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CARLO DAPUETO and LAURA PARENTI

Contributions and Obstacles of Contexts in the Development of Mathematical Knowledge

ABSTRACT. A framework for discussing (through the concepts of "model" and "field of experience") the nature of the relationships between contexts and the formation of mathematical knowledge is proposed, and some educational, cultural and cognitive problems of situated teaching-learning are tackled within it. This framework is supported and illustrated with references to the curricular innovation and educational research carried out by the Genoa group, of which the authors are members.

Over the past two decades, the failure of "new mathematics" has contributed, together with other factors, to the development of a "movement", grounded in theory and practice, which has focused renewed attention (in the planning of mathematics curricula and in the study of concept formation) on the uses of mathematics and its out-of-school applications. The Genoa Group for Research in Mathematics Education, has taken part in this movement.

In Boero et al. (1995) we and other researchers of the Genoa Group explored the question of teaching in/with contexts as part of the relationship between "mathematics" and "culture". In this paper we take the viewpoint of curricular research, and explore issues in epistemology and cognitive psychology which we regard as being intertwined with it:

- forms of connection between mathematics and contexts,
- normative and descriptive aspects of mathematization,
- risk of predominance of mathematical point of view,
- control over shifts between different layers of meaning, and other problems.

We will propose a framework based on the epistemological concept of *model* and the didactical one of *field of experience* in order to tackle these issues, which we will illustrate with some brief examples based on the projects Genoa Group has carried out at various school levels.

1. TEACHING IN CONTEXT: GENERAL ISSUES

1.1. *Introduction*

When speaking about "teaching/learning in context" (also known as "situated teaching/learning") referred to a given set of disciplinary concepts, the word "context" is used to indicate a situation or activity in which these concepts are *introduced* and applied in a *meaningful* manner (that is, meaningful as regards to the situation or activity itself). This way of teaching contrasts with other ones where the separate concepts are introduced in an abstract exposition, while their combination and application to situations are postponed to example phases, to the teaching of other subjects, or to post-scholastic experience.

To clear the epistemological and didactical problems of the sense to attribute to the adjective "meaningful" we shall use two conceptual tools: model and field of experience.

1.2. *Models ...*

Modelling is usually a form of metaphor (Boyd & Kuhn, 1979), i.e. a shift between semantic spaces with a non-empty intersection. It can involve abstraction or idealization. In the case of logic, model means particularisation of a theory (shift from the semantic space of "syntactical" objects to that of mathematical structures); for instance, the structure of whole numbers is a model of the theory of groups. Both meanings of "model" (abstraction-idealization and particularisation-exemplification) exist even in common language.

We shall concentrate on the former use, that is the *model* as a simplified representation of a portion of reality (of interest in a certain problem) used for: improving the "visualisation" of some aspects (using a scale reproduction instead of the original diagram), generalising properties (grammar rules), allowing comparisons (comparing different regions using population density instead of population and area), etc.. From this point of view the theory of groups is a model for the structure of whole numbers. The model concept can be represented as in the *diagram* in Fig. 1, which makes the representations generally used in the literature on mathematical modelling (e.g. see Gilchrist, 1984) more detailed.

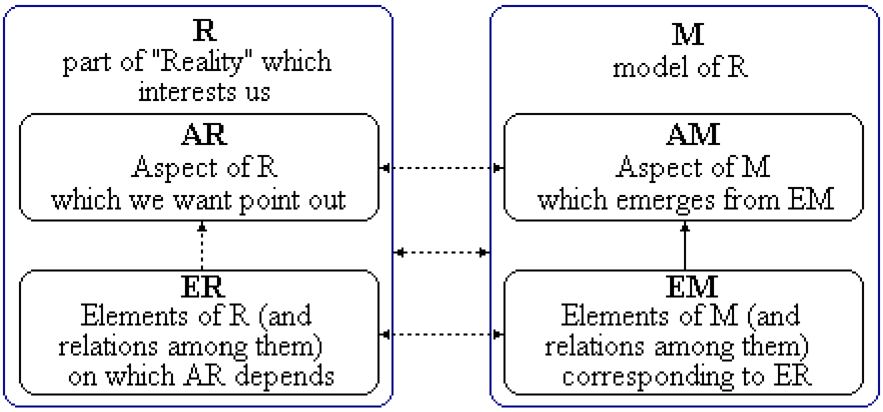

Fig. 1

For example, let R be a quadrangular field that we wish to evaluate economically and AR the extent of that field. If we note that the field is more or less rectangular, we can consider the lengths of two consecutive

