration. One is also able to spell out the conditions for the existence of global action-angle variables (topological triviality of the fibration $\pi\colon M \to B$ and exactness of the symplectic form of M). One should notice that in general these action-angle variables *are not* global coordinates.

Additional topics we consider are the construction of the complex manifold "mirror dual" to the Lagrangian torus fibration and the Gauss-Manin connection in its various personifications.

Acknowledgements. I am grateful to Marco Pedroni for pointing out some inaccuracies and to Fabio Pioli for discussions on these topics.

2 The basic constructions

Let (M, ω) be a connected symplectic manifold of dimension $2n$ which supports a fibration $\pi\colon M \to B$ such that for every $b \in B$ the fibre $F_b = \pi^{-1}(b)$ is a compact connected Lagrangian submanifold of M. We may assume that π is surjective.

Proposition 2.1. *Let $B_0 \subset B$ be an open subset supporting a map $\chi\colon B_0 \to \mathbb{R}^n$ which is a coordinate system. Then the functions $f_i = \chi_i \circ \pi$ are in involution.*

Proof. The 1-forms df_i are normal to the tangent spaces T_xF_b for all $x \in \pi^{-1}(B_0)$. The associated Hamiltonian vector fields X_{df_i} lie in the symplectic orthogonal spaces, which, since the fibres are Lagrangian, coincide with the tangent spaces. Therefore,

$$\{f_i, f_j\} = L_{X_{df_i}} df_j = X_{df_i}(f_j) = 0.$$

□

Lemma 2.2. *The map π is submersive.*

Proof. Since the duals of the vector fields X_{df_i} generate the cotangent spaces to the fibres, for every $p \in M$ one has

$$T_p^*M \simeq T_p^*F_b \oplus T_b^*B$$

with $b = \pi(p)$, and the map π^* is the inclusion of the second summand, so that it is injective. □

By the local Liouville-Arnold theorem, the fibres F_b are tori. The flows of the vector fields X_{df_i} define an action of $\mathbb{R}^n$ on every fibre F_b. This action depends of the choice of the map χ, but the (finite-dimensional) vector space C_b formed by the vertical vector fields invariant under this action does not depend on the choice of χ. This defines a vector bundle C on B. For every b, let Γ_b be the space of vector fields in C_b which are periodic with period 1; Γ_b is a lattice. By considering the sections of C which restricted to every fibre satisfy this condition, we obtain a sheaf Γ, called the *period lattice* of the fibration $\pi\colon M \to B$. For every $\lambda \in \Gamma_b$ let $\gamma(\lambda)$ be the integral curve of λ which is periodic of period 1. Then by taking the homology class of $\gamma(\lambda)$ we obtain a homomorphism

$$\begin{aligned} \Gamma_b &\to H_1(F_b, \mathbb{Z}) \\ \lambda &\mapsto [\gamma(\lambda)]. \end{aligned}$$

This is easily shown to be an isomorphism.[2]

We have isomorphisms

$$\Gamma \otimes_{\mathbb{Z}} \mathcal{C}^\infty_B \simeq C, \qquad \pi^* C \simeq \mathrm{vert}(TM). \tag{1}$$

Let us fix a basis $\{e_j\}$ of sections of Γ over B_0 (possibly after shrinking B_0); so for every $b \in B_0$, $\{e_j(b)\}$ is a basis of $H_1(F_b, \mathbb{Z})$. Every e_j may be regarded as a vertical vector field X_j on $\pi^{-1}(B_0)$. Moreover, choose in every fibre F_b angle coordinates ϕ_j, smoothly depending on b, such that

$$\frac{1}{2\pi} \int_{X_j} d\phi_k = \delta^j_k.$$

The following is then a restatement of a classical result (cf. [2]).

Proposition 2.3. *1. The vector fields X_j are Hamiltonian, and their Hamiltonian functions are constant along the fibres F_b, so that they can be written as $I_j \circ \pi$, where the functions I_j are coordinates in B_0;*

2. the collection $\{I_1, \ldots, I_n, \phi_1, \ldots, \phi_n\}$ is a system of local Darboux coordinates, i.e.

$$\omega = \sum_j dI_j \wedge d\phi_j$$

on suitable open subsets of M.

