“LOOP QUANTIZATION OF DIFFEOMORPHISM INVARIANT GAUGE THEORIES”

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CONTENTS OF THE SEMINAR

1. Overview of Ashtekar's program of "loop quantization";
2. A brief account on the mathematical formalism of gauge theories;
3. The reformulation of general relativity in terms of Ashtekar's "new variables";
4. The Wilson functions and their use in gauge theory;
5. The holonomy C^*-algebra, its spectrum and its characterizations;
6. Diffeomorphism and gauge invariant cylindrical measures on the spectrum of the holonomy C^*-algebra;
7. The quantization of the holonomy C^*-algebra and the algorithm of the loop quantization in detail;
8. The "loop representation" of Rovelli and Smolin;
9. Some recent developments of the program and open problems.

1. OVERVIEW OF ASHTEKAR'S PROGRAM OF "LOOP QUANTIZATION"

The program of loop quantization of gauge theories is a project which has the intent to construct a self-consistent and mathematically rigorous model of canonical (Hamiltonian) non-perturbative quantization of pure and constrained gauge theories.

The most important example of constrained gauge theory is the general relativity written in terms of Ashtekar's 'new variables', another important example is the Chern-Simon theory in 3-D.

The fact that the quantization procedure is non-perturbative is essential to study the gravity at a quantum level, in fact the perturbation models assumes the existence of a base geometry (the Minkowski geometry) and then perturbate it in such a way to introduce the spacetime distortions, but this violates Einstein's equivalence postulate!

To develop the loop quantization no new physical structures (like strings of supersymmetry) are postulated, the idea is to use new techniques rather than new concepts, i.e. to stay as close as possible to conventional canonical quantum field theory.

The first step of the program is the individuation of a set of gauge-invariant functions on the configuration space of the classical theory which is *complete* in the sense that it contains all the physically distinct (i.e. gauge-inequivalent) degrees of freedom of the theory. This set happens to be that of Wilson's loop functions.

Then one constructs the C^*-algebra generated by these functions and reaches a quantum formulation by a GNS representation of this C^*-algebra carried on a certain Hilbert space, which becomes the **kinematical state space** $\mathcal{H}_{kin}$ of the quantum theory.

Finally the configuration observables and their conjugate momenta are promoted to self-adjoint operators on $\mathcal{H}_{kin}$ which are required to satisfy the canonical commutation rules.

If the theory incorporates some other constraints, such as the diffeomorphism and the Hamiltonian constraints like in general relativity, then these constraints are imposed in the quantum theory by means of operator equations which has to be satisfied by the vector states belonging to $\mathcal{H}_{kin}$. The subspace $\mathcal{H}_{phys} \subset \mathcal{H}_{kin}$ containing the states satisfying the constraint equations is taken to be the true (physical) state space of the quantum theory.

I anticipate that, at the moment, the kinematical part of the program has been completed, while the dynamical part (i.e. the solution to the Hamiltonian constraint) is not yet well understood and remains the most difficult open problem of the loop quantization program.

2. A BRIEF ACCOUNT ON THE MATHEMATICAL FORMALISM OF GAUGE THEORIES

The features of gauge theories are fully encoded in the mathematical concept of principal fiber bundle $P(M, G)$ once the following identifications are made:

- M represents the spacetime of the theory;
- G is the group of the **internal symmetries** of the theory, also called **gauge group**;
- P is a super-imposed structure, an auxiliary space containing the fibers over the points of M (copies of G) which identifies with the internal states of the fields (or particles, in the quantum vision) described in the theory;
- the connections on the principal bundle are identified with the gauge potentials, i.e. the fields representing the force which makes the matter fields interact, these ones are introduced in the theory as sections of vector bundles associated to the fixed principal bundle.

In local expressions the gauge potentials are $\mathfrak{g}$-valued 1-forms A_i^a on an open subset of M, where $\mathfrak{g}$ is the Lie algebra of G, i is a spacetime index and a is a coordinate in $\mathfrak{g}$ and so ranges from 1 to $dim(\mathfrak{g})$.

The momentum conjugated to A_i^a is a tensorial density of weight $+1$ indicated by $\tilde{E}_a^i$ which takes values in the dual of $\mathfrak{g}$.

The Poisson algebra is:

$$\{A_i^a(t, \mathbf{x}), A_j^b(t, \mathbf{y})\} = 0;$$

$$\{\tilde{E}_a^i(t, \mathbf{x}), \tilde{E}_b^j(t, \mathbf{y})\} = 0;$$

$$\{A_i^a(t, \mathbf{x}), \tilde{E}_j^b(t, \mathbf{y})\} = \delta_i^j \delta_a^b \delta^3(\mathbf{x}, \mathbf{y}).$$

It is worth noting that the characteristic invariance of gauge theories, i.e. the gauge-invariance, imposes that the physically distinct configurations of a gauge theory are not labelled by the set of all connections, denoted with $\mathcal{A}$, but by the set of connections modulo gauge transformations $\mathcal{A}/\mathcal{G}$, where $\mathcal{G}$ is obviously the group of gauge transformations on P, i.e. the G-equivariant automorphisms of P inducing the identity on M.

$\mathcal{A}/\mathcal{G}$ is called the **classical configuration space** of the gauge theory.

From a physical point of view the most interesting gauge theories are the unitary ones, i.e. those having as gauge group a subgroup of $U(N)$, $N \geq 1$.

In fact it is well known that the **standard model**, i.e. the quantum description of the electromagnetic, strong and weak nuclear interactions, is the collections of two quantized gauge theories, precisely:

1. the **electroweak theory**, which unifies the electromagnetic and weak nuclear interactions, is a quantized gauge theory with gauge group $SU(2) \times U(1)$; the splitting of the two forces at small energy levels is due to a phenomenon called **spontaneous symmetry breaking**;

2. the **quantum chromodynamics**, which describes the strong nuclear forces that makes the quarks interact in the hadrons, is a quantized gauge theory with gauge group $SU(3)$.

In the quantum theory of gauge fields the interactions are described by exchange of quanta of the gauge fields, which are bosonic particles: the photon γ for the electromagnetic interactions, the three vector bosons $W^{\pm}$, Z^0 for the weak nuclear interactions and the eight gluons for the strong nuclear interactions.

The last interaction known in nature, i.e. the gravity, admits a self-consistent and physically predictive formulation only at a classical level by means of Einstein's general relativity.

A fundamental work due to Ashtekar shows that the Euclidean version of general relativity can be reformulated as a constrained gauge theory relative to the gauge group $SU(2)$.

This very important result has introduced in the community of theoretical physicists the hope that gravity can finally be quantized by adapting the well established techniques of quantization of gauge theories to Ashtekar's reformulation of general relativity.

In the next section it is briefly explained how this reformulation can be obtained.

3. THE REFORMULATION OF GENERAL RELATIVITY IN TERMS OF ASHTEKAR'S "NEW VARIABLES"

Einstein's general relativity is the physical theory which describes how the distribution of matter and energy curves the geometry of the spacetime in which it is immersed. The way in which it says this is expressed by the Einstein equations.

This equations relate the **stress-energy tensor**, a symmetric $(0, 2)$ tensor (usually written as $T_{\mu\nu}$ in local components) which express the flow of energy and momentum through a given point of spacetime, with the curvature of the Levi-Civita connection ∇ associated to the metric $g_{\mu\nu}$ of the spacetime manifold.

This curvature is expressed by means of the **Riemann tensor**, defined by: $R(X, Y)Z := (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z$, or, in local coordinates:

$$R^{\mu}_{\nu\lambda\gamma} = \partial_\nu \Gamma^{\mu}_{\lambda\gamma} - \partial_\lambda \Gamma^{\mu}_{\nu\gamma} + \Gamma^{\sigma}_{\lambda\gamma}\Gamma^{\mu}_{\nu\sigma} - \Gamma^{\sigma}_{\nu\gamma}\Gamma^{\mu}_{\lambda\sigma}.$$

The trace of this tensor gives rise to the **Ricci tensor**: $R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu}$ and the contraction of the Ricci tensor gives the **scalar curvature** $R = R^{\mu}{}_{\mu}$.

This objects appear, with the metric itself, in **the Einstein equations**:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}$$

in units where Newton's gravitational constant is fixed to be 1.

$G_{\mu\nu}$ are the components of a symmetric $(0, 2)$ tensor named **Einstein's tensor**.

