Linearizing flows on Jacobians [1]

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1. Completely integrable systems

We briefly review the definition of what a completely integrable system is, starting from the basics of hamiltonian mechanics.

If M is an n-dimensional differentiable manifold, the cotangent bundle $p\colon T^*M \to M$ carries on its total space a tautological 1-form θ, called the *Liouville form,* and defined as

$$\theta_u(X) = u(p_*(X))$$

if $u \in T^*M$ and $X \in T_uT^*M$. If $(q^1, \dots, q^n)$ are local coordinates in open set $U \subset M$, one defines coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$ in $p^{-1}(U)$ by letting

$$q_i(u) = q_i(p(u)), \qquad u = \sum_{i=1}^n p_i(u)\,(dq^i)_{p(u)}\,.$$

In these coordinates the Liouville form is written as

$$\theta = \sum_{i=1}^n p_i\,dq^i\,.$$

The 2-form $\omega = d\theta$ is a canonically defined symplectic form on T^*M, for which the previously introduced coordinates are Darboux coordinates:

$$\omega = \sum_{i=1}^n dp_i \wedge dq^i\,.$$

Being nondegenerate the symplectic form establishes an isomorphism

$$\iota\colon T^*T^*M \to TT^*M.$$

For every function $f\colon T^*M \to \mathbb{R}$ one usually writes

$$X_f = \iota(df)$$

[1] These are expanded notes of a talk I gave at the Department of Mathematics of the University of Genova on November 29, 2000. The main bulk of these notes is formed by material taken from the classical work of Griffiths [**?**].

and X_f is a vector field on T^*M. In this way one can define the *canonical Poisson bracket* on the cotangent bundle: for every pair f, g of functions on T^*M, one gets the function

$$\{f, g\} = \omega(X_f, X_g)\,.$$

Suppose now that we fix a function $H\colon T^*M \to \mathbb{R}$. We associate to it the differential equation

$$\dot\gamma = X_H$$

where γ is a curve in T^*M. In the coordinates (q, p), this equation splits into two sets of equations,

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}\,. \tag{1}$$

These are the *Hamilton equations.* The first set of equations says that γ is a lift of a curve on M to T^*M.

We say that the specification of the datum (M, H) is a *Hamiltonian system.* Moreover, we say that a Hamiltonian system is completely integrable if there exists n functions $F_i\colon T^*M \to \mathbb{R}$, $i = 1, \ldots, n$, such that

1. $F_1 = H$;
2. the F_i's are mutually in involution, namely, $\{F_i, F_j\} = 0$ for all $i, j = 1, \ldots n$;
3. the F_i's are independent, in the sense that the differential forms dF_i are linearly independent in an open dense subset of T^*M.

The condition $\{F_i, H\} = \{F_i, F_1\} = 0$ implies that the F_i's are constant along the curves in T^*M which solve the Hamilton equations, namely, they keep constant during the evolution of the system. For this reason they are called *integrals of the motion* of the system (**??**).

Example 1.1. (The Toda lattice.) This is a mechanical system formed by n particles moving on a straight line subject to a "first neighbourhood" potential interaction. The Hamiltonian function for this system is

$$H(x^1, \ldots, x^n, y_1, \ldots, y_n) = \tfrac12 \sum_{i=1}^n p_i^2 + \sum_{i=1}^n e^{q^i - q^{i+1}}\,, \qquad q^{n+1} = q^1\,.$$

The equations of motion are conveniently written in terms of the *Flaschka variables*

$$\begin{cases} a_i &=& \tfrac12 e^{(q^i - q^{i+1})} \\ b_i &=& -\tfrac12 p_i \end{cases}$$

and in these variables they read

$$\begin{cases} \dfrac{db_i}{dt} &=& 2(a_i^2 - a_{i-1}^2)\,, \qquad a_0 = a_n \\ \dfrac{da_i}{dt} &=& a_i(b_{i+1} - b_i),\,, \qquad b_{n+1} = b_1\,. \end{cases} \tag{2}$$

The first equation implies that

$$\frac{d}{dt}\sum_{i=1}^{n} b_i = 0$$

so one can normalize the b_i's by letting

$$\sum_{i=1}^{n} b_i = 0\,.$$

This is an integral of the motion; to show that the Toda lattice is a completey integrable system one should find additional $n-1$ integrals.

The equations of motion may be written in terms of a Lax pair by introducing the matrices

$$A = \begin{pmatrix} b_1 & a_1 & & & & a_n \\ a_1 & b_2 & & & & \\ & & \cdot\cdot & & & \\ & & & \cdot\cdot & & \\ & & & & \cdot\cdot & \\ & & & & b_{n-1} & a_{n-1} \\ a_n & & & & a_{n-1} & b_n \end{pmatrix}$$

$$B = -\begin{pmatrix} 0 & a_1 & & & & -a_n \\ -a_1 & 0 & & & & \\ & & \cdot\cdot & & & \\ & & & \cdot\cdot & & \\ & & & & \cdot\cdot & \\ & & & & 0 & a_{n-1} \\ a_n & & & & -a_{n_1} & 0 \end{pmatrix}$$

The diagonal is A contains $b_1 \ldots b_n$ while the subdiagonal and the superdiagonal in B contain $a_1 \ldots a_{n-1}$. Then one proves that the Lax equation

$$\dot{A} = [B, A]$$

is equivalent to the equations of motion (**??**).