sides as the factors ER that determine the extent. The arrow ER $\rightarrow$ AR representing this dependence is dotted, as we overlook other factors: protuberances and depressions, the fact that the shape is only roughly rectangular, etc..
Let us associate the lengths ER to two numbers, *a* and *b*, which express the lengths in a particular unit *U*; ER $\leftrightarrow$ EM is also dotted as the measurements are not exact.
Let us associate to AR $a \cdot b$; this is AM. The arrow EM $\rightarrow$ AM is not dotted; the value of $a \cdot b$ depends exactly on *a* and *b*; AR $\leftrightarrow$ AM is dotted because it is affected by the simplifications and approximations used in the choice of ERs and the association of EMs.
Hence, the model M is the multiplication (or rather the formula $S=a \cdot b$, if *S* expresses the extent in squares whose side is *U*).
Let us take another example. Let R be the relationship that exists in the English language between how a noun is used for expressing one object and how it is used for expressing more than one, and AR the way the noun changes in appearance passing from one use to the other. We shall consider as the meaningful elements (ER) the final parts of the nouns and their changes. The model is made up of a collection of statements such as: «nouns ending in -ch, -s, -sh, -x form the plural by rewriting: *noun* $\rightarrow$ *noun*+"es"»; EMs are the final segments of the strings that constitute the nouns and the rewriting rules; AM is the formal procedure for string transformation obtained by applying the rules. This is another example of simplified representation: the "exceptions" (epoch $\rightarrow$ epochs, basis $\rightarrow$ bases, fish $\rightarrow$ fish, codex $\rightarrow$ codices, etc.) are overlooked. By considering only the strings, we have lost aspects related to pronunciation (if -ch is pronounced -k, -s is used instead of -es), the origin of nouns (foreign words follow different rules), changes in use, etc..

The preliminary phase of *modelling* involves focusing on essential aspects of the problem (the delimitation of AR, which applied mathematicians often call specification of the "prototype"; cf. Aris, 1978); then comes the choice of ERs and finally their association to EMs. Once the shift R $\rightarrow$ M has been completed, the model can be *processed* (the calculation of $a \cdot b$, the choice and application of rewriting rule, etc.) and it is possible to carry out a *reinterpretation* M $\rightarrow$ R of the resulting processing in R (the extent of the field, how to form the plural of a particular word, etc.), which must take account of the simplifications employed.

Different types of modelling are often considered simultaneously: if R represents weather conditions in locality X in the month Y of the year Z, we may consider R $\rightarrow$ M_1 and R $\rightarrow$ M_2, where M_1 is a graph of the daily maximum temperature related to the date and M_2 a distribution histogram of suitably classified daily temperatures. *More sophisticated* modelling may subsequently take place: when modelling the extent of the rectangular field, one can represent the lengths as indeterminacy intervals rather than as exact numbers.

The "model" *diagram* in Fig. 1 is itself a model. It summarises and illustrates only some aspects of the relationships between situations and their representation, and is not easy to apply to all cases (e.g. the distinction between AM and EM may be interpreted differently, in some cases the arrow EM $\rightarrow$ AM should be dotted, etc.).

Some apply the term "model" to the triple (M, R, {arrows}) (particularly the arrows from R to M), or to the triple (M,R,S), where S is the subject that constructs and/or interprets the arrows (for references, see Blum& Niss, 1991 and Norman, 1993). We have adopted the more common usage of applying it to the outcome of "modelling", i.e. the pair (EM, AM), a structure which can be associated to different pairs (ER, AR): $S=a \cdot b$ can be applied not only to the field of the previous example.