Due to the symmetries of the Ricci tensor, the Einstein equations are 10 second order hyperbolic non-linear equations in the components of the metric tensor for every 4-dimensional spacetime.

If one imposes the Cauchy problem on these equations suddenly understand that not all of them are evolutory equations, in fact 4 equations are constraints and the remaining 6 equations are evolutory equations.

The reason why this happens is better understood if one consider the variational formulation of general relativity.

For the sake of simplicity, the next discussion of the actions for gravity will be focused only on the vacuum situation.

The first action for gravity is the Einstein-Hilbert action, i.e. a functional S on the space of all Lorentzian metrics on a 4-D spacetime M given by:

$$S(g) := \int_M R\, vol$$

where vol is the volume form induced by g, which can be written, in local coordinates, as $vol = \sqrt{|det(g)|}dx^0 \wedge \ldots \wedge dx^3$.

The variation of S is minimized precisely when the Einstein vacuum equations hold.

The important thing to note is that this action is invariant under the action of the orientation preserving diffeomorphisms ϕ of M, i.e.:

$$\int_M (\phi^* R)\phi^* vol = \int_M R\, vol.$$

In general, the presence of such local symmetries implies that the Euler-Lagrange equations deduced from the minimization of the variation of the actions (the vacuum Einstein equations when the action is $S(g)$) are not independent and the theory, both in the Lagrangian and in the Hamiltonian formulation, is submitted to constraints.

The Hamiltonian formalism of general relativity is encoded in the ADM (Arnowitt-Deser-Misner) formulation, its discussion will show explicitly the constraints.

In the ADM formulation one assumes that the spacetime M is diffeomorphic to the cartesian product $\mathbb{R} \times \Sigma$, where Σ is a 3-D space-like slice embedded in M. This assumption is called a **splitting** of the spacetime M.

Roughly speaking, in the ADM formalism, general relativity becomes a theory which says how the curvature of Σ evolves in time.

To make this assertion rigorous one has to define the so-called **extrinsic curvature** K of Σ, which is the $(0, 2)$ symmetric tensor given by

$$K(u, v) := -g(\nabla_u v, n)$$

where u, v are tangent vectors on Σ, ∇_u is the covariant derivative defined by g, n is a unit time-like vector normal to Σ, i.e.:

$$g(n, n) = -1,\ g(n, v) = 0\ \ \forall v \in T_p\Sigma.$$

K says how much Σ is curved in the way it sits in M, since it measures how much the unit normal vector n rotates in the direction v when it is parallel translated in the direction u.

In this formalism one can derive the so-called Gauss-Codazzi equations:

$$G^0_0 = -\frac{1}{2}({}^3R^m_{ijk} + K_{jk}K^m_i - K_{ik}K^m_j) = 0$$

$$G^0_i = {}^3\nabla_i K_{jk} - {}^3\nabla_j K_{ik} = 0,\ i = 1, \ldots, 3$$

which says that 4 Einstein's equations are indeed constraints involving the extrinsic metric.

The objects which appear with the left suffix 3 are constructed by the **intrinsic metric** of Σ, i.e. the restriction of the metric g of M on Σ, usually written ${}^3g$.

By introducing the **shift vector field** $\vec{N}$ and the **lapse function** N one can show that the remaining 6 equations are evolutory equations which says how Σ evolves in time, in fact they contain second order time derivatives of the intrinsic curvature ${}^3g$ of Σ.

The last purpose of this section is to describe Ashtekar's action for general relativity, a formulation in which the constraints significantly simplifies.

The fundamental idea behind Ashtekar formulation is to use the peculiarities of the 4-dimensional spacetime to write down the self-dual part of the Palatini action for gravity, this action turns out to induce the same equations of general relativity.

Before showing the peculiarities of the 4-dimensional spacetime used by Ashtekar it is worth remembering the most important issues of Palatini's action for gravity.

In **Palatini's formalism** one consider a parallelizable oriented 4-D manifold M, i.e. it assumes that there exists a vector bundle isomorphism

$$e : \tau \equiv M \times \mathbb{R}^4 \to TM$$

inducing the identity on M. $\mathbb{R}^4$ here is called the **internal space** and capital letters $I, J, \ldots$ are used to denote its coordinates.

If $\{\xi_I\}_{I=0,\ldots,3}$ is the standard base of sections of τ then the corresponding base of vector fields on M is $\{e_I \equiv e \circ \xi_I\}_{I=0,\ldots,3}$ and e_I is locally expressed as: $e_I = e_I^\alpha \partial_\alpha$.

The Minkowski metric on each fiber defines on τ the so-called **internal metric** η.

In general the map e is called a **frame** and if the basis $\{e_I\}$ is orthonormal with respect to a given Lorentzian metric g on M, i.e. if $g(e_I, e_J) = \eta_{IJ}$, then the map e is called a **tetrad** or a **vierbein** for g.

Conversely, e defines a metric g on M by the formula above.

The inverse map $e^{-1} : TM \to M \times \mathbb{R}^4$ has local coordinates e_I^α satisfying $e_I^\alpha e_\alpha^J = \delta_I^J$ and is called a **cotetrad**.

The important thing to stress now is that if M is parallelizable then its frame principal bundle $\mathcal{R}M$ is also trivializable and every vierbein generates a trivialization by

$$\begin{array}{rccc} T: & M \times GL(4) & \longrightarrow & \mathcal{R}M \\ & (x, G) & \mapsto & T(x, G) := \{G_I^J e^I(x)\}. \end{array}$$

By considering in particular the sub-bundle $M \times SO(3,1)$ of $M \times GL(4)$ one can construct the Palatini action:

$$S(e,A) := \int_M e_I^\alpha e_J^\beta F_{\alpha\beta}^{IJ}\, vol(e)$$

where e is a vierbein, A is a principal connection on $M \times SO(3,1)$, F is its curvature and $vol(e)$ is the volume form defined by the Lorentzian metric g expressed as a function of e, i.e. $g_{\alpha\beta} = \eta_{IJ} e_\alpha^I e_\beta^J$.

Thus the Palatini action is a functional of a connection A and a vierbein e, and it can be shown that varying S, with respect to both A and e, the equation $\delta S = 0$ implies that the metric $g_{\alpha\beta}$ satisfies Einstein's vacuum equations.

The Palatini formulation of general relativity has the remarkable feature to encode this theory in the framework of gauge theories. In Palatini's formalism there are both gauge and diffeomorphism constraint, due to the invariance of $S(e,A)$ under gauge transformations and diffeomorphisms.

Even though the form of these constraints is much simpler than in the Einstein-Hilbert approach (since they have a polynomial character), these constraints are not closed under Poisson brackets and this creates many difficulties in the canonical quantization of the theory.

Ashtekar's reformulation of Palatini's action is a clever way to eliminate this problem and also to simplify even more the constraints.

The starting point of Ashtekar's work is the recognition that, on the 4-dimensional Minkowski space $M \equiv (\mathbb{R}^4, \eta = diag(-1,+1,+1,+1))$, the linear endomorphism given by the Hodge star operator $* : \bigwedge^2 M \to \bigwedge^2 M$ defined on the antisymmetric $(0,2)$ tensors as:

$$*F_{IJ} = \frac{1}{2}\epsilon_{IJ}^{KL} F_{KL}$$

where ϵ_{IJ}^{KL} is the Levi-Civita symbol, given by:

$$\epsilon_{IJKL} = \begin{cases} +1 & \text{if IJKL is an even permutation of 1234} \\ -1 & \text{if IJKL is an odd permutation of 1234} \\ 0 & \text{otherwise} \end{cases}$$

doesn't admit eigenvalues, but if one complexifies M to $\mathbb{C}^4$, the Hodge star operator has eigenvalues $\pm i$ and the space $\bigwedge^2 \mathbb{C}^4$ decomposes into the direct sum of its **self-dual** and **antiself-dual** subspaces:

$$\bigwedge^2 \mathbb{C}^4 = \bigwedge^2 (\mathbb{C}^4)^+ \oplus \bigwedge^2 (\mathbb{C}^4)^-$$

which are the eigenspaces relatives to the eigenvalues $\pm i$.