The functions

$$F_k = \frac{1}{k}\,\mathrm{tr}\,A^k$$

are integrals of the motion:

$$\dot{F}_k = \mathrm{tr}(\dot{A}A^{k-1}) = \mathrm{tr}([B,A]A^{k-1}) = \mathrm{tr}(BA^k - ABA^{k-1}) = 0\,.$$

The integral F_1 is the one we already know. One proves that all these integrals are in involution and that for $k = 1, \ldots, n$ they are independent.

2. Linearizing flows on Jacobians: introduction

In this section we wish to state the problem and provide a first example. Let us consider a dynamical system (i.e., a system of first order ordinary differential equations) which admits a *Lax pair,* i.e., it can be written in the form

$$\dot{A} = [B, A]\,, \tag{3}$$

where A and B are $n \times n$ matrix depending on time t, and a dot denotes differentiation w.r.t. t. One can study a more general situation where the matrices A and B depend on a parameter ξ. We shall actually assume that A and B are polynomials in ξ,

$$A(\xi) = \sum_{k=1}^{m} A_k\, \xi^k\,, \qquad B(\xi) = \sum_{k=1}^{m} B_k\, \xi^k\,, \tag{4}$$

where the coefficients A_k, B_k are matrices depending on t. We shall suppose that the equation (**??**) and the equation

$$\dot{A}(\xi) = [B(\xi), A(\xi)]\,,$$

assumed to hold for every value of ξ, are equivalent. We also introduce the *spectral polynomial*

$$Q(\xi, \eta) = \det(\eta\, I - A(\xi))$$

where η is another variable, and I is the $n \times n$ identity matrix.

Proposition 2.1. *If the matrix A evolves according to eq. (**??**), then the spectral polynomial Q does not depend on t.*

Proof. The matrix $M = \eta\, I - A$ evolves according to the equation $\dot{M} = [B, M]$. We have

$$\begin{aligned} \dot{Q} &= \frac{d}{dt} \det M = \det M\ \operatorname{tr}(M^{-1}\, \dot{M}) \\ &= \det M\ \operatorname{tr}(M^{-1}\, [B, M]) = \det M\ \operatorname{tr}(M^{-1}\, B\, M - B) = 0\,. \end{aligned}$$

□

As a consequence the polynomial Q defines a fixed affine curve C' in $\mathbb{C}^2$. We shall make two assumptions:[2]

1. The curve C' can be completed to a smooth projective algebraic curve C;
2. for every $(\xi, \eta) \in C'$ the corresponding eigenspace of A is one-dimensional.

Because of the second assumption one has a (time-dependent) map

$$f_t \colon C \to \mathbb{P}V$$

[2]We shall study the spectral curve in more detail in the next section.

where V is an n-dimensional vector space. Let $L_t = f_t^*\mathcal{O}(1)$; by continuity, the degree of L_t does not vary with time, and since

$$\deg L_t = \int_C f_t^* H = \int_{f_t(C)} H = \mathrm{P.D.}([C]) \cdot H = \deg(C)$$

it coincides with the degree d of C (here "P.D." denotes the Poincaré dual, and H is the hyperplane class of $\mathbb{P}V$). While t varies, L_t moves in $\mathrm{Pic}^d(C)$, and, if we fix a line bundle $L_0 \in \mathrm{Pic}^d(C)$, the line bundle $\tilde{L}_t = L_0^{-1} \otimes L_t$ moves in the Jacobian variety $J(C)$. As we shall see, the motion of the line bundle $\tilde{L}_t$ depends on the choice of the matrix B. *In particular, the condition for the linearity of the motion may be expressed as a cohomological condition on B.*

Example 2.2. (The n-dimensional rigid body.[3]) Let

$$\Upsilon = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

be the matrix representing the tensor of inertia of a rigid body in a principal axis system, and let $\Omega(t) \in \mathrm{so}(n)$ be the skew-symmetric matrix associated to the angular velocity vector of the rigid body in the usual way. Define $M = \Omega\Upsilon + \Upsilon\Omega$; the equations of motion of the rigid body can be written as

$$\dot{M} = [M, \Omega]\,.$$

These equations are equivalent to

$$\frac{d}{dt}(M + \Upsilon^2\xi) = [M + \Upsilon^2\xi, \Omega + \Upsilon\xi]$$

so that we get a Lax pair with parameter.

Since $M = -\tilde{M}$, one has

$$\begin{aligned} Q(-\xi, -\eta) &= \det(-\eta\, I - M + \Upsilon^2\,\xi) \\ &= \det(-\eta\, I + M + \Upsilon^2\,\xi) = (-1)^n Q(\xi, \eta)\,, \end{aligned}$$

and the spectral curve has an involution,

$$j\colon C \to C, \qquad j(\xi, \eta) = (-\xi, -\eta)\,.$$

As one might suspect, the flow linearizes on the associated Prym variety [?]. If $n = 3$ the polynomial Q is cubic,

$$Q(\xi, \eta) = \det(\eta\, I - M - \Upsilon^2\xi) = \eta^3 - (\det\Upsilon^2)\xi^3 + \dots$$

and C is a cubic curve. □

[3] Some generalities about the rigid body are described in the Appendix.