1.3. *... and Contexts*

1.3.1. *Reality*

Models are built by using material *artifacts* (objects, signs, etc., like squared paper, the computer,"$\rightarrow$", "·", histogram rectangles, colours, etc.) or mental artifacts (the concepts "noun", "multiplication", "frequency", etc.) which in turn are often models themselves (histogram rectangles to indicate amounts, "noun" to indicate particular words, etc.). We use the term artifacts after Norman (1993): this seems to us more expressive than other terms like "prosthesis tools" used by Bruner.

The *reality* R (see 1.2) is not necessarily a material object or a natural or social phenomenon. It may be a situation that is presented in a modelled form (e.g. a graph of a certain phenomenon), or a reality made up of a set of models ("being a group", "being increasing", "being singular", "being vectorial", etc. are models which have, like R, structures, functions, names, physical entities, etc.).

1.3.2. *$R \rightarrow M$ and $M \rightarrow R$*

The use of models as artifacts for the construction of other models and the further modelling of models (i.e. the existence of metalevels of representation) characterises higher, human forms of *reasoning*. In our mental activities we frequently pass, more or less consciously, from one level to another, not only in the sense of R $\rightarrow$ M (reasoning based on pure numbers to find the area of a surface, employing grammatical rules to orient oneself in linguistic behaviour, etc.) but also in the sense of M $\rightarrow$ R (considering multiplication as a model for calculating the area of a rectangle in order to remind oneself of its commutativity, thinking up an example in order to reconstruct a specific rule for forming the plural of nouns, etc.).

1.3.3. *Disciplines*

Currently, the processes of modelling, reasoning at different levels of abstractions and so on are for the most part organised into *disciplines* or refer to them; they are supplied with specialised languages and procedures that standardise the forms of models, their internal processing, the links between different types of models, etc.. Furthermore, there are socially shared forms of *knowledge* which are not "accredited" but developed and passed down at popular level, into crafts, etc. (with their own models: proverbs, procedures, etc.).

Essentially, disciplines differ in that they organise models that refer to different types of phenomena. *Mathematics* is an exception: currently, it is characterized not by its area of application (it is no longer the language of physics) but by the kind of artifacts it employs; basically, its organisation into sectors is linked to artifact type, not to application environments (even though there are some borderline sectors - mathematical physics, information theory, econometrics, etc. - characterised in part by relationships with certain disciplines or technologies, and there is logic, which has mathematical activity as its area of application).

In mathematical modelling (i.e. when artifacts are mathematical objects), the process ER $\rightarrow$ EM is called *mathematization.* But also the use of mathematical artifacts in building a physical, biological, … model (for instance a physical law or a bionic model) is a mathematization. In a modelling activity which includes mathematization, the selection of ERs and the choice of EMs can involve assumptions, intentions, perceptions, intuitions, etc. which *interlace* mathematics' and other disciplines' aspects. With regard to this interaction and the nature of R, our "working definitions" of modelling and mathematization differ from the ones of Blum& Niss (1991): their R and AR are a *real world* and a *real model* which are outside mathematics, and they consider only mathematical models, whereas we want take into account also the mathematization of mathematics (see 1.3.1; cf. Freudental, 1968) and the presence of interferences with other disciplines.

1.3.4. *Contexts*

In a modelling activity, R is the *context of AR*, i.e. the setting or the conditions in which the aspects we want consider are found or take place.

The *context of modelling* includes the *reasons* for the choice of AR and the *artifacts* that one has at his disposal and means to use to build M. In Blum & Niss (1991) this context is described as a problem to solve, but the usual "didactical" meaning of *problem solving* can run down the complexity of the interests of the various "solvers "involved in a real modelling activity.

Contexts can be internal to a discipline (see 1.3.1). For an example, the context R of derivative concept can be the rate of the change of a function, while the context of formulating this concept can be the need of a tool for building the differential equation concept.

1.4. *Situated teaching-learning*

Among *arguments* in favour of including real situations of use and modelling activities in mathematics instruction (cf. Blum & Niss, 1991, de Lange, 1996) we can distinguish the ones that emphasize how referring to reality offers opportunities for *forming* critical and socially and culturally aware people, and the ones that focus on its potentialities in promoting an effective and balanced *construction* of mathematical *concepts*. In this paper we are considering especially topics that are linked to the latter ones.