The important thing to observe now is that there exists the isomorphism $\bigwedge^2 \mathbb{C}^4 \simeq \mathfrak{so}(3,1) \otimes \mathbb{C}$ and, thanks to the existence of the double cover $\rho : SL(2,\mathbb{C}) \to SO_0(3,1)$, the above splitting of $\bigwedge^2(\mathbb{C}^4)$ into self-dual and antiself-dual part corresponds to the splitting

$$\mathfrak{so}(3,1) \otimes \mathbb{C} \simeq \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}).$$

Since a Lorentz connection A on $M \times \mathbb{C}^4$ is just an $\mathfrak{so}(3,1) \otimes \mathbb{C}$-valued 1-form on M, the self-dual part of this connection, written usually as ${}^+A$, is a $\mathfrak{sl}(2,\mathbb{C})$-valued 1-form on M.

Thus Ashtekar modifies Palatini's formalism by introducing:

- the vector bundle $\mathbb{C}\tau = M \times \mathbb{C}^4$;
- the complexified tangent bundle $\mathbb{C}TM = \coprod_{x \in M} \mathbb{C} \otimes T_x M$;
- complex frame fields, i.e. vector bundle isomorphisms $e : \mathbb{C}\tau \to \mathbb{C}TM$;

and then define an action, the so-called **Ashtekar's self-dual action for gravity** simply by taking the complexified Palatini action written in terms of the self-dual connection ${}^+A$ and the complex vierbein e:

$$S(e, {}^+A) := \int_M e_I^\alpha e_J^\beta {}^+F_{\alpha\beta}^{IJ} vol(e)$$

miraculously, by varying S both with respect to e and ${}^+A$, one gets the vacuum Einstein equations for the complex valued metric $g_{\alpha\beta} = \eta_{IJ} e_\alpha^I e_\beta^J$.

To obtain the usual (real) gravitation one has two possibilities:

1. impose reality conditions on the complex frame fields in terms of which the metric is expressed, to get a real-valued metric;
2. start from an Euclidean self-dual action, defined by a volume form $vol(e)$ on $\mathbb{R}^4$ induced by the (real) Riemannian metric

   $$g_{\alpha\beta} = \delta_{IJ} e_\alpha^I e_\beta^J$$

   and $\mathfrak{su}(2)$-valued self-dual connections ${}^+A$. From the fact the $\mathfrak{su}(2)$ is the compact real form of $\mathfrak{sl}(2,\mathbb{C})$, one obtains again the (real) Einstein's equations. The relation between the Euclidean formulation and the Lorentzian formulation is then obtained with a generalized Wick transform, called **coherent state transform**, constructed from Ashtekar and others.

This Euclidean formulation of general relativity in terms of Ashtekar's new variables $(e, {}^+A)$ is the most important constrained gauge theory with compact gauge group $(SU(2))$ to which the program of loop quantization applies.

4. THE WILSON FUNCTIONS AND THEIR USE IN GAUGE THEORIES

For the reason discussed in 2., it is not reductive (from a physical perspective) to make the following initial assumption: **in the sequel the gauge group will be assumed to be a compact connected Lie subgroup of** $U(N)$.

Fixed a gauge theory relative to the principal bundle $P(M,G)$, the Wilson functions form a complete set of gauge invariant functions, complete in the sense that they encode all the gauge inequivalent (i.e. physically distinct) configurations of the theory.

The rigorous introduction of the Wilson functions needs the concepts of **holonomy, holonomy map** and **group of loop**.

Fixed a loop α in the base space M based on the point $\star$, a connection A on P and a point $p_0 \in P_\star$, the fiber over $\star$, it can be shown that there exists one and only one *A-horizontal lift* of the loop which starts in p_0, by definition of lift it follows that its ending point, call it p_1, also belongs to $P_\star$.

This point is said to be obtained from p_0 by **parallel transport associated to** A **along** α, by varying p_0 in $P_\star$ one gets a map $\wp_{\alpha,A} : P_\star \to P_\star$ which happens to be a G-equivariant diffeomorphism. Moreover the freedom and the transitivity of the action of the gauge group on the fibers of P imply that there exists exactly one element $H_A(\alpha) \in G$ such that $p_1 = \wp_{A,\alpha}(p_0) = p_0.H_A(\alpha)$, this element of G is called the **holonomy of the loop** α **associated to the connection** A.

It can be shown that $H_A(\alpha)$ is the solution (in the final value of the parameter t) of a non-autonomous linear differential equation in G, precisely

$$\dot{g}_t = -[A(\dot{\alpha_t})]g_t$$

hence its explicit form is given in terms of the path-ordered exponential:

$$H_A(\alpha) = \mathcal{P}\exp \oint_\alpha A\,.$$

The map H_A which assigns to the loop α its holonomy associated to the connection A is called **holonomy map**.

Since the image of H_A relies in G, one would like to give the structure of a group to the set of loops in such a way to transform the holonomy map in a homomorphism.

This can be done by introducing in the semigroup of loops in M an equivalence relation and then by taking the quotient with respect to it.

Here I consider three equivalence relations between loops, the technical definitions and requests common to every equivalence are the following:

- **the loop considered are taken to be piecewise analytic**;
- their interval of parameterization is taken to be $[0, 1]$;
- two loops α and β are said to differ from an **orientation-preserving reparameterization** if there is a growing diffeomorphism $\tau : [0, 1] \rightarrow [0, 1]$ such that $\alpha(t) = \beta(\tau(t))$ for every $t \in [0, 1]$;
- an **oriented loop** is a orientation-preserving reparameterization class of loops.

The three equivalence relations between loops are described below.

1. The **elementary equivalence**: a loop α is said to be **immediately retraced** if it can be written as $\alpha = \prod_i \gamma_i {\gamma_i}^{-1}$, for some paths γ_i in M. Two oriented loops α and β are said to be **elementary equivalent** if one is obtained from the other by composition with an immediately retraced loop γ, i.e. $\alpha = \beta\gamma$. The quotient of the set of oriented loops with respect to the elementary equivalence is called the **group of loops** and indicated with $L_\star(M)$;
2. The **thin equivalence**: a loop $\alpha \in L_\star(M)$ is said to be **thin** if it is homotopic to the constant loop $\star$ with a homotopy having image entirely contained in $\alpha^* \equiv Im(\alpha)$. α,$\beta \in L_\star(M)$ are said to be **thin equivalent** if there exists a thin loop γ such that $\alpha = \beta\gamma$. The quotient of $L_\star(M)$ with respect to this equivalence relation is indicated by $\mathcal{L}_\star(M)$;
3. The **holonomic equivalence**: two loops $\alpha, \beta \in L_\star(M)$ are said to be **holonomy equivalent** if they have the same holonomy with respect to every connection, i.e.
$$H_A(\alpha) = H_A(\beta) \qquad \forall A \in \mathcal{A}\,.$$
The quotient of $L_\star(M)$ with respect to the holonomic equivalence gives rise to a group called the **hoop group** and indicated with $\mathcal{H}_\star(M, G)$.

The relation between the three loop groups introduced above is contained in the next theorem.

Theorem. The following assertions hold:

- $L_\star(M)$ and $\mathcal{L}_\star(M)$ are always isomorphic;
- if G contains a subgroup isomorphic to $SU(2)$ then $\mathcal{H}_\star(M, G)$ is isomorphic to $L_\star(M)$.

The fact that there is a geometric condition which has to be satisfied in order to have the equality between the group of loops and the hoop group is not surprising: while the elementary and thin equivalence have a topological nature, the holonomic one has a geometrical character!

The difference between $L_\star(M) \simeq \mathcal{L}_\star(M)$ and $\mathcal{H}_\star(M,G)$ is substantial only in the Abelian case, where $\mathcal{H}_\star(M,G)$ is an Abelian group.

It can be shown that $L_\star(M)$ can be endowed with a topology which makes it a topological Hausdorff group.

The proof of the previous theorem is not easy and relies on a fundamental property of piecewise analytic loops, i.e. the existence of the **independent loops**, i.e. collections $\{\beta_i\}(i = 1, \dots, n) \subset L_\star(M)$ such that every β_i admits an arc l_i such that $l_i \cap {\beta_j}^* = \emptyset \ \forall j \neq i$, i.e. every loop of the family has an arc which doesn't intersect the images of the other loops of the same family.