3. Spectral curves

The curve described by the polynomial Q can be compactified in a standard way as follows. We start with a short digression, showing how the total space of a vector bundle can be compactified by passing to the *projectivization* of the vector bundle. Let E be a rank r holomorphic vector bundle on a complex manifold X, described by transition functions $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to Gl(r,\mathbb{C})$. By composing the functions $g_{\alpha\beta}$ with the projection

$$Gl(r,\mathbb{C}) \to \mathbb{P}Gl(r,\mathbb{C}) = Gl(r,\mathbb{C})/\mathbb{C}^*\, I$$

one obtains transition functions for a holomorphic bundle $\mathbb{P}E$ on X whose standard fibre is the projective space $\mathbb{P}_{r-1}$. This is called the projectivization of E, and its total space is compact when X is so. Moreover, the total space of the bundle $\mathbb{P}(E \oplus \underline{\mathbb{C}})$ is a compactification of the total space of E: the latter sits in the former as a dense open subset.

Let Y be the total space of the line bundle $H^m = \mathcal{O}_{\mathbb{P}_1}(m)$ for some $m > 0$, and denote $\pi\colon Y \to \mathbb{P}_1$ the projection. Let us denote by $\bar{Y}$ the compactification described above. A point in Y is a pair (ξ,η), with $\xi \in \mathbb{P}_1$ and $\eta \in H^m_\xi$, so that one can define the *tautological section*

$$\sigma \in H^0(Y,\pi^* H^m), \qquad \sigma(\xi,\eta) = \eta$$

We assume that the $n \times n$ matrix A can be written in the form given in eq. (**??**), where ξ is regarded as an affine coordinate in $\mathbb{P}_1$. The characteristic polynomial

$$Q(\xi,\eta) = \det(\eta I - A(\xi,t))$$

is an element in $H^0(Y,\pi^*\mathcal{O}_{\mathbb{P}_1}(mn))$ and therefore corresponds to a divisor. Let C_0 be the corresponding curve in $\bar{Y}$. This is a projective curve, but need not be smooth, and therefore we introduce its normalization C. If we forget about η we obtain a map

$$\pi\colon C \to \mathbb{P}_1 \tag{5}$$

which exhibits C as a branched cover of $\mathbb{P}_1$ of degree n (since the characteristic polynomial has n roots). The curve C is called the *spectral curve.* It is compact, and is a compactification of the affine plane curve introduced in the previous section. Let us compute its genus when the compactified curve is smooth (so one does not need to normalize).

For every ξ let $\eta_1,\dots,\eta_n$ be the roots of $Q(\xi,\eta) = 0$, and define the discriminant

$$\Delta = \prod_{i<j} (\eta_i - \eta_j)^2 \,.$$

Let us denote $\tilde{\xi} = (\xi_0,\xi_1)$ the homogeneous coordinates in $\mathbb{P}_1$, and write

$$A = \sum_{j=0}^{m} A_j\, \xi_1^j\, \xi_0^{m-j}\,.$$

Δ is homogeneous as a function of $\tilde{\xi}$,

$$\Delta(\lambda\tilde{\xi}) = \lambda^{2m\binom{n}{2}}\,\Delta(\tilde{\xi})$$

and therefore

$$\Delta \in H^0(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(mn(n-1)))\,.$$

Let us suppose that at $\bar{\xi} \in \mathbb{P}_1$ exactly k eigenvalues η coincide and take the value $\bar{\eta}$, and that no other eigenvalues coincide. Then $\bar{\xi}$ is a ramification point of order $k-1$ for $\pi\colon C \to \mathbb{P}_1$. We may take a coordinate t centred in $\bar{\xi}$ such that

$$\eta_j = e^{\frac{2\pi i}{k}j}\,t^{1/k} + \bar{\eta}, \qquad 1 \le j \le k$$

(here "i" is the imaginary unit). Then the behaviour of Δ at $\bar{\xi}$ is given by the contribution

$$\prod_{1\le j<\ell\le k} (e^{\frac{2\pi i}{k}j}\,t^{1/k} - e^{\frac{2\pi i}{k}\ell}\,t^{1/k})^2 = c\,t^{k-1}$$

for some complex number c. So the vanishing order of Δ at $\bar{\xi}$ is the ramification index, which, taking account of all possibile ramifications, means that the degree of the ramification divisor of π is the degree of Δ, i.e., $mn(n-1)$. From the Riemann-Hurwitz formula we get

$$g(C) = \frac{mn(n-1)}{2} - n + 1\,. \tag{6}$$

For instance for the 3-dimensional rigid body one has $m = 1$, $n = 3$ and we get $g = 1$.