In *traditional* teaching *disciplinary concepts* are not presented as *models*, but only as things to study. In example phases, if present, there is only AR as context of AM; it is usually introduced verbally, and in most cases is (**1**) a make-believe context used as an alternative to symbols for setting out a stereotyped disciplinary problem, or a quiz-type question (as in puzzle-magazines). Rarely, especially at secondary level, AR is (**2**) explicitly presented as "educational" simplification of real-life problem; type (2) contexts are sometimes used also at the beginning, as a springboard for the abstract introduction of concepts. Case (1) presents merely a caricature of the phase ER $\rightarrow$ EM; case (2) involves the phase of choosing ERs, but the delimitation of AR (i.e. its context, R) is usually neglected.

When discussing *situated teaching*, by *context* we usually mean the context of the modelling activity (see 1.3.4) in which the concept is introduced (it is introduced as *M* or a model used as an *artifact* to build M).

By referring the introduction of concepts to meaningful contexts, situated teaching aims to develop students' skills in managing the interaction between the various linguistic and semantic levels through which knowledge is organised and applied: in particular, the significant aspect of teaching how to employ mathematics for solving problems does not lie so much in pupils' internal mastery of the symbols used to build the model of a situation, but rather in their understanding of or confidence with the process by which the *problem is transformed* into a (more) formalised situation and subsequently further transformed (through internal processing and/or transfer to other - more formalised or less formalised - situations that are easier to solve) until the solution becomes clear.

Both the choice of the contexts and their didactic management (teacher's interaction with the students, organisation of learning material, etc.) are important to this effect. A context can be presented in various ways: (partially) experienced, evoked by drawing on student experience, etc. or described verbally and graphically as it happens for examples in traditional teaching. What's important is to exploit the *potential cognitive resources* of working with a non-prepacked situation for a prolonged period. By reorganizing (and referring to mathematics learning) arguments from Norman (1993), we can say it is well suited for:

– • facilitating cohabitation, interaction and contribution of students who have different *styles* of exploration/understanding/use of concepts, different *levels* of formalised knowledge, etc.; • permitting the teacher to scale and differentiate the pace at which pupils move from work on ERs to work on EMs, thus giving *everyone* the chance to *follow* the main aspects of the activity; • offering the students a chance to *think back over* things when the work has given rise to new *motivations*;

– • helping more balanced development of *reflective learning* and *experiential learning* (hereafter, we use these two poles as a simplified model of human cognition in the sense of Norman, 1993; for an example, think of learning Latin and pre-school learning mother tongue); • focusing better on the role and process of formalization without placing undue importance (in activities, pupil evaluation, etc.) on the *harder*, more mechanical and less important aspects of mathematics (attention, patience, precision), the ones that school often stresses, to the detriment of *softer* aspects: creativity, critical sense, managing the shift from R to M, using R to reflect on M (building oneself situations of use; see 1.3.2);

– facilitating the design and restructuring of the *schemata* through which knowledge is organised (according to cognitive psychology's usage, by "schemata" we mean mental entities into which perceptual stimuli and internal representations are organised; cf. Bartelett, 1932; Scholnik, 1983).

1.5. *Fields of experience*

When the Genoa group was set up in the late seventies and tackled the context problem, it was pursuing a desire for school reform linked to the experience of its members, and drew on a varied background (study and reflection on the history of culture, skills in mathematical logic and applied mathematics which clearly revealed the cultural and technical shortcomings of "new mathematics", etc.). In teaching, it was natural for us to extend the definition of "mathematics" and "culture" beyond the knowledge that today's mathematicians and "learned men" possess to include the ways of thinking and reasoning by which this knowledge is developed and applied more or less explicitly in various human endeavours. Only at a later point was this outlook coupled with cognitive considerations (see 1.4).

"Educational theories" were approached in a non-rigid manner, considering them as "models" (for instance, we adopted some concepts developed by the French school of mathematics education - like "didactical contract" or "tool-object dialectics" - even though we did not share their reference to Piaget's constructivism; cf. Boero, 1993). When, to provide a framework for planning and describing the evolution of situated teaching in the classrooms, a specific "model" of our own was established, we chose a flexible one in order to avoid the risk of *tunnel vision* inherent in all modelling activities, particularly in cognitive research (cf. Norman, 1993, Ch.5): persisting in interpreting R on the grounds of a short-sighted or prejudicial selection of ERs, in trying to fit one's own model overlooking or disregarding other factors that would put the whole modelling process in doubt.