The importance of the independent loops relies in the fact that, under the initial hypothesis for the gauge group, one can prove that the so-called **interpolation property** holds: let $\{\beta_1, \dots, \beta_n\} \subset L_\star(M)$ be a finite independent family of loops and $\{g_1, \dots, g_n\} \subset G$ a finite family in G with the same cardinality. Then there exists a connection A such that

$$g_i = H_A(\beta_i) \qquad i = 1, \dots, n.$$

By using the interpolation property and some other tools one can show that the quotient which defines $\mathcal{L}_\star(M)$ is always trivial and that the one which defines $\mathcal{H}_\star(M,G)$ is trivial when G contains $SU(2)$. This fact is very useful in the applications since one can use both geometrical and topological tools to prove important results.

The important thing now is that the holonomy map factorize to a homomorphism between every loop group and the gauge group G, i.e. H_A **realizes an unitary representation of the loop groups**.

Once recognized this fact the definition of the Wilson functions is very easy: for every loop α belonging to one of the loop groups, the Wilson function associated to α, denoted by T_α, is the function which associates to a connection A the value of the normalized character of the unitary representation H_A of the loop group calculated in the loop itself, i.e.

$$\begin{array}{rccl} T_\alpha : & \mathcal{A} & \longrightarrow & \mathbb{C} \\ & A & \mapsto & T_\alpha(A) := \frac{1}{N} Tr(H_A(\alpha)) \end{array}$$

Tr means the trace operator taken in the fundamental representation of the gauge group.

The Wilson functions factorize on the quotient $\mathcal{A}/\mathcal{G}$ thanks to the so-called **representation theorem** which states that two connections A and

A' are gauge-equivalent if and only if their holonomy maps are conjugated, i.e. if there exists $g \in G$ such that $H_{A'}(\alpha) = gH_A(\alpha)g^{-1}$, for every loop α.

Thanks to the cyclic property of the trace one immediately sees that, if A and A' are gauge-equivalent, then $T_\alpha(A) = T_\alpha(A')$, for every loop α. Moreover, since the trace is a continuous operation and the entries of an unitary matrix are bounded, it follows that $T_\alpha \in \mathcal{C}_b(\mathcal{A}/\mathcal{G})$, for every loop α.

This fundamental fact is resumed by saying that **the Wilson functions are gauge-invariant**, i.e. they form a set of observables for gauge theories. The outstanding fact is that *every* observable of a given gauge theory can be obtained from the Wilson functions by linear combination of products of Wilson functions!

This result is obtained by extending the well known result of one-to-one correspondence between equivalence classes of unitary representations of a compact group and their characters to the (non compact) loop groups, which can be done with some topological tools.

The details are quite boring, while the final result is very precious, because it allows to choice **the Wilson functions as the set of the observables of a unitary gauge theory**.

This choice has a triple advantage:

1. it allows to implement a manifestly gauge-invariant formalism of gauge theories in which the constraint due to the invariance under gauge transformations is automatically solved already at a classical level. This avoids the use of the **gauge fixing** procedure, which leads to the well know problems of **Gribov ambiguities**;

2. it leads to the quantum configuration space of the gauge theories in a natural way;

3. it allows to implement a representation of the quantum theory in which the diffeomorphism contraints can easily be solved.

5. THE HOLONOMY C^*-ALGEBRA, ITS SPECTRUM AND ITS CHARACTERIZATIONS

The central step of the loop quantization procedure is the promotion of Wilson's classical observables to quantum observables, i.e. self-adjoint operators on a suitable Hilbert space.

This step can be realized in an algebraic way by completing the algebra generated by the Wilson functions to a C^*-algebra and by implementing a Gelfand-Naimark-Segal (GNS) representation.

The C^*-algebra generated by the Wilson functions is an unital Abelian C^*-algebra[1] usually called 'holonomy C^*-algebra' and indicated with $Hol(M,G)$, since it depends on M and G but (it can be proved) not on the entire structure of the principal fiber bundle $P(M,G)$. The product is the punctual multiplication, the involution is the complex conjugation and the norm is $\| \ \|_\infty$.

It is worth remembering a few facts about C^*-algebras. In what follows $\mathfrak{A}$ will denote an Abelian C^*-algebra with unit u. An element $a \in \mathfrak{A}$ is said to be **positive** if there exists an element $b \in \mathfrak{A}$ such that $a = b^*b$; a linear functional φ on $\mathfrak{A}$ is positive if $\varphi(a) \geq 0$ for every positive element $a \in \mathfrak{A}$, such a functional is always continuous and its norm is the value assumed in the unit of $\mathfrak{A}$: $\|\varphi\| = \varphi(u)$. The positive linear functionals on $\mathfrak{A}$ of unit norm are called the **states** of $\mathfrak{A}$ and they form a compact convex subset of the dual space $\mathfrak{A}^*$.

A **character** of $\mathfrak{A}$ is a non-identically zero homomorphism φ from $\mathfrak{A}$ to $\mathbb{C}$, i.e. $\varphi \in Hom(\mathfrak{A}, \mathbb{C}^*)$, it is always *continuous* and has *unit norm* so that the characters of $\mathfrak{A}$ are precisely its multiplicative states. The **spectrum** of $\mathfrak{A}$, $\sigma(\mathfrak{A})$, is the set of all its characters; endowed with the w^*-topology[2] this is a **compact Hausdorff space**.

$\mathfrak{A}$ is isometrically isomorphic to the unital Abelian C^*-algebra of continuous complex-valued functions on its spectrum by means of the **Gelfand isomorphism**:

$$\begin{array}{rccc} \hat{} : & \mathfrak{A} & \longrightarrow & \mathcal{C}(\sigma(\mathfrak{A})) \\ & a & \mapsto & \hat{a} \end{array}$$

with $\hat{a}(\varphi) := \varphi(a)$. The Gelfand isomorphism preserves the positivity.

Identifying $\mathfrak{A}$ with $\mathcal{C}(\sigma(\mathfrak{A}))$ and using the Riesz-Markov theorem one has that *there is an isomorphism between positive linear functionals on* $\mathfrak{A}$ *and*

[1] A unital Banach $*$-algebra such that $\|a\, a^*\|_\infty = \|a\|_\infty{}^2$, for every element a of the algebra.

[2] A sequence of characters $\{\varphi_n\}$ converges to φ in the w^*-topology if and only if $\lim_{n\to\infty} < \varphi_n, a >=< \varphi, a >$ for every $a \in \mathfrak{A}$.

positive regular Borel measures on $\sigma(\mathfrak{A})$. The representation of the positive linear functional φ_μ associated to the positive regular Borel measure μ is given by:

$$\varphi_\mu(a) = \int_{\sigma(\mathfrak{A})} \hat{a}\, d\mu\,.$$

Furthermore $\|\varphi_\mu\| = \|\mu\|$, hence **the states on $\mathfrak{A}$ are in bijection with the probability measures on** $\sigma(\mathfrak{A})$.

Finally, to every positive measure μ on $\sigma(\mathfrak{A})$ (alias to every positive functional φ_μ on $\mathfrak{A}$) one can associate the **GNS representation**, which is given by the correspondence $a \mapsto M_{\hat{a}}$, where $M_{\hat{a}}$ is the multiplication operator on $L^2(\sigma(\mathfrak{A}),\mu)$ defined by $M_{\hat{a}}\psi := \hat{a}\psi$, for every $a \in \mathfrak{A}$ and $\psi \in L^2(\sigma(\mathfrak{A}),\mu)$.

All these considerations and results apply to the holonomy C^*-algebra $Hol(M,G)$, whose compact Hausdorff spectrum $\sigma(Hol(M,G))$ has a very important property:

Theorem. $\mathcal{A}/\mathcal{G}$ is densely and injectively embedded in the spectrum of the holonomy algebra.

For this reason $\sigma(Hol(M,G))$ is usually denoted as $\overline{\mathcal{A}/\mathcal{G}}$, its elements are written as $\bar{A}$ and called **generalized connections**.

The Gelfand isomorphism specialized to the holonomy C^*-algebra can be written as:

$$\begin{array}{rccl} \hat{} : & Hol(M,G) & \longrightarrow & \mathcal{C}(\overline{\mathcal{A}/\mathcal{G}}) \\ & f & \mapsto & \hat{f}, \quad \hat{f}(\bar{A}) := \bar{A}(f). \end{array}$$

The isometric isomorphism $Hol(M,G) \simeq \mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$ is usually used to identify Wilson functions with their Gelfand transformed.