These are of course the *action-angle coordinates.* A further restatement of this is that the 1-forms $\tilde{\xi}_j = \omega(X_j)$ are horizontal (indeed $\tilde{\xi}_j(X_{df_i}) = df_i(X_j) = 0$ since the df_i are horizontal and the X_j vertical), and are constant along the fibres, so that

$$\tilde{\xi}_j = \pi^* \xi_j$$

[2] Globally this means that the sheaf $\Gamma^* = \mathcal{H}om_{\mathbb{Z}}(\Gamma, \mathbb{Z})$ is isomorphic to $R^1\pi_*\mathbb{Z}$.

for some well-defined 1-forms ξ_j on B. Locally one has

$$\xi_j = dI_j.$$

At every $b \in B$ the 1-forms $\xi_j(b)$ are a basis of $T^*_b B$ (over $\mathbb{R}$), and generate over $\mathbb{Z}$ a lattice $\Lambda_b \subset T^*_b B$.

Remark 2.4. A consequence of this fact is that if I'_j is another set of local action coordinates, the transition function has the form

$$I'_j = \sum_{k=1}^{n} a_{jk}\, I_k + b_j,$$

where the b_j are real numbers, while the matrix a is in $Gl(n,\mathbb{Z})$ (the (group of integer-valued matrices with determinant ± 1). As a consequence, the base manifold B is *affine* (it may be given an atlas whose transition functions are affine transformations). $\square$

Moreover, identifying the X_j with the forms ξ_j we obtain an isomorfism $C_{|B_0} \simeq T^*B_0$, but since this is independent of the choices made, one actually has an isomorphism of vector bundles $C \simeq T^*B$.[3] The image of Γ under this isomorphism is a submanifold Λ of T^*B which covers B, and is such that $\Lambda \cap T^*_b B = \Lambda_b$. Let $p\colon T^*B \to B$ be the bundle projection. The sheaf of sections of the restriction $p_{|\Lambda}\colon \Lambda \to B$, that we shall again denote by Λ, is a sheaf of abelian groups on B, and it has an action on T^*B (which is simply the addition of differential forms). As a sheaf, Λ is a subsheaf of the sheaf $\tilde{\mathcal{L}}$ of Lagrangian sections of T^*B. The quotient T^*B/Λ is a torus bundle on B.

Lemma 2.5. *The Lagrangian sections of T^*B are the closed 1-forms on B.*

In other terms, $\tilde{\mathcal{L}}$ is the sheaf of closed 1-forms on B.

Proof. Let us denote by τ the canonical symplectic form of T^*B. If $(x^1, \ldots, x^n, y_1, \ldots, y_n)$ are fibred coordinates on T^*B, we have $\tau = dy_j \wedge dx^j$. A section of T^*B is a 1-form $\eta = \eta_i(x)\, dx^i$, i.e., in local coordinates, $y_i = \eta_i(x)$. The Lagrangian condition is

$$0 = \tau_{|\eta} = \frac{\partial \eta_i}{\partial x^j}\, dx^j \wedge dx^i = d\eta.$$

$\square$

Proposition 2.6. *1. Λ is a Lagrangian submanifold of T^*B.*

2. The vertical transformation induced in the fibres of T^*B by a section are symplectic if and only if the section is Lagrangian.

[3]Together with the isomorphisms (1) and $\Gamma \otimes_{\mathbb{Z}} \mathcal{C}^\infty_B \simeq C$, we also obtain an isomorphisms $TB \simeq R^1\pi_*\mathbb{R} \otimes_{\mathbb{R}} \mathcal{C}^\infty_B$.

Proof. 1. Λ is formed by the sections of T^*B corresponding to the closed 1-forms ξ_i, so that it is Lagrangian.

2. Given a section of T^*B, i.e. a 1-form $\eta = \eta_i(x)\,dx^i$, this induces the transformation $y_i \mapsto y_i + \eta_i(x)$. The symplectic form of T^*B is preserved exactly when η is closed. $\square$

As a consequence of 2, the canonical symplectic structure of T^*B is invariant under the action of the lattice Λ, and therefore induces a symplectic structure on T^*B/Λ.

We have a canonical section on T^*B/Λ, image of the zero section of T^*B, and this is Lagrangian. This induces a structure of Abelian group on the space of Lagrangian sections of $T^*B/\Lambda \to B$, and allows us to introduce the sheaf $\mathcal{L}$ of Lagrangian sections of $T^*B/\Lambda \to B$, which is a sheaf of Abelian groups.