4. Deformations

Refining the assumption that we made in the previous section, for a generic point in C, corresponding to a generic (smooth) point $(\xi, \eta) \in C_0$, we assume that

$$\dim\ker(\eta\, I - A(\xi)) = 1\,.$$

Then the map which assigns to (ξ, η) the one-dimensional eigenspace of $A(\xi)$ belonging to the eigenvalue η is a time-dependent holomorphic map $f_t\colon C - \{p_1 \dots p_N\} \to \mathbb{P}V$ for some points $p_1 \dots p_N \in C$ (which are exactly the points where the covering map π ramifies). Since the matrix A is polynomial in ξ, f_t extends to a meromorphic map on the whole of C, and hence also to a *holomorphic* map [**?**], which we denote by the same letter. We call this the *eigenvector map.*

Let us consider $t = 0$, and let $f_0 = f$. We have an exact sequence

$$0 \to T_C \xrightarrow{f_*} f^*T_{\mathbb{P}V} \to N_f \to 0 \tag{7}$$

where T_C, $T_{\mathbb{P}V}$ are the tangent bundles, and N_f is the normal sheaf. The latter may not be locally free because f is not an embedding (it ceases to be such at the points where C is ramified over its image). The space $H^0(C, N_f)$ may be regarded as the space of *infinitesimal deformations* of the pair (C, f): its elements describe simultaneous deformations of the curve and its map into the projective space $\mathbb{P}V$. Let us consider the associated long cohomology sequence

$$0 \to H^0(C, T_C) \to H^0(C, f^*T_{\mathbb{P}V}) \to H^0(C, N_f) \xrightarrow{\bar{\delta}} H^1(C, T_C).$$

For every $p \in C$ one has a curve $f_t(p)$ in $\mathbb{P}V$, and taking derivative at $t = 0$ (or at any other given time) we obtain an element in $H^0(C, f^*T_{\mathbb{P}V})$. We denote $\dot{f}$ its image in $H^0(C, N_f)$. The kernel

$$\ker \bar{\delta} = \frac{H^0(C, f^*T_{\mathbb{P}V})}{H^0(C, T_C)}$$

parametrizes the deformations of f when C is kept fixed. This is the situation we are interested in, since the time evolution of our dynamical system does not change the characteristic polynomial of A, so that the curve C remains fixed.

We shall denote $L = f^*\mathcal{O}_{\mathbb{P}V}(1)$ and shall consider the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}V} \xrightarrow{i} V \otimes \mathcal{O}_{\mathbb{P}V}(1) \xrightarrow{p} T_{\mathbb{P}V} \to 0\,. \tag{8}$$

This is an exact sequence of vector bundles, so that it remains exact after pulling back to C via f^*. We have then a diagram of exact sequences

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathcal{O}_C \arrow[d, "\nu"'] & & \\
 & & V \otimes L \arrow[d] & & \\
0 \arrow[r] & T_C \arrow[r] & f^*T_{\mathbb{P}V} \arrow[r] \arrow[d] & N_f \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd} \tag{9}$$

The associated cohomology diagram contains the piece

$$\begin{tikzcd}
 & & H^0(C, V \otimes L) \arrow[d, "\tau"'] & & \\
0 \arrow[r] & H^0(C, T_C) \arrow[r] & H^0(C, f^*T_{\mathbb{P}V}) \arrow[r, "j"] \arrow[d, "\delta"'] & H^0(C, N_f) \arrow[r, "\bar{\delta}"] & H^1(C, T_C) \\
 & & H^1(C, \mathcal{O}_C) & &
\end{tikzcd} \tag{10}$$

All the morphisms here involved, and the line bundle L, depends actually on t, and it makes sense to take their derivatives with respect to t and evaluate them at $t = 0$, or at any other fixed time, as we did to define $\dot{f}$. Again, we shall denote this operation by a superimposed dot. For instance, we have $\dot{L} \in H^1(C, \mathcal{O}_C)$. We may understand this by a direct computation: if $g_{\alpha\beta}$ are the transition functions of the line bundle $L = L_0$, the transition functions of an "infinitesimally deformed" line bundle L_t may be written as

$$g'_{\alpha\beta} = g_{\alpha\beta} + \epsilon_{\alpha\beta}\, g_{\alpha\beta},$$

and the cocycle conditions for $g'_{\alpha\beta}$ imply

$$\epsilon_{\alpha\beta} + \epsilon_{\beta\gamma} - \epsilon_{\alpha\gamma} = 0$$

so that we get indeed an element in $H^1(C, \mathcal{O}_C)$. On the other hand, by infinitesimally varying a line bundle we get an element of the tangent space of the Jacobian variety $J(C)$ of C, and this may obviously identified with $H^1(C, \mathcal{O}_C)$ (recall that $J(C) \simeq H^1(C, \mathcal{O}_C)/H^1(C, \mathbb{Z})$).

Let ν_t be a lift of f_t to $V - \{0\}$; this corresponds to a choice of an eigenvector for every eigenvalue of the matrix A. So ν_t is a time-dependent map $C \to V - \{0\}$. This lift is not canonical and exists only locally, but we are going to use it to define an object (that we will denote $\dot{\nu}$) which will be independent of the lift and therefore will be globally well defined.

We are using the same name as the morphisms ν in diagram (**??**) because the latter may also be represented as follows. Since $\mathcal{O}_{\mathbb{P}V}$ is the tautological bundle of $\mathbb{P}V$, the fibre of $f_t^*\mathcal{O}_{\mathbb{P}V}(-1)$ at a point $p \in C$ may be identified with the space $\mathbb{C}\,\nu_t(p)$; this defines a map $f_t^*\mathcal{O}_{\mathbb{P}V}(-1) \to V \otimes \mathcal{O}_C$, and therefore a map

$$\begin{array}{rcl} \nu_t \colon \mathcal{O}_C & \to & V \otimes L_t \\ \varphi & \mapsto & \varphi\, \nu_t \end{array}$$

(notice that ν_0 coincides with the map ν in the diagram (**??**)).