The established model is the *field of experience* concept (Boero et al., 1995), seen as a sector of human culture which the teacher and pupils (or similar characters involved in a learning situation) can recognise and consider as sufficiently unitary and homogeneous (from the standpoint of their schemata).

We can identify three main *components* through which references to a field of experience evolve in teaching/learning situations (we use "component" in the sense of *polarity* - by extension of statistics and physics uses - rather than in the sense of "part"): an "external" component made up of the portions of "reality" (objects, phenomena, information, documents, procedures, etc.) that are the subject of reflection, discussion and analysis; a component "internal to the pupil" constituted by his/her schemata, conceptions (including the unconscious ones), emotions, expectations, etc.; and a corresponding component "internal to the teacher", which also includes learning goals.

Fields of experience can be related to *contexts* in different ways.

The field of experience related to the way production costs form may be the context for a learning activity, but may also be recalled in an activity involving other contexts, e.g. activities concerning the limitations of mathematical modelling (under what conditions and up to what production levels can a linear model be used), or the set of linear functions (for recalling an application example, for consolidating the roles of two parameters, etc.).

Even in cases where the field of experience is the context of a learning activity, it does not necessarily need to be handled comprehensively in the classroom. It can be *evoked,* if, in terms of the pupil's schemata, it really is already a field of experience (so that the modes of learning examined in 1.4 can be activated).

A field of experience can also be gradually *built up* at school within work on other contexts (e.g. the field of experience of linear functions dealt with in activities related to economics, technology, etc.). This may then become a direct context for a learning activity (for introducing the definition of the linear function concept, studying the properties of the structure of linear functions, etc.).

1.6. *Examples of the use of fields of experience as tools to organize situated teaching*

We shall briefly illustrate through a few examples how the choice (and the construction) of fields of experience and contexts (internal and external) have been developed in our projects (see Boero, 1997, 1996 and Gruppo Didattico MaCoSa, 1997 for documentation about primary, lower secondary and upper secondary school projects, respectively).

At the beginning of primary school, *money* and *calendars* are already fields of experience for the children. Actual money transactions in school and the use of the calendar to take notes on classroom activity, weather conditions and other facts (to be gone over again later and analysed "statistically") are learning contexts that interact with pupils' *out-of-school experience,* enriching these fields and helping to build the field of experience of *numbers*.

The *shadows* phenomenon is a field of experience in which the child's point of view may be affected by various misconceptions. In primary and lower secondary school it is adopted as a context that is enriched and *rationalised* through new experiences, exchanges of opinion, experiments etc., as well as the development of suitable mathematical models. Work in this field of experience contributes to the construction of the *geometry* and *astronomy* fields of experience

Bodily development is a *problem* of great significance in *adolescence*, involving comparison with others and monitoring of progress. This field of experience is tackled in lower and upper secondary school as a learning context for the meaningful introduction of mathematical models that allow pupils to rationalise, depersonalise and socialise their problems. It also helps in the building of the *statistics* and *probability* fields of experience.

We have concentrated on wide-ranging contexts, presented to students in a manner that integrates verbal and graphical forms, references to life experience or non-scholastic knowledge (handling money, technology, observation of nature, etc.), and, in many cases, activities that deal directly with the situation itself. We choose wide-ranging contexts not only for the cognitive reasons we discussed in 1.4, but in order to avoid giving the idea that the mathematization of a phenomenal area is a mere summation of many (only) mathematical models and historically developed as a summation of problem solving activities (see 1.3.3, 1.3.4).

References to aspects of *daily life* are not reduced to the reproduction of realistic problem (such as "make a cake" or "plan a trip"): "bodily development" is a real matter which interests pupils but its mathematization is not a real life problem. The "*external*" *components* are related to the "*internal*" ones: pupils' needs/knowledge (adolescence problems, biological conceptions, mathematical background, etc.) and teacher's goals/knowledge (relating to pupils' problems, introducing to statistics, biology and statistics knowledge, etc.). In the mathematics education literature (in both examples and general considerations) this kind of interplay with students' life is usually left out of account (see for instance Burkhardt, 1994 and de Lange, 1996).