Lemma 2.7. *The sections of $\mathcal{L}$ act fibrewise on M. If U is small enough, given two Lagrangian sections $\sigma, \sigma' \colon U \to M$, there is a unique Lagrangian section $\mu \in \mathcal{L}(U)$ such that $\sigma'(b) = \mu(b)(\sigma(b))$.*

3 The Chern class of the fibration $\pi\colon M \to B$

We may cover the base manifold B with open sets $\{B_\alpha\}$ which support some fixed coordinate systems. Once these data are fixed, specifying angle coordinates (coordinates on the fibres F_b) is equivalent to specifying Lagrangian sections $\sigma_\alpha \colon B_\alpha \to \pi^{-1}(B_\alpha) \subset M$, and on the overlaps by Lemma 2.7 one has

$$\sigma_\alpha(b) = \mu_{\alpha\beta}(b)(\sigma_\beta(p)) \tag{2}$$

where the Lagrangian sections $\mu_{\alpha\beta} \in \mathcal{L}(B_\alpha \cap B_\beta)$ define a 1-cocycle of the sheaf $\mathcal{L}$.

We shall not enter into details about this issue, but one can easily check that the class of the cocycle μ determines the structure of the bundle $M \to B$ as a symplectic manifold with Lagrangian fibres; locally, $M \to B$ is just isomomorphic to the bundle T^*B/Λ, but its "local pieces" are glued in a different way, as prescribed by μ.

Let again $\tilde{\mathcal{L}}$ denote the sheaf of Lagrangian sections of T^*B. One has the commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r] \arrow[d] & \tilde{\mathcal{L}} \arrow[r] \arrow[d] & \mathcal{L} \arrow[r] \arrow[d] & 0\,; \\
0 \arrow[r] & \Lambda \arrow[r] & T^*B \arrow[r] & T^*B/\Lambda \arrow[r] & 0
\end{tikzcd}$$

here T^*B, T^*B/Λ denote the sheaves of smooth (not necessarily Lagrangian) sections of the corresponding bundles. By going to the associated long exact cohomology sequences[4] we obtain a commutative diagram[5]

$$\begin{array}{ccccccc}
 & & H^1(B,\tilde{\mathcal{L}}) & \longrightarrow & H^1(B,\mathcal{L}) & \xrightarrow{\delta} & H^2(B,\Lambda) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 = & & H^1(B,T^*B) & \longrightarrow & H^1(B,T^*B/\Lambda) & \xrightarrow{\delta'} & H^2(B,\Lambda) & \longrightarrow & 0
\end{array}$$

where $\delta\colon H^1(B,\mathcal{L}) \to H^2(B,\Lambda)$ and δ' are the so-called connecting morphisms. The rightmost vertical arrow in the last diagram is an isomorphism, and δ' is an isomorphism as well. This means that as a smooth bundle (forgetting the symplectic structure), M is just classified by the class ν. In particular we have:

Lemma 3.1. $\nu = 0$ *if and only if* $M \simeq T^*B/\Lambda$ *as fibre bundles (regardless of the symplectic structures).* □

The image

$$\nu = \delta([\mu]) \in H^2(B,\Lambda)$$

is called the *Chern class* of the fibration $\pi\colon M \to B$.

Remark 3.2. In view of Lemma 2.5 there is an isomorphism

$$H^k(B,\tilde{\mathcal{L}}) \simeq H^{k+1}(B,\mathbb{R}), \qquad k \geq 1$$

(cf. e.g. [8]). □ □

Theorem 3.3. *The following statements are equivalent.*

1. $M \simeq T^*B/\Lambda$ *as fibre bundles.*

2. *The bundle* $\pi\colon M \to B$ *admits a global section* $s\colon B \to M$*.*

3. *The Chern class* $\nu = \delta([\mu])$ *vanishes.*

Proof. (1⇔2) If $M \to B$ has a section, by identifying this with the zero section of T^*B/Λ and using the torus actions on the two sides, we get an isomorphism $M \simeq T^*B/\Lambda$. The converse is obvious.