If ν' is another lift, we have $\nu' = \rho\, \nu$, where ρ is a function, so that

$$\dot{\nu}' = \rho\, \dot{\nu} + \dot{\rho}\, \nu\,.$$

The quantity

$$\dot{\nu} = \left(\frac{\partial \nu_t}{\partial t}\right)_{t=0} \mod \nu$$

is a well-defined element in $H^0(C, V \otimes L/\mathcal{O}_C) \simeq H^0(C, f^*T_{\mathbb{P}V})$, independent of the choice of the lift. One can also note that if one fixes a point $p \in C$ then $\nu_t(p)$ describes a curve in $V - \{0\}$. By projecting the tangent vector field to $\mathbb{P}V$ we obtain a section $\dot{\nu} \in H^0(C, f^*T_{\mathbb{P}V})$.

It is quite clear that $j(\dot{\nu}) = \dot{f}$.

Lemma 4.1. *$\delta(\dot{\nu}) = \dot{L}$.*

Proof. Let $(w^0, \dots, w^n)$ be homogeneous coordinates in $\mathbb{P}V$ associated with a fixed basis of V, and let $\{U_i\}$ be the corresponding open cover of $\mathbb{P}V$. We shall write $w^k \circ \nu = \nu^k$. The quantity $\dot\nu$ as a vector field on $\mathbb{P}V$ may be written as

$$\sum_{i=0}^{n} \dot\nu^i \, \frac{\partial}{\partial w^i} \,.$$

In order to calculate the action of the connecting morphism δ on $\dot\nu$ we must take a counterimage of $\dot\nu$ under the morphism p of eq. (**??**): this is the n-ple $(\dot\nu^0, \dots, \dot\nu^n)$. Now we must apply the the Čech differential associated with the cover $\{f^{-1}(U_i)\}$ of C and invert the map i. Since the latter is

$$f \mapsto (f w^0, \dots, f w^n)$$

we get the cocycle

$$\left\{ \frac{\dot\nu^k}{\nu^k} - \frac{\dot\nu^i}{\nu^i} \right\}$$

which describes a class in $H^1(C, \mathcal{O}_C)$.

On the other hand, the transition functions of the line bundle L_t may be written as

$$g_{ik}(t) = \frac{w^k}{w^i} \circ f_t = \frac{\nu^k}{\nu^i} \,;$$

the cocycle corresponding to $\dot L$ is therefore

$$\varepsilon_{ik} = \dot g_{ik} \, g_{ik}^{-1} = \frac{\dot\nu^k}{\nu^k} - \frac{\dot\nu^i}{\nu^i} \,.$$

□

Corollary 4.2. *$\dot L = 0$ if and only if $\dot\nu = \tau(w)$ for some $w \in H^0(C, V \otimes L)$ (τ is the map in diagram (**??**).*

Proof. From the exactness of the column in diagram (**??**). □

5. The main results

We write also

$$B(\xi, t) = \sum_{i=0}^{m} B_i(t)\, \xi^i = \sum_{i=0}^{m} B_i(t)\, \xi_0^{m-i}\, \xi_1^i$$

where we have regarded ξ as an affine coordinate in the $\mathbb{P}_1$ which is the base of the covering $\pi \colon C \to \mathbb{P}_1$, while ξ_0, ξ_1 are homogeneous coordinates. Let D be the divisor

$$D = m\, \pi^{-1}(\infty)$$

in C. We may think of B as an element in $H^0(C, \mathrm{Hom}(V, V(D))$ (here V denotes the sheaf of sections of the trivial bundle $C \times V$, and $V(D) = V \otimes \mathcal{O}_C(D)$. Moreover, since $\nu \in H^0(C, V \otimes L)$, we have

$$B \cdot \nu \in H^0(C, V \otimes L(D)) \,.$$

Theorem 5.1. $\dot{\nu} = \tau(B \cdot \nu)\,.$

Proof. We consider the diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_C \arrow[r] \arrow[d, "\nu"'] & \mathcal{O}_C(D) \arrow[r] \arrow[d, "\nu"'] & \mathcal{O}_D(D) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V \otimes L \arrow[r] \arrow[d, "\tau"'] & V \otimes L(D) \arrow[r] \arrow[d, "\tau"'] & V \otimes L \otimes \mathcal{O}_D(D) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f^*T_{\mathbb{P}V} \arrow[r] \arrow[d] & f^*T_{\mathbb{P}V}(D) \arrow[r] \arrow[d] & f^*T_{\mathbb{P}V} \otimes \mathcal{O}_D(D) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

whence we get the cohomology diagram

(11)

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & H^0(C, \mathcal{O}_C(D)) \arrow[r, "\rho"] \arrow[d, "\nu"'] & H^0(C, \mathcal{O}_D(D)) \arrow[r, "\delta_1"] \arrow[d, "\sigma"'] & {} \\
0 \arrow[r] & H^0(C, V \otimes L) \arrow[r, "i"] \arrow[d, "\tau"'] & H^0(C, V \otimes L(D)) \arrow[r, "j"] \arrow[d, "\tau"'] & H^0(C, V \otimes L \otimes \mathcal{O}_D(D)) \arrow[d, "\tau"] & \\
0 \arrow[r] & H^0(C, f^*T_{\mathbb{P}V}) \arrow[r, "i"] \arrow[d, "\delta"'] & H^0(C, f^*T_{\mathbb{P}V}(D)) \arrow[r, "j"] \arrow[d, "\delta"'] & H^0(C, f^*T_{\mathbb{P}V} \otimes \mathcal{O}_D(D)) & \\
{} \arrow[r, "\delta_1"] & H^1(C, \mathcal{O}_C) \arrow[r] & H^1(C, \mathcal{O}_C(D)) & &
\end{tikzcd}
$$

Notice that the map δ_1 in the second row is the same as the one in the bottom row; it is the first connecting morphism of the long cohomology sequence of the first horizontal sequence in diagram (**??**).