To understand how much the spectrum $\overline{\mathcal{A}/\mathcal{G}}$ of $Hol(M,G)$ is big, it is useful to cite the first characterization of $\overline{\mathcal{A}/\mathcal{G}}$, i.e. the so-called theorem of Ashtekar-Lewandowski-Baumgärtel (ALB):

$$\sigma(Hol(M,G)) \simeq Hom(L_\star(M),G)/Ad\,.$$

The proof of the result above uses in an essential way the interpolation condition.

The ALB theorem shows that a generalized connection can be characterized in an easier way as an algebraic homomorphism from the group of loops to the gauge group up to Ad-equivalence, without any topological request.

While this algebraic characterization of $\overline{\mathcal{A}/\mathcal{G}}$ is very useful to understand *what* the generalized connections are, there is a second characterization of $\overline{\mathcal{A}/\mathcal{G}}$ which has an enormous importance both to understand the intrinsic structure of this space and to construct measure of it, which is an essential tool to implement the GNS representation of the holonomy C^*-algebra.

This second characterization uses as fundamental tool the theory of projective and inductive limits, briefly resumed below.

Def. *A* **projective family** *of topological Hausdorff spaces is a triple* $\{\Omega_j, \pi_{ij}, J\}$ *where:*

- Ω_j *is a topological Hausdorff space for every* $j \in J$*;*
- J *is a* **directed** *set of indexes, i.e. it is endowed with a partial order relationship* $\leq$ *such that*

$$\forall i, j \in J \ \exists k \in J \text{ such that } i \leq k \text{ and } j \leq k;$$

- *if* $i \leq j$ *then the maps* $\pi_{ij} : \Omega_j \to \Omega_i$ *are continuous surjective projections such that:*
  1. $\pi_{jj} = id_{\Omega_j} \ \forall j \in J$*;*
  2. *if* $i \leq j \leq k$ *then* $\pi_{ij} \circ \pi_{jk} = \pi_{ik}$ *(***"consistency relation"***).*

An element $\{\omega_j\}_{j \in J}$ *of the cartesian product* $\prod_{j \in J} \Omega_j$ *is called* **wire** *if it satisfies the condition*

$$\pi_{ij}\omega_j = \omega_i \qquad \forall i < j$$

i.e. if every element of the ordered sequence is obtained from one of the previous via projection.

The **projective limit** *of* $\{\Omega_j, \pi_{ij}, J\}$ *is the subset of the cartesian product* $\prod_{j \in J} \Omega_j$ *given by all its wires, this space is indicated by*

$$\Omega \equiv \varprojlim_{j \in J} \Omega_j \, .$$

The maps

$$\begin{array}{rccl} \pi_j : & \Omega & \longrightarrow & \Omega_j \\ & \{\omega_i\}_{i \in J} & \mapsto & \pi_j(\{\omega_i\}_{i \in J}) := \omega_j \end{array}$$

are called the projections of Ω*.*

The projective limit Ω carries a natural topology, called **initial topology**, which is the smallest topology w.r.t. the projections π_j of Ω are continuous.

A base of this topology is given by the sets $\prod_{j \in J} U_j$, where $U_j \in \Omega_j$ is an open set such that $U_j = \Omega_j \ \forall j \in J$ but for a finite number of indexes.

In the initial topology *the projections are open maps* and *the projective limit is closed.*

It is easy to proof that if I is a **cofinal** subset of J, i.e. $\forall j \in J \ \exists i \in I$ such that $j \leq i$, then

$$\varprojlim_{j \in J} \Omega_j = \varprojlim_{i \in I} \Omega_i \, .$$

Furthermore, *if the spaces Ω_j are all compact then the projective limit Ω is a non-empty compact Hausdorff space.*

The most important class of functions associated to the projective limit of topological spaces is the class of the **cylindrical functions**.

Def. *The space $Cyl(\Omega)$ of the cylindrical functions on the projective limit Ω of the family $\{\Omega_j, \pi_{ij}, J\}$ is the quotient of the disjoint union $\coprod_{j\in J}\mathcal{C}(\Omega_j)$ modulo the equivalence relation defined by: $f \in \mathcal{C}(\Omega_j)$, $g \in \mathcal{C}(\Omega_{j'})$, $f \sim g$ if there exists an index j'' such that $\pi_{jj''}(f) = \pi_{j'j''}(g)$.*

Note that, in particular, the cylindrical functions are continuous, by converse it can be easily proved that *a continuous function f on Ω is cylindrical if and only if there exists a function $f_j \in \mathcal{C}(\Omega_j)$ such that $f = f_j \circ \pi_j$*, if this is the case then f is said to be cylindrical w.r.t. the index j and one writes $f \in Cyl_j(\Omega)$. Obviously

$$Cyl(\Omega) = \coprod_{j\in J} Cyl_j(\Omega)\,.$$

The map

$$\begin{array}{rccc} i: & Cyl(\Omega) & \longrightarrow & \mathcal{C}(\Omega) \\ & f_j & \mapsto & i(f_j) := f_j \circ \pi_j \end{array}$$

is an injective homomorphism which embeds $Cyl(\Omega)$ in $\mathcal{C}(\Omega)$.

The final result I want to cite about projective limits is the celebrated A.Weil's theorem which says that every compact group is the projective limit of compact Lie groups.

The dual construction of the projective limit is the inductive limit. For the later purposes it is worth introducing the definition of inductive limit directly on C^*-algebras, the same definition extends to general linear spaces and algebras.

Def. *An* **inductive family** *of C^*-algebras is a triple $\{\mathfrak{A}_\alpha, i_{\beta\alpha}, A\}$ where $\mathfrak{A}_\alpha$ are C^*-algebras and A is a directed set of indexes such that, for every $\alpha \le \beta$, there exist continuous injective inclusions $i_{\beta\alpha} : \mathfrak{A}_\alpha \to \mathfrak{A}_\beta$ satisfying:*

1. $i_{\alpha\alpha} = id_{\mathfrak{A}_\alpha}$;
2. $i_{\gamma\beta} \circ i_{\beta\alpha} = i_{\gamma\alpha}$, *whenever $\alpha \le \beta \le \gamma$.*

The **inductive limit** *of $\{\mathfrak{A}_\alpha, i_{\beta\alpha}, A\}$ is, set-theoretically, the quotient of the disjoint union $\coprod_{\alpha\in A}\mathfrak{A}_\alpha$ modulo the following equivalence relation: $a \in \mathfrak{A}_\alpha$, $b \in \mathfrak{A}_\beta$, $a \sim b$ if there exists $\gamma \ge \alpha, \beta$ such that $i_{\gamma\alpha}(a) = i_{\gamma\beta}(b)$.*

The symbol used to represent the inductive limit is

$$\mathfrak{A} \equiv \varinjlim_{\alpha\in A}\mathfrak{A}_\alpha\,.$$

The canonical inclusion of $\mathfrak{A}_\alpha$, α fixed in A, in the disjoint union defines, by quotient, the inclusion map in the inductive limit $\mathfrak{A}$, $i_\alpha : \mathfrak{A}_\alpha \hookrightarrow \mathfrak{A}$, which satisfies $i_\beta \circ i_{\beta\alpha} = i_\alpha$ for every $\alpha \leq \beta$.

To endow $\mathfrak{A}$ with an algebraic structure it is necessary to use the following lemma.

Lemma *Let $\{\mathfrak{A}_\alpha, i_{\beta\alpha}, A\}$ be an inductive family of C^*-algebras with inductive limit $\mathfrak{A}$. Then, fixed n elements $\{a_1, \ldots, a_n\} \subset \mathfrak{A}$, there exist an index β and n elements $\{b_1, \ldots, b_n\} \subset \mathfrak{A}_\beta$ such that*

$$a_i = i_\beta(b_i) \qquad i = 1, \ldots, n.$$

Thanks to the previous lemma one can define the $*$-algebraic structure of $\mathfrak{A}$ using that of the $*$-algebras appearing in the family:

$$\begin{cases} \lambda a := i_\beta(\lambda b) \\ a_1 + a_2 := i_\beta(b_1 + b_2) \\ a_1 a_2 := i_\beta(b_1 b_2) \\ a^* := i_\beta(b^*) \end{cases}$$

where $\lambda \in \mathbb{C}$, $a, a_1, a_2 \in \mathfrak{A}$ and $b, b_1, b_2 \in \mathfrak{A}_\beta$ satisfy $i_\beta(b) = a$, $i_\beta(b_1) = a_1$ and $i_\beta(b_2) = a_2$.