(2⇔3) This is Lemma 3.1. □

[4]The cohomology groups we are introducing are Čech cohomology groups, cf. e.g. [8, 7].

[5]In the bottom line we have $H^1(B,T^*B) = H^2(B,T^*B) = 0$ because, due to the existence of partitions of unity, the sheaf T^*B is acyclic, i.e., it has nonvanishing cohomology only in degree zero.

Theorem 3.4. *The following statements are equivalent.*

1. *$M \simeq T^*B/\Lambda$ as symplectic manifolds fibred over B in Lagrangian submanifolds.*

2. *The bundle $\pi\colon M \to B$ admits a global Lagrangian section $\sigma\colon B \to M$.*

3. *The Chern class $\nu = \delta([\mu])$ vanishes, and for any section $s\colon B \to M$, the 2-form $s^*\omega$ on B is exact.*

Proof. (1⇔2) As in the previous theorem, but noting that since the section is Lagrangian, the identification is by symplectomorphisms.

(2⇒3) The existence of the Lagrangian section σ implies $[\mu] = 0$, hence $\nu = 0$. As for the exactness of $s^*\omega$, we may identify M with T^*B/Λ, so that the section s is represented by a 1-form on B, $\tilde{s} = \sum_j s_j\, dI^j$. Then

$$s^*\omega = s^*\left(\sum_j dI_j \wedge d\phi_j\right) = \sum_j dI_j \wedge ds_j = -d(\sum_j s_j\, dI_j).$$

(3⇒2) The vanishing of ν implies that we may identify M with T^*B/Λ as fibre bundles. Moreover, $[\mu]$ lies in the image of the map $H^1(B, \tilde{\mathcal{L}}) \to H^1(B, \mathcal{L})$, i.e., $[\mu] = [\Omega] \bmod \Lambda$ where the cocycle $\Omega = \{\Omega_{\alpha\beta}\}$ is a collection of closed 1-forms on the intersections $B_\alpha \cap B_\beta$. In action-angle coordinates the section s provides maps $s_\alpha\colon B_\alpha \to (\mathbb{R}/\mathbb{Z})^n$ such that $\Omega_{\alpha\beta} = s_\alpha - s_\beta \bmod \mathbb{Z}^n$ (cf. equation (2). The differentials ds_α piece together to a closed 2-form ρ on B which represents $[\Omega]$ under the isomorphism $H^1(B, \tilde{\mathcal{L}}) \simeq H^2(B, \mathbb{R})$. But we also have $ds_\alpha = s_\alpha^*\omega$, so that

$$\rho = s^*\omega. \tag{3}$$

So, $[\mu]$ is the image in $H^1(B, \mathcal{L})$ of the class in $H^1(B, \tilde{\mathcal{L}})$, which, under the isomorphism $H^1(B, \tilde{\mathcal{L}}) \simeq H^2(B, \mathbb{R})$, corresponds to the de Rham class $[s^*\omega]$. Since by hypothesis the latter vanishes, we have $[\mu] = 0$, i.e., $\pi\colon M \to B$ admits a global Lagrangian section. □

4 Monodromy

Whether the fibration $M \to B$ is topologically trivial, and global action-angle variables exist, depend on the *monodromy* of the covering $\Lambda \to B$. For every $b \in B$ one has a representation, called the *monodromy* of the covering,

$$M_b\colon \pi_1(B, b) \to \mathrm{Aut}(\Lambda_b) \simeq \mathrm{Aut}(H_1(F_b, \mathbb{Z})). \tag{4}$$

This is defined as follows. Given $Z \in \pi_1(B,b)$, represent Z by a loop γ in B based at b, and lift it to a continuous path $\tilde{\gamma}$ in M starting at a point $p \in \Lambda_p$. Assume that γ is parametrized so that $\gamma(0) = \gamma(1) = b$. We have a map

$$\begin{array}{rcl} \tilde{M}_b \colon \pi_1(B,b) \times H_1(F_b, \mathbb{Z}) & \to & H_1(F_b, \mathbb{Z}) \\ (Z,p) & \mapsto & \tilde{\gamma}(1). \end{array}$$

This also defines a map as in (4). If we choose a basis in Λ_p, the representation M_b maps $\pi_1(B,b)$ to the group $Gl(n,\mathbb{Z})$.