In more precise term, the statement of the theorem is

$$i(\dot{\nu}) = \tau(B\,\nu)\,.$$

The proof is a simple computation. From

$$A(\xi, t)\,\nu(p, t) = \eta\,\nu(p, t)$$

we have $\dot{A}\,\nu + A\,\dot{\nu} = \eta\,\dot{\nu}$ and using the Lax equation

$$A(\dot{\nu} - B\,\nu) = \eta(\dot{\nu} - B\,\nu)$$

Since generically the eigenvalues have multiplicity 1 we have

$$\dot{\nu} - B\,\nu = -\lambda\,\nu \tag{12}$$

for some λ. But regarding $\dot{\nu}$ as a section of $V \otimes L/\mathcal{O}_C$ this means $\tau(B\,\nu) = \dot{\nu}$. □

Corollary 5.2. *$\dot{L} = 0$ if and only if there is a section $\phi \in H^0(C, \mathcal{O}_C(D))$ such that $B\,\nu + \nu(\phi) = i(b)$ for some $b \in H^0(C, V \otimes L)$. (Notice that ϕ is a meromorphic function on C, and that $\nu(\phi) = \phi\,\nu$).*[4]

Proof. Since

$$i(\dot{\nu}) = \tau(B\,\nu) = \tau(B\,\nu + \nu(\phi)) = \tau \circ i(b) = i \circ \tau(b)$$

we have $\dot{\nu} = \tau(b)$ so that

$$\dot{L} = \delta(\dot{\nu}) = \delta \circ \tau(\nu) = 0.$$

□

Referring to eq. (**??**) we note that, since ν and $\dot{\nu}$ are holomorphic, while $B\,\nu$ has poles on the divisor D, the function λ has poles on D, i.e., it defines a section of $\mathcal{O}_D(D)$. This section will be denoted $\rho(B)$ and will be called the *residue* of B; it is the basic invariant of the Lax equation, and will allow us to state a criterior for the linearity of the flow L_t.

Theorem 5.3. *$\dot{L} = \delta_1(\rho(B))$.*

Proof. Let $E = B\,\nu$. We know that $\tau(E) = i(\dot{\nu})$, so that

$$\tau \circ j(E) = j \circ \tau(E) = j \circ i(\dot{\nu}) = 0$$

and there exists $\lambda \in H^0(C, \mathcal{O}_D(D))$ such that $\sigma(\lambda) = j(E)$. From diagram (**??**) we obtain $\delta(\dot{\nu}) = \delta_1(\lambda)$, which is a restatement of the claim. □

We shall say that the flow L_t is *linear* if there exists a complex number c such that $\ddot{L}_t = c\dot{L}_t$ (this equality holds in the fixed vector space $H^1(C, \mathcal{O}_C)$).

A digression on residues. To state the linearity condition for L_t we need a few facts about residues and the Mittag-Leffler problem. We consider the exact sequence

$$H^0(C, \mathcal{O}_C(D)) \xrightarrow{\text{res}} H^0(C, \mathcal{O}_D(D)) \xrightarrow{\delta_1} H^1(C, \mathcal{O}_C). \tag{13}$$

The elements in $H^0(C, \mathcal{O}_C(D))$ are meromorphic functions on C, having poles along D, and the image under res of such an element is the collection of its Laurent tails. We recall

[4] We shall not use this corollary in the following.

that given a meromorphic differential ψ on C and a pole $p \in C$ of ψ, the *residue* of ψ at p is the number

$$\mathrm{Res}_p(\psi) = \frac{1}{2\pi i} \int_\gamma \psi$$

where γ is a loop around p which does not include any other pole of ψ. If ψ is locally written as $\psi = f\, dz$ with $z(p) = 0$ and

$$f(z) = \sum_{j=1}^{N} \frac{b_j}{z^j} + \text{a holomorphic function}$$

for some N, then $\mathrm{Res}_p(\psi) = b_1$. Moreover, we know that the sum of the residues of a meromorphic differential is always zero.

Lemma 5.4. *(The Mittag-Leffler problem.) Let $D = \sum_i a_i\, p_i$. Given an element $\Phi = \{\phi_i\} \in H^0(C, \mathcal{O}_D(D))$ there is on C a meromorphic function ϕ, holomorphic out of D, such that $\rho(\phi) = \Phi$ if and only if*

$$\sum_i \mathrm{Res}_{p_i}(\phi_i\, \omega) = 0$$

for every holomorphic differential ω on C.