The various contexts, combined with "traditional" example-contexts (see 1.4) and practice activities (mixed in different proportions depending on the school level: from primary to secondary school we pass from an "interdisciplinary integrated approach" to a "mathematics curriculum integrated approach": see Blum & Niss, 1991), are arranged within pathways which take account of mathematical, cultural and motivational background and priorities. Contexts are referred not only to non-scholastic experience and knowledge (see 1.3.4).

The *fields of experience within mathematics* whose construction was touched upon above are gradually taken on as contexts. For example, as early as grade one of primary school, numbers constitute a subject of direct study. In upper secondary school, the entire field of experience of numbers becomes a context (reflection on the various uses of numbers in society, the concept of numerical structure, calculations with real numbers, etc.).

In upper secondary school the *mathematical model* concept as well as the entire field of experience of mathematics repeatedly become contexts for learning activities.

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2. SOME ASPECTS OF SITUATED TEACHING

2.1. *An example*

Having examined several general aspects of the rationale behind context-based teaching and the way it can be organised, we shall explore in greater detail a few issues concerning how to choose and deal with contexts. In order to introduce these issues we refer to an example: the *money* and *calendar* fields of experience (see 1.6), the first ones considered in our primary school project.

These fields are used as starting contexts for building the field of experience of arithmetic. Behind this choice there also lies the goal of balanced development and integration, from the outset of primary school, of: • ordinal, cardinal and vectorial aspects of numbers; • various calculation strategies; • lexicographic and polynomial aspects of writing numbers; • working with "big" and "small" numbers; • different uses of numbers (for sorting, labelling, measuring physical or economic entities both directly and indirectly, etc.); • the procedural and declarative meaning of "=" and its commutativity; • different representations of numbers and operations (strings, the number line, histograms, graphs, Venn diagrams, etc.) (cf. Dapueto et al., 1986).

These contexts, chosen for a *suitable introduction* of arithmetic, in turn suggested *re-examining/rearranging objectives* of teaching: they led to the introduction from grade one of primary school of negative numbers (temperature), numbers in the order of magnitude of thousands (in Italy an ice-cream costs over 1000 lire), histograms, and so on.

In the organisation of learning material and classroom activities, we (researchers and teachers involved in projects) took care the fields of experiences were handled paying special attention to substantial integration with *out-of-school* life, where mathematics occurs in both *normative* (incorporated) and *descriptive* forms. This attention has the purpose the activities foster effective formation of *object-prototypes* (the number line, histograms, etc.) and *process-prototypes* (strategies, transcriptions, etc. for transforming problems), which the pupils are able to reconstruct mentally to tackle other situations. See *2.2* and *2.3*.

According to our experience (see Boero et al., 1995 for references), the development of these prototypes can take place if the teacher is able to make explicit in the classroom the *mental strategies* that children use in out-of-school practices without untimely translation into abstract form or connection to "scholastic" strategies: • when adding 2 hundred lire and 3 hundred lire, the child has little difficulty in considering "2 hundred + 3 hundred" as "(2+3) hundred"; this distribution procedure in standard teaching is obscured, while in money context the teacher, with appropriate (linguistic, graphics, …) activities, can focus on and *transfer* it to numerical problems concerning days, minutes or degrees, in "calendar" field of experiences; • the child easily finds two days back starting from the 21st day, using an ordinal procedure which the teacher can gradually transpose on abstract numbers line and, then, indirectly, to money field of experience.

Consolidating prototypes related to the two fields and transition from a context form of a problem to abstract form and vice versa is useful for confronting the *conceptual obstacles* connected with reference to situations of use (interpreting 300·15 as "3 hundred lire times 15" may hinder its transformation into 15·3·100, interpreting 1/4 as "1 hour divided by 4" may hinder its transformation into 25 hundredths). See *2.4*.