We say that the monodromy is *trivial* if $M_b(\pi_1(B,b)) = 1$ for all $b \in B$. If the covering $\Lambda \to B$ is trivial, then its monodromy is trivial as well; indeed, since the fibres of the covering are discrete, the curve $\tilde{\gamma}$ cannot but go back and terminate at its starting point. The covering Λ is indeed trivial if and only if so is its monodromy.

Theorem 4.1. *The fibration $M \to B$ is topologically trivial if and only if its Chern class and its monodromy are trivial.*

Proof. If the fibration $M \to B$ is topologically trivial, the claim is obvious. Conversely, if the Chern class is zero, we may identify M with T^*B/Λ. The triviality of the monodromy implies the triviality of Λ as a bundle over B, and since $T^*B \simeq \Lambda \otimes_{\mathbb{Z}} C^\infty_B$, also the cotangent bundle to B is trivial. $\square$

One should notice that T^*B may be trivial, and Λ not be so: the phase space of the spherical pendulum [6] provides an example of such a situation.

Let us assume the triviality of the monodromy and of the Chern class. A chosen basis in a fibre Λ_{b_0} extends globally, therefore (in conformity with Theorem 4.1) one gets a trivialization of T^*B by n closed 1-forms ξ_i. We wish to show that if the symplectic form ω of M is exact then the forms ξ_i are exact as well. Under these conditions, the fibrations $M \to B$ and T^*B/Λ are symplectic isomorphic. Let γ be a loop in B based at b, and let ϵ a lift to M. This can be completed to a closed curve ϵ_0 by joining a curve completely contained in F_b. Let Ψ^i_s be the flow of the locally Hamiltonian vector field $\omega^{-1}(\pi^{-1}(\xi_i))$, which leaves the fibre F_b invariant and is periodic. One can define a 2-cycle β_i on F_b by letting

$$\begin{array}{rcl} \beta \colon (\mathbb{R}/\mathbb{Z})^2 & \to & M \\ \beta_i(t,s) & = & \Psi^i_s(\epsilon_0(t)). \end{array}$$

Then,

$$\langle [\xi_i], [\gamma] \rangle = \langle [\omega], [\beta_i] \rangle$$

so that, if ω is exact, we have $[\xi_i] = 0$.

So, if ω is exact, there exist globally defined functions I_j on B such that $\xi_j = dI_j$ *(global action coordinates)*. These define a map $B \to \mathbb{R}^n$ which is a "local diffeomorphism" (i.e., its tangent map is bijective at every point). Moreover, there exists functions ϕ^j on the fibres, having values in $(\mathbb{R}/\mathbb{Z})^n$, such that

$$X_{\pi^*\xi_j} = \frac{\partial}{\partial \phi^j}.$$

All this can be summarized as follows.

Theorem 4.2. *The following two conditions are equivalent.*

1. *The fibration $M \to B$ is topologically trivial and the symplectic form ω is exact.*

2. *There exist a smooth map $(I, \phi)\colon M \to \mathbb{R}^n \times (\mathbb{R}/\mathbb{Z})^n$ such that*

   - $\omega = dI_j \wedge d\phi^j$;
   - *the functions I_j are constant on the fibres of $\pi\colon M \to B$;*
   - *the functions ϕ^j are injective on the fibres of $\pi\colon M \to B$.*

5 The Gauss-Manin connection

We consider the fibration $M \to B$ in the general case where it has no sections. We may show that the manifolds B and M are naturally endowed with a flat, torsion-free linear connection, called the *Gauss-Manin connection.* One possible way for defining this connection is to use the isomorphism

$$TB \simeq R^1\pi_*\mathbb{R} \otimes \mathcal{C}^\infty_B. \tag{5}$$

The connection

$$\nabla^{GM}\colon TB \to TB \otimes T^*B$$

is defined by letting $\nabla^{GM} = 1 \otimes d$ under the isomorphism (5). This connection is obviously flat. Let us compute its coefficients in action coordinates. The vector fields $\frac{\partial}{\partial I_j}$ provide a $\mathbb{Z}$-basis of $\Gamma^* \simeq R^1\pi_*\mathbb{Z}$ and hence an $\mathbb{R}$-basis $\{e_j\}$ of $R^1 f_*\mathbb{R}$; in this basis, the isomorphism (5) is

$$\frac{\partial}{\partial I_j} \mapsto e_j \otimes 1.$$

Therefore we have

$$\nabla^{GM} \frac{\partial}{\partial I_j} = 0$$

and the connection coefficients in the action coordinates vanish. As a consequence, ∇^{GM} is torsion-free.