Proof. "Only if": if ϕ is such a function, then $\phi - \phi_i$ is a holomorphic function defined in a neighbourhood of p_i,

$$\sum_i \mathrm{Res}_{p_i}(\phi_i\, \omega) = \sum_i \mathrm{Res}_{p_i}(\phi\, \omega) = 0\,.$$

"If": From the isomorphism $H^1(C, \mathcal{O}_C) \simeq H^0(C, K_C)^*$ (Serre duality) we obtain a pairing $< \delta_1(\Phi), \omega >$. Let us show that

$$< \delta_1(\Phi), \omega >= -2\pi\, i \sum_i \mathrm{Res}_{p_i}(\phi_i\, \omega)\,. \tag{14}$$

Let us take an open cover $\{U_i\}$ of C such that every U_i contains at most one p_i, and such that on every U_i there is a meromorphic function f_i with $\mathrm{Res}_{p_i}(f_i) = \phi_i$ if U_i contains p_i; if U_i contains no point, take the zero function. Then the cocycle $\{\delta_1(\Phi)_{ik} = \{f_i - f_k\}$ represents an element in $H^1(C, \mathcal{O}_C)$. The corresponding form of type (0,1) under the Dolbeault isomorphism $H^1(C, \mathcal{O}_C) \simeq H^{0,1}_{\bar\partial}(C)$ is

$$h = \sum_i \bar\partial(\chi_i\, f_i)$$

where $\{\chi_i\}$ is a partition of unity subordinated to $\{U_i\}$ such that $\chi_i = 1$ in a neighbourhood of p_i, and we also define $h(p_i) = 0$. One then has

$$\begin{aligned} < \delta_1(\Phi), \omega > &= \int_C h \wedge \omega = \int_C \sum_i \bar\partial(\chi_i\, f_i\, \omega) \\ &= \int_C \sum_i d(\chi_i\, f_i\, \omega) = \lim_{\epsilon_i \to 0} \int_{C - B(\epsilon_i)} \sum_i d(\chi_i\, f_i\, \omega) \\ &= - \lim_{\epsilon_i \to 0} \int_{\partial B(\epsilon_i)} \sum_i (\chi_i\, f_i\, \omega) = -2\pi\, i \sum_i \mathrm{Res}_{p_i}(f_i\, \omega) \\ &= -2\pi\, i \sum_i \mathrm{Res}_{p_i}(\phi_i\, \omega) \end{aligned}$$

So the formula (**??**) has been proved. This shows that under our hypotheses $\delta_1(\Phi) = 0$, and since the sequence (**??**) is exact, we obtain $\Phi \in \mathrm{Im}\, \rho$. □

Corollary 5.5. *The flow L_t in $\mathrm{Pic}^d(C)$ is linear if and only if*

$$\rho(\dot{B}) = 0 \quad \mathrm{mod}\{\mathrm{Im}\, \mathrm{res},\ \rho(B)\} \tag{15}$$

Proof. “If”: From $\rho(\dot{B}) = c\, \rho(B) + \mathrm{res}(\phi)$ for some $\phi \in H^0(C, \mathcal{O}_C(D))$ then

$$\delta_1(\rho(\dot{B})) = c\, \delta_1(\rho(B))$$

so that $\ddot{L} = c' \dot{L}$.

“Only if:” From $\ddot{L} = c' \dot{L}$ we get

$$\delta_1(\rho(\dot{B})) - c\, \rho(B)) = 0$$

so that there exists $\phi \in H^0(C, \mathcal{O}_C(D))$ such that

$$\rho(\dot{B}) - c\rho(B) = \mathrm{res}(\phi)\,.$$

□

Example 5.6. (The rigid body strikes back.) It was shown in Example **??** that the motion of the n-dimensional rigid body with a fixed point is described by a Lax pair with parameter, where

$$B = -\Omega - \Upsilon\, \xi.$$

This has a simple pole at infinity, so that $D = \sum_{i=1}^n p_1$, where the p_i's are the pre-images of the point at infinity under the covering map $C \to \mathbb{P}_1$. If z_i is a local coordinate on C around p_i then from equation (**??**) we obtain

$$\rho(B) = -\sum_{i=1}^n \frac{\lambda_i}{z_i}$$

if $\Upsilon = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Since the λ_i's are constant one clearly has $\rho(\dot{B}) = 0$ so the flow on $J(C)$ is linear.

Example 5.7. Let us work out the condition on the residue $\rho(B)$ in the case of the Toda lattice. We may choose

$$A(\xi) = A_{-1}\,\xi^{-1} + A_0 + A_1\,\xi$$

with

$$A_0 = \begin{pmatrix} b_1 & a_1 & & & & \\ a_1 & b_2 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & b_{n-1} & a_{n-1} \\ & & & & a_{n-1} & b_n \end{pmatrix}$$

$$A_{-1} = \text{all zero but } (A_{-1})_{nn} = a_n, \qquad A_1 = \widetilde{A_{-1}}.$$

In this case the matrix A is meromorphic rather than polynomial in ξ, but the theory may easily be adapted to this situation.

The matrix $A(\xi)$ has the symmetry $\widetilde{A(\xi)} = A(\xi^{-1})$ so that $Q(\xi,\eta) = Q(\xi^{-1},\eta)$ and the spectral curve has an involution

$$j\colon C \to C, \qquad j(\xi,\eta) = (\xi^{-1},\eta)\,.$$

The explicit form of the spectral polynomial is

$$Q = a_1 \cdot \dots \cdot a_{n-1}(\xi + \xi^{-1}) + P(\eta)$$

where P is a polynomial of degree n. The resulting curve is singular at the point at infinity for $n \geq 4$.