By endowing $\mathfrak{A}$ of the finest locally convex topology which makes the homomorphisms i_α continuous, called **final topology**, $\mathfrak{A}$ becomes a topological $*$-algebra.

It is essential to observe that **an inductive family of C^*-algebras always induces a projective family**, in fact if $\{\mathfrak{A}_\alpha, i_{\beta\alpha}, A\}$ is such a family then a projective family is obtained by associating to every $\mathfrak{A}_\alpha$ its spectrum $\sigma(\mathfrak{A}_\alpha)$ and to every inclusion $i_{\beta\alpha}$, $\alpha \leq \beta$, the restriction of its transposed map to the spectrum of $\mathfrak{A}_\beta$, $\pi_{\alpha\beta} \equiv {}^t i_{\beta\alpha}|_{\sigma(\mathfrak{A}_\beta)}$, where:

$$\begin{array}{rccc} {}^t i_{\beta\alpha} : & {\mathfrak{A}_\beta}^* & \longrightarrow & {\mathfrak{A}_\alpha}^* \\ & \varphi & \mapsto & {}^t i_{\beta\alpha}(\varphi), \end{array}$$

is defined in the usual way, i.e. $({}^t i_{\beta\alpha}(\varphi))(f) := \varphi(i_{\beta\alpha}(f))$.

It is easy to verify that the family $\{\sigma(\mathfrak{A}_\alpha), \pi_{\alpha\beta}, A\}$ is a well defined projective family.

If the $\mathfrak{A}_\alpha$ are also unital and Abelian then the spectra $\sigma(\mathfrak{A}_\alpha)$ are compact Hausdorff spaces, hence the projective limit $\varprojlim_{\alpha \in A} \mathfrak{A}_\alpha$ is a non-void compact Hausdorff space.

The most remarkable fact about this family, which will be used in the next section to obtain the projective characterization of $\overline{\mathcal{A}/\mathcal{G}}$, is expressed by the following result.

Theorem (•) *Let* $\{\mathfrak{A}_\alpha, i_{\beta\alpha}, A\}$ *be an inductive family of Abelian* C^**-algebras with unit. Then its inductive limit* $\mathfrak{A}$ *is an Abelian topological algebra with unit (in the final topology) whose spectrum* $\sigma(\mathfrak{A})$ *is a compact Hausdorff space homeomorphic to the projective limit of* $\{\sigma(\mathfrak{A}_\alpha), \pi_{\beta\alpha}, A\}$*:*

$$\mathfrak{A} = \varinjlim_{\alpha\in A} \mathfrak{A}_\alpha \quad \Rightarrow \quad \sigma(\mathfrak{A}) \simeq \varprojlim_{\alpha\in A} \sigma(\mathfrak{A}_\alpha)\,.$$

Now that the basic facts about projective and inductive limits has been remembered, the construction of the projective family which characterize $\overline{\mathcal{A}/\mathcal{G}}$ as projective limit can be implemented.

The most important concept is that of 'graph' in M, this needs the concepts of edge and vertex, which are introduced below.

Def. *An* **edge** *in* M *is a continuous map* $e : [0,1] \to M$ *such that its restriction* $\tilde{e} \equiv e|_{(0,1)}$ *is an analytic embedding*[3] *of* $(0,1)$ *in* M*.*

The **vertexes** *of an edge are its starting and ending point, that is* $e(0)$ *and* $e(1)$*, also called* **source** *and* **target***, respectively.*

A **graph** *in* M *is the union of a finite family of images of edges intersecting only in their vertexes.*

The usual symbol for a graph is Γ; the number of edges and vertexes of Γ will be indicated by E_Γ and V_Γ, respectively.

A simple (but significant) example of graph in M is the image of a piecewise analytic graph γ in M.

To construct the projective family which induces $\overline{\mathcal{A}/\mathcal{G}}$, first of all fix the directed set of indexes to be the set of all graphs Γ in M ordered w.r.t. the natural inclusion and denote it by L. This set is directed because if Γ and Γ' belong to L then also $\Gamma \cup \Gamma'$ belongs to L and $\Gamma \le \Gamma \cup \Gamma'$, $\Gamma' \le \Gamma \cup \Gamma'$.

Now the idea is to use this directed set to construct an inductive family of C^*-algebras whose inductive limit is dense in the holonomy C^*-algebra, then, by using theorem (•), the desired result will be reached.

To every graph Γ associate the unital Abelian C^*-algebra $A(\Gamma)$ generated by the Wilson functions T_α such that $\alpha^* \subset \Gamma$.

It is obvious that if $f \in A(\Gamma)$ then $f \in A(\Gamma')$ for every $\Gamma' \ge \Gamma$ so that the

[3]This means that $\tilde{e}$ is analytic and injective, with injective tangent map and $\tilde{e}^*$ is a sub-manifold of M w.r.t. the topology induced by M.

inclusions $i_{\Gamma'\Gamma}$ are naturally defined by:

$$\begin{array}{rccl} i_{\Gamma'\Gamma}: & A(\Gamma) & \hookrightarrow & A(\Gamma') \\ & f & \mapsto & i_{\Gamma'\Gamma}(f) := f\,. \end{array}$$

These inclusions satisfy the consistency relations, hence $\{A(\Gamma), i_{\Gamma'\Gamma}, L\}$ is an inductive family of unital Abelian C^*-algebras whose inductive limit is continuously included in $Hol(M,G)$.

By comparing the definition of inductive limit of the $A(\Gamma) \subset \mathcal{C}(\mathcal{A}/\mathcal{G})$ with the definition of the algebra of the cylindrical functions on $\mathcal{A}/\mathcal{G}$ one immediately recognizes that the two algebras agree:

$$\varinjlim_{\Gamma\in L} A(\Gamma) = Cyl(\mathcal{A}/\mathcal{G})\,.$$

Observe now that the polynomial algebra $\mathcal{W}$ generated by the Wilson functions is contained in $Cyl(\mathcal{A}/\mathcal{G})$ hence:

$$\overline{Cyl(\mathcal{A}/\mathcal{G})} = Hol(M,G)\,.$$

If $\sigma(\Gamma)$ denotes the (compact, Hausdorff) spectrum of $A(\Gamma)$, then the theorem ($\bullet$) implies that

$$\varprojlim_{\Gamma\in L} \sigma(\Gamma) = \sigma(Cyl(\mathcal{A}/\mathcal{G}))$$

where the projective limit is referred to the family $\{\sigma(\Gamma), \pi_{\Gamma\Gamma'}, L\}$, with $\pi_{\Gamma\Gamma'} := {}^t i_{\Gamma'\Gamma}|_{\sigma(\Gamma')}$.

Finally the theorem of bounded extension of bounded functionals implies that $\sigma(Cyl(\mathcal{A}/\mathcal{G})) = \sigma(Hol(M,G))$, hence one the following result holds.

Projective characterization of $\overline{\mathcal{A}/\mathcal{G}}$. *The spectrum of $Hol(M,G)$ is isomorphic to the projective limit of the family $\{\sigma(\Gamma), \pi_{\Gamma\Gamma'}, L\}$:*

$$\overline{\mathcal{A}/\mathcal{G}} \simeq \varprojlim_{\Gamma\in L} \sigma(\Gamma)\,.$$

Pictorially, the duality between the inductive family of C^*-algebras $A(\Gamma)$ and the projective family of their spectra $\sigma(\Gamma)$ can be represented as follows:

$$\ldots \subseteq A(\Gamma) \subseteq \ldots \subseteq A(\Gamma') \subseteq \ldots \longrightarrow \varinjlim_{\Gamma\in L} A(\Gamma) \equiv Cyl(\mathcal{A}/\mathcal{G})$$

$$\ldots \supseteq \sigma(\Gamma) \supseteq \ldots \supseteq \sigma(\Gamma') \supseteq \ldots \longleftarrow \varprojlim_{\Gamma\in L} \sigma(\Gamma) \equiv \overline{\mathcal{A}/\mathcal{G}}\,.$$

The spectra $\sigma(\Gamma)$ can be explicitly characterized by means of the following isomorphisms:

$$\sigma(\Gamma) \simeq Hom(L_\star(\Gamma), G)/Ad \simeq G^{n_\Gamma}/Ad\,.$$

where $L_\star(\Gamma)$ is the subgroup of $L_\star(M)$ given by the loops α such that $\alpha^* \subset \Gamma$ and n_Γ is the **connectivity** of Γ, i.e. the integer:

$$n_\Gamma = E_\Gamma - V_\Gamma + 1.$$

n_Γ *is a topological invariant* of the graph Γ which represents the highest number of edges that can be deleted from the graph without it fails to be connected.

To prove this result one has to use some topological tools (such as the properties of the first homotopy group associated to a graph and the Siefer-Van Kampen theorem) and also the results about the independent loops.

6. DIFFEOMORPHISM AND GAUGE INVARIANT CYLINDRICAL MEASURES ON THE SPECTRUM OF THE HOLONOMY C^*-ALGEBRA

Now the constructions of diffeomorphism and gauge invariant measures on $\overline{\mathcal{A}/\mathcal{G}}$ can be implemented.

First of all, if J is a directed set and there is a family of probability spaces $\{\Omega_j\}(j \in J)$ which has measurable projections $\pi_{jj'}$, defined for every $j \leq j'$ and satisfying the axioms of a projective family, then the triple $\{\Omega_j, \pi_{jj'}, J\}$ is said to be a **projective family of probability spaces**.

Suppose now to have a measure μ on the projective limit Ω of this family, then the push-forward of μ via the canonical projection $\pi_j : \Omega \to \Omega_j$, i.e. $\mu_j := {\pi_j}_* \mu \equiv \mu \circ \pi_j$, is a measure on Ω_j, for every $j \in J$.

Furthermore the family of measures $\{\mu_j\}(j \in J)$ satisfies the **consistency condition**

$$\mu_j = (\pi_{jj'})_* \mu_{j'} = \mu_{j'}|_{\Omega j} \circ \pi_{jj'}$$

which guaranties that there is no ambiguity when a portion of Ω_j is measured directly by μ_j or by the restriction of $\mu_{j'}$ to Ω_j.

A family of measures $\{\mu_j\}(j \in J)$ satisfying the consistency condition is said to be a **promeasure**.

A classical problem of measure theory is to study when it is possible to construct a measure μ on Ω starting from a promeasure, i.e. when it is possible to obtain a representation theorem for measures on projective limits, since the inverse process is always possible, as just discussed.

Luckily, when the probability spaces are compact the extension of a promeasure to a measure on the projective limit is always possible.

To simplify the notation a regular Borel probability measure will be simply called "probability measure".

Theorem. *Let $\{\Omega_j, \pi_{jj'}, J\}$ be a projective family of compact Hausdorff spaces with projective limit Ω.*

Then there is a bijective correspondence between probability measures on Ω and probability promeasures $\{\mu_j\}(j \in J)$.

The proof of the theorem uses the fact that, when Ω is compact, the Stone-Weierstrass theorem implies that $\overline{Cyl(\Omega)} = \mathcal{C}(\Omega)$.

This result can be specialized to the projective family of the compact Hausdorff spaces $\{\sigma(\Gamma)\}(\Gamma \in L)$, which gives rise to the the compact Hausdorff space $\overline{\mathcal{A}/\mathcal{G}}$, to obtain the following important result.

Theorem. *There is a bijection between the probability measures on $\overline{\mathcal{A}/\mathcal{G}}$ and the probability promeasures $\{\mu_\Gamma\}(\Gamma \in L)$ on the spectra $\sigma(\Gamma)$.*

Thanks to the characterization $\sigma(\Gamma) \simeq G^{n_\Gamma}/Ad$ an explicit (and natural) promeasure which gives rise to a probability measure on $\overline{\mathcal{A}/\mathcal{G}}$ is given by the family of the normalized Haar measures dg^{n_Γ} on the groups G^{n_Γ}, which are Ad-invariant (thanks to the assumption of compactness for G) and thus passes unaffected to the quotient G^{n_Γ}/Ad.

The probability measure obtained from the promeasure $\{dg^{n_\Gamma}\}(\Gamma \in L)$ is called the **uniform measure** on $\overline{\mathcal{A}/\mathcal{G}}$ and denoted by μ_0.

If a function $f \in Hol(M, G) \simeq \mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$ is cylindrical w.r.t. the index-graph Γ, i.e. it exists $f_\Gamma \in \mathcal{C}(G^{n_\Gamma}/Ad)$ such that $f = f_\Gamma \circ \pi_\Gamma$, then its explicit integral w.r.t. the uniform measure is given by:

$$\int_{\overline{\mathcal{A}/\mathcal{G}}} f(\bar{A})\, d\mu_0(\bar{A}) = \int_{G^{n_\Gamma}} f_\Gamma(g_1, \ldots, g_{n_\Gamma})\, dg^{n_\Gamma}(g_1, \ldots, g_{n_\Gamma})\,.$$

Thanks to the density of $Cyl(\overline{\mathcal{A}/\mathcal{G}})$ in $\mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$, the formula above extends (by uniform limit) to all the functions of $\mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$.

The most important properties of the uniform measure are the following:

1. μ_0 is gauge-invariant: this follows from the bi-invariance of the Haar measure on compact groups;

2. μ_0 is invariant under diffeomorphisms: this follows from the fact that the only possible dependence of μ_0 on the diffeomorphisms of M is contained in the connectivity n_Γ, but this is a topological invariant and so it is unaffected by them (more rigorously it can be proved that the so-called 'covariance condition' is satisfied);

3. μ_0 is faithful, i.e. $f \in \mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$, $f \geq 0$ and $\int_{\overline{\mathcal{A}/\mathcal{G}}} f\, d\mu_0 = 0$ implies $f \equiv \mathbf{0}$;

4. μ_0 is concentrated on the generalized connections, i.e $\mu_0(\mathcal{A}/\mathcal{G}) = 0$.

Beside the uniform measure there are also many other gauge and diffeomorphism invariant measures on $\overline{\mathcal{A}/\mathcal{G}}$, they can be constructed starting from G-valued random variables using a procedure implemented by Baez which contemplates the uniform measure as a particular case.

7. THE QUANTIZATION OF THE HOLONOMY ALGEBRA AND THE ALGORITHM OF THE LOOP QUANTIZATION IN DETAIL

What usually happens in the quantization of gauge theories is that on the classical configuration space, denoted generically with X, there is a cylindrical but not σ-additive measure μ which enables to construct the pre-Hilbert space $L^2_{cyl}(X,\mu)$ of the square-integrable cylindrical functions on X; if μ admits an extension to a Borel measure $\bar{\mu}$ on X then the completion of $L^2_{cyl}(X,\mu)$ leads to the Hilbert space $L^2(X,\bar{\mu})$.

However, if this extension is not available, the quantum theory is implemented by extending (on the base of physical and/or mathematical considerations) the classical configuration space X to a wider space $\bar{X}$ on which a genuine measure ν is available, in order to construct the Hilbert space $L^2(\bar{X},\nu)$.

The space $\bar{X}$ is called **the quantum configuration space** and the Hilbert space $L^2(\bar{X},\nu)$ is taken to be **the space of the quantum kinematical states** of the theory.