We may look at this connection in a different way. A local section of the lattice Λ over an open set $U \subset B$ provides an identification of the cotangent spaces to B at all $b \in U$. This identification does not depend on the chosen section, and therefore defines a connection on T^*B. In action coordinates the geodesics of this connection are straight lines, and therefore this connection is the Gauss-Manin connection.

Proposition 5.1. *The holonomy of the connection ∇^{GM} coincides with the monodromy of the covering Λ.*

Proof. The monodromy representation of the fundamental group of B defines a bundle with a flat connection. The holonomy of this connection is the monodromy of Λ by construction. On the other hand, the bundle we have obtained in this way is isomorphic to the cotangent bundle T^*B, and the connection coincides with the Gauss-Manin connection, in view of the new description we have given of the latter. □

If $\pi \colon M \to B$ has a section (not necessarily Lagrangian), the Gauss-Manin connection can be lifted to a connection on M. The Gauss-Manin connection provides a splitting of the Atiyah sequence of the bunle T^*B

$$0 \to \mathrm{vert}(TT^*B) \to TT^*B \to p^*TB \to 0$$

where $p \colon T^*B \to B$ is the bundle projection. We have therefore an isomorphism

$$TT^*B \simeq \mathrm{vert}(TT^*B) \oplus p^*TB\,.$$

If $\pi \colon M \to B$ has a section, we have an isomorphism $M \simeq T^*B/\Lambda$. Let $\rho \colon T^*B \to M$ be the induced projection. One has a canonical isomorphism $\rho^*TM \simeq TT^*B$, so that the previous splitting, together with the equality $p^* = \rho^* \circ \pi^*$ and the isomorphism $\mathrm{vert}(TM) \simeq \pi^*T^*B$, yields

$$\rho^*TM \simeq \rho^*\mathrm{vert}(TM) \oplus (\rho^* \circ \pi^*)\, TB \simeq (\rho^* \circ \pi^*)\, T^*B \oplus (\rho^* \circ \pi^*)\, TB.$$

This implies a splitting

$$TM \simeq \pi^*T^*B \oplus \pi^*TB, \tag{6}$$

so that the Gauss-Manin connection induces a connection on TM.

6 The mirror manifold

The relative Jacobian of M — i.e., the manifold obtained by taking the fibrewise dual of M — may be endowed with a complex structure. Given a real torus T, the dual torus may be defined as $H^1(T,\mathbb{R})/H^1(T,\mathbb{Z})$. The relative analogue of this is the fact that the sheaf of sections of the relative Jacobian of $f\colon M \to B$ is the sheaf $R^1\pi_*\mathbb{R}/R^1\pi_*\mathbb{Z}$. We therefore define the relative Jacobian, or dual fibration, as

$$\hat{M} = TB/\Lambda^*$$

(whose sheaf of sections is exactly $R^1\pi_*\mathbb{R}/R^1\pi_*\mathbb{Z}$ due to the isomorphisms discussed in section 2). Irrespective of the fact that $\pi\colon M \to B$ may have a section or not, the relative Jacobian always has a section. We denote $\hat{\pi}\colon \hat{M} \to B$ the projection. From (6) we have

$$T\hat{M} \simeq \hat{\pi}^*TB \oplus \hat{\pi}^*TB. \tag{7}$$

The map $T\hat{M} \to T\hat{M}$ given by $(\alpha,\beta) \mapsto (-\beta,\alpha)$ (with reference to the splitting (7)) is a complex structure on $\hat{M}$. We call $\hat{M}$ with this complex structure the *mirror manifold* to M. If $\{\psi_j\}$ are angle coordinates on $\hat{M}$, fibrewise dual to the angle coordinates $\{\phi_j\}$ of M, the functions

$$z_j = I_j + i\,\psi_j$$

are local holomorphic coordinates on $\hat{M}$.

The splitting (7) also tells us that the Gauss-Manin connection induces a connection on $\hat{M}$ (regardless of the existence of a section of $f\colon M \to B$). This is compatible with the complex structure.

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