Behind the choice of money and calendar fields lie also *historical considerations*: most basic knowledge in arithmetic was developed in the fields of measurement of time and commercial trading. Reference to historical development of mathematical number concept (considering the technical and foundational contexts of employing set-theory, for instance) was useful also in order to critically examine the didactic introduction of it.

2.2. *Mathematics incorporated into contexts or used for rationalising them*

When choosing a non-mathematical context in which a specific mathematical concept is to be introduced, consideration must be given to which aspects of the mathematical concept (**1**) are *incorporated* in the context and/or (**2**) have a meaningful *role* in understanding it. Mutatis mutandis, similar considerations can be made about mathematics contexts (see 1.3.4).

As far as (**1**) is concerned, we note mathematics is usually incorporated in everyday *objects* which are not merely "things" but include functions of use, knowledge, etc. (e.g. money, calendars, rulers, thermometers, scales, watches, speedometers, standard containers, diagrams, playing cards, calculators, radios, cameras, and so on); it is also incorporated in the *social relations* that are inherent in these objects (cf. Bishop, 1988). The teaching effectiveness of references to these objects depends heavily on two factors (as we saw for money and calendar contexts): the extent to which the aspects of function and social relations have been taken into account, and the links that the class-work maintains with *out-of-school modes of learning and knowledge transmission* (see the end of 1.4). For further investigation, problematic aspects and references see 2.4.2, Walkerdine (1988), Nunes (1992), Boero et al. (1995) and Fitz Simons et al. (1996).

Also in case (**2**), the effectiveness of references to contexts depends on the success in reproducing in the classroom forms of training and communication (collective discussion, mutual correction, etc.) common in out-of-school environments (work, play, sport, etc.; for further investigation and references see Norman, 1993), so that children (with teacher's help) can use, explicit and compare their own mental strategies (which are often richer and more directly linked to pure mathematical concepts than standard scholastic ones) and build *prototypes* that they can use as a support for abstract mathematical reasoning.

In *shadows* field of experience (see 1.6) mathematics has especially a rationalising, so a type (2), role. These didactic activities give the chance of building important prototypes for geometrical reasoning: *figures* are approached genetically (the half-line as a unidirectional trajectory, angles generated by a rotating half-line, etc.) and dynamically (a square that becomes a parallelogram, right-angled or not); this makes it possible to avoid (or overcome) misconceptions arising from interpretation of figures as limited parts of space and from the limitations and static nature of figures confined to the page (imagining two lines that do not intersect on the page is different from the visualisation of sun rays using a perforated rod), etc.. In addition, this field of experience permits an early, meaningful approach to *three-dimensional* geometry (not limited to the study of solid figures), brings to light the ideal nature of geometric *models*, and provides an opportunity for educating in *argumentation* and introducing *proof*: transferring reasoning from the situation to the geometric model; thinking up experiments and engaging in hypothetical reasoning (What would happen if the sun were in this position?); reasoning beyond the bounds of experience (Is there a position from which a bulb projects a rectangle into a non-rectangular parallelogram?); making explicit and using the things we are now (or assume be) sure of as facts that form the basis of reasoning; etc..

This transition from context to abstraction, from individual ideas to shared knowledge, is successful if the teacher is able to motivate the pupils, help them to render their ideas explicit and understandable, support these ideas or put them in a state of crisis, relate them to the teacher's inner and the external components (make tasks and constraints clear, …),etc..

Connections between the components (external and internal ones) change according to the features of the field of experience. In "money", didactic activities especially concern the external component. In "shadows", they especially concern the relationship between the phenomenon (external component) and the pupils' conceptions (their internal components). In other fields the components have further ways of linking together.

In the primary school project numerous learning activities (drawing pathways, machinery, human activities, etc., linked to shadow field too) interact with fields of experience concerning *plane representation* of spatial situations and *vision*. In lower secondary school, these fields become an explicit learning context: the pupils' *critical analysis* of *their own* drawings gradually moves on to the first steps in mathematical modelling of perspective (and contributes to the construction of *proof* fields of experience, in a similar way as shadows field does). In pupils' works the external component (the subject and its representation) and the internal one (the observing and drawing skills evidenced by the representation) overlap; in these activities, rationalising refers not only to "reality", but also to