This is precisely what happens in the loop quantization of gauge theories: the lack of a measure on $\mathcal{A}/\mathcal{G}$ leads to search an extension of this space, the major candidate to this role is $\overline{\mathcal{A}/\mathcal{G}}$ for the following reasons:

- first of all $\mathcal{A}/\mathcal{G}$ is injectively and densely embedded in $\overline{\mathcal{A}/\mathcal{G}}$, hence the classical theory is, in a sense, contained in the quantum theory;
- $\overline{\mathcal{A}/\mathcal{G}}$ is an infinite-dimensional compact Hausdorff space endowed with a *natural* probability measure, the uniform measure μ_0. Associated to this (faithful) measure there is one and only one faithful representation of the holonomy C^*-algebra $Hol(M,G)$ supported by the Hilbert space $L^2(\overline{\mathcal{A}/\mathcal{G}},\mu_0)$, i.e. the GNS representation:

$$\begin{array}{ccc} Hol(M,G) & \longrightarrow & \mathcal{B}(L^2(\overline{\mathcal{A}/\mathcal{G}},\mu_0)) \\ f & \mapsto & M_{\hat{f}} \end{array}$$

  $M_{\hat{f}}(\psi) := \hat{f}(\bar{A})\psi(\bar{A})$, $\forall \psi \in L^2(\overline{\mathcal{A}/\mathcal{G}},\mu_0)$, $\hat{f} \in \mathcal{C}(\overline{\mathcal{A}/\mathcal{G}}) \subset L^2(\overline{\mathcal{A}/\mathcal{G}},\mu_0)$ is the Gelfand transformed of f. Hence the elements of the holonomy C^*-algebra are promoted to bounded multiplication operators on the Hilbert space $L^2(\overline{\mathcal{A}/\mathcal{G}},\mu_0)$, they are bounded because the Wilson functions (which generate $Hol(M,G)$) are bounded and the Gelfand isomorphism is isometric. The real part of the Wilson functions are thus promoted to bounded self-adjoint operators on $L^2(\overline{\mathcal{A}/\mathcal{G}},\mu_0)$, i.e. observables in the quantum theory, this is way the GNS representation on $L^2(\overline{\mathcal{A}/\mathcal{G}},\mu_0)$ is called the **quantum representation of the**

holonomy C^*-algebra. It is worth noting that the real parts of the Wilson functions generate the same C^*-algebra thanks to the identity ${T_\alpha}^* = T_{\alpha^{-1}}$;

- it can be proved that $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$ supports even the quantum version of the strip-moments T_S conjugated to the Wilson functions, which are gauge-invariant linear combinations of the densitized triads $\tilde{E}^i_a$. These observables becomes self-adjoint derivative operators on $\overline{\mathcal{A}/\mathcal{G}}$ satisfying the canonical commutation rules with the operators $\hat{T}_\alpha$. The differential structure on $\overline{\mathcal{A}/\mathcal{G}}$ which allows the definition of the quantum version of the strip moments is induced again by the projective nature of this space. This structure is too wide to explain here and so I avoid also to write down the specifical form of the quantum version of the strip moments;

- while the previous are mathematically rigorous motivations for the choice of $\overline{\mathcal{A}/\mathcal{G}}$ as the quantum configuration space, there is a further heuristic motivation based on a physical intuition: in a lattice gauge theory, with lattice given by a graph Γ, the configuration space is G^{n_Γ}/Ad, hence, being $\overline{\mathcal{A}/\mathcal{G}}$ the projective limit of the family $\{G^{n_\Gamma}/Ad\}_\Gamma$, a gauge field theory which has $\overline{\mathcal{A}/\mathcal{G}}$ as quantum configuration space is suitable to be interpreted as the continuous limit of the lattice gauge theories corresponding to every fixed graph, which are approximated (or regularized) theories. The fact that a graph deformed by a diffeomorphism is again a graph, i.e. that the set of graphs is closed under diffeomorphisms, is an important property when the diffeomorphism invariance is taken into account. For the reasons discussed above, a graph Γ is interpreted in the formalism of the loop quantization as a *floating lattice in M*.

The compactification of the configuration space is not a characteristic feature of this procedure, but it often appears in the quantization of the systems with an infinite number of degrees of freedom, such as field theories. For example in the quantization of the scalar field in d-dimensions the classical configuration space, i.e. the Schwartz space $S(\mathbb{R}^d)$, is substituted by $S'(\mathbb{R}^d)$, the space of the tempered distributions on $\mathbb{R}^d$, in which it is densely embedded.

I stress that the compactification $\mathcal{A}/\mathcal{G} \hookrightarrow \overline{\mathcal{A}/\mathcal{G}}$ is highly non-trivial, since the uniform measure μ_0 restricted to $\mathcal{A}/\mathcal{G}$ is the null measure. This fact has put in evidence the important role of the generalized connections in the loop quantization in the same way as the quantum field theory has put in evidence the role of the operator-valued distributions.

The assumption of the Wilson functions as configuration observables solves already at the classical level the Gauss constraint generated by the invariance under gauge transformations.

If the gauge theory is also invariant under diffeomorphisms (as the general relativity in Ashtekar's formulation), then the constraints generated by this invariance are imposed at the quantum level by selecting a suitable subspace of the kinematical state space $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$ given by the states satisfying this constraint.

In the canonical quantization one operates the splitting of the space-time in space+time, hence there are two kind of constraints generated by the invariance under diffeomorphism: the constraint depending on the spatial part and that depending on the temporal evolution, called **Hamiltonian constraint** or Wheeler-De Witt equation.

While the Hamiltonian constraint is not yet well understood, the spatial diffeomorphism constraints can be solved passing implementing the so-called 'loop representation', which is described in the next section.

8. THE "LOOP REPRESENTATION" OF ROVELLI AND SMOLIN

The quantum theory which one reaches following the prescriptions of the algorithm described above is called the 'connection representation', because the states of this theory, i.e. the unit vectors of $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$, are functions of (generalized) connections.

In 1990 Rovelli and Smolin constructed an important instrument to pass from this quantum representation to another description in which the states are functions of loops and for this reason called '**loop representation**'.

The major advantage of this representation is that the diffeomorphism constraint admits explicit solutions in terms of 'knot invariant states'.

The map which takes a state $\psi \in L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$ in the connection representation into a state ℓ_ψ in the loop representation is the **loop transform**, whose expression is this:

$$\ell_\psi(\alpha) := \int_{\overline{\mathcal{A}/\mathcal{G}}} T_\alpha(\bar{A})\overline{\psi}(\bar{A})d\mu_0(\bar{A}).$$

If the measure $\psi d\mu_0$ is gauge and diffeomorphism invariant (and such measures exist, as shown above) then ℓ_ψ is a knot invariant, i.e. it assumes the same value on every loop β obtained by α by means of a diffeomorphism $\varphi \in Diff_0(M)$, in more explicit words: if $\beta = \varphi \circ \alpha$ then $\ell_\psi(\alpha) = \ell_\psi(\beta)$.

It can be proved that the knot invariant states solve the diffeomorphism constraint in the loop representation.

Since the expectation values of an observable $\mathcal{O}$ in the state $|\psi>$ (i.e. the measured mean value of the observable after many measurements conducted on the system prepared in the state $|\psi>$) is contained in the inner product $<\psi|\mathcal{O}\psi>$, the loop representation gives the same physical information of the connection representation if and only if the transform which connects the two representations, i.e. the loop transform, is a unitary operator.

If this is the case, then its range is the space of the quantum states in the loop representation.

The proof of the unitary character of the loop transform for non-Abelian gauge theories is still lacking, but it can be proved that, if the gauge group is $U(1)$ as in the electromagnetism, then the loop transform is a unitary operator between the Hilbert spaces $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$ and $L^2(\mathcal{H}_\star(M, U(1)), \mu_d)$, where μ_d is the discrete measure on the hoop group.

The unitary character of the loop transform in the Abelian case follows from the fact that in this case the loop transform is an inductive limit of Fourier-Plancherel transforms on n-dimensional tori.

The loop transform can serve as a useful instrument to define the 'momentum observables' in the quantum theory: in fact one can *postulate* them to be the self-adjoint operators unitary equivalent to the $\hat{T}_\alpha$ via the loop transform, in the same way the momentum operators in quantum mechanics are related to the position operators via unitary equivalence inducted by the Fourier transform.

9. SOME RECENT DEVELOPMENTS OF THE PROGRAM AND OPEN PROBLEMS

The loop quantization program is still incomplete at a dynamical level since the Hamiltonian constraint has not been solved yet, this is of course the most important open problem of this theory.

The most promising works in this direction are those related to the use of the Jones polynomials to solve this constraint.

Another important open problem is the rigorous proof of the fact that the loop transform is a unitary operator and the discovery of the characterization of its range, i.e. the quantum kinematical space of the loop representation.

There is an hope that the inductive construction of the loop transform can be extended to the non-Abelian case in order to obtain the solution of these problems.

Also, there is no common agreement on how the matter (spinorial) fields has to be introduced in the theory.

Even though there are still this foundational problems, in the program of loop quantization, in the last 90's there has been some interesting developments of the theory, at least at a heuristic level. The perhaps more surprising and interesting are the following:

1. the construction of volume and area operators with discrete spectrum, which would indicate the outstanding result of a discretization of the spacetime structure at the Planck scale;

2. the treatment of the Hawking-Bekenstein black-holes entropy in a more systematic way then in the perturbative treatment of this problem in quantum gravity.