To study the spectral curve C we think of it as a ramified double cover $C \to \mathbb{P}_1$ of the η-sphere. It is ramified where $\xi = \pm 1$, and these are $2n$ points. By Riemann-Hurwitz we get $g(C) = n-1$. So the Jacobian variety has exactly the dimension corresponding to the number of the integrals of the motion, minus the trivial one.

The divisor (ξ) may be written as $(\xi) = np - nq$, where p and q lie on two distinct sheets of the cover. In this case the divisor D is $D = np + nq$.

We should compute a set of (holomorphic) eigenvectors so as to be able to extract the residues from B. We briefly describe a method due to van Moerbeke and Mumford [**?**]. We compute the residue at q but the same method applies for the residue at p. Let $E = r_1 + \dots + r_g$ be a generic divisor of degree g on C such that

$$h^0(E + (k-1)p - kq) = 0 \qquad \text{for all } k.$$

Then one easily shows that $h^0(E + kp - kq) = 1$ for all k. Let us pick a generator $f_k \in H^0(C, \mathcal{O}_C(E + kp - kq))$ for every $k = 1, \dots, n$, with $f_n = \xi$. If we organize the

f_i's into a column vector $\tilde{\nu}$, then it is possible to choose E in such a way that $\tilde{\nu}$ is an eigenvector of A:

$$A\,\tilde{\nu} = \eta\,\tilde{\nu}\,.$$

Moreover, $\nu = \xi^{-1}\,\tilde{\nu}$ is an eigenvector as well and is holomorphic.

We work out in some detail the case $n = 3$. The condition $A\,\tilde{\nu} = \eta\,\tilde{\nu}$ reads

$$\left\{\begin{array}{r} b_1 f_1 + a_2 f_2 + a_3 = \eta\, f_1 \\ a_1 f_1 + b_2 f_2 + a_2\xi = \eta\, f_2 \\ a_3\xi f_1 + a_2 f_2 + b_3\xi = \eta\,\xi \end{array}\right.$$

Multiplying by ξ^{-1} everythings turns holomorphic but the last line

$$a_3 f_1 = \eta + \text{holomorphic terms}$$

Comparing with

$$B\nu = \rho(B)\nu + \text{holomorphic terms}$$

we get

$$\begin{pmatrix} a_1 f_2 \xi^{-1} - a_3 \xi^{-1} \\ -a_1 f_1 \xi^{-1} + a_2 \\ a_3 f_1 - a_2 f_2 \xi^{-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \eta \end{pmatrix} + \text{holomorphic terms}$$

whence

$$\rho(B) = \frac{\eta}{\xi}\,.$$

This is time-independent and, since a similar condition holds for the residue at p, we obtain that the flow on the Jacobian is linear.

Appendix: the equations of motion of a rigid body

The motion of a (3-dimensional) rigid body with a fixed point O may be described in terms of the motion of a triple of orthogonal cartesian axes XYZ, centred in O and at rest with respect to the body, with respect to another triple of orthogonal axes xyz, again centred in O, but this time at rest in an inertial reference system. The motion of XYZ with respect to xyz is described by a time-dependent matrix R in $SO(3)$. The matrix

$$\Omega = \dot{R}\,\tilde{R}$$

(where ˜ denotes transposition) is a time-dependent element in the Lie algebra $o(3)$, i.e., a time-dependent skew-symmetric matrix. We may associate to it a vector ω in the usual way. This is called the *angular velocity vector.*

Another ingredient we need to describe the motion of the rigid body is its *tensor of inertia* Υ. It is a symmetric rank-two tensor which depends on the mass distribution of the rigid body. Due to its symmetry we may always choose the axes XYZ in such a way that the matrix representing the tensor is symmetric,

$$\Upsilon = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)\,.$$

In the absence of external momenta the equations of motion of the rigid body are the *Euler equations*

$$\lambda_1\,\dot\omega_1 = (\lambda_3 - \lambda_2)\,\omega_2\,\omega_3, \qquad \lambda_2\,\dot\omega_2 = (\lambda_1 - \lambda_3)\,\omega_1\,\omega_3, \qquad \lambda_3\,\dot\omega_3 = (\lambda_2 - \lambda_1)\,\omega_1\,\omega_2\,.$$

If we set

$$M = \Omega\Upsilon + \Upsilon\Omega$$

a straightforward calculation shows that the Euler equations may be written in the form

$$\dot M = [M,\,\Omega]\,.$$

This equation makes sense in any dimension and therefore we call it the equation of motion of the n-dimensional rigid body (if $\Omega \in o(n)$ and Υ is an $n \times n$ diagonal matrix).

References

[1] M. Adler & P. van Moerbeke, *Linearization of Hamiltonian systems, Jacobi varieties, and representation theory,* Adv. Math. **38** (1980), 318–379.

[2] Ph.A. Griffiths, *Linearizing flows and a cohomological interpretation of Lax equations,* Amer. J. Math. **107** (1985) 1445–1483.

[3] P. A. Griffiths & J. Harris, *Principles of Algebraic Geometry*, Wiley, New York 1994.

[4] P. van Moerbeke & D. Mumford, *The spectrum of difference operators and algebraic curves,* Acta Math. **143** (1979), 93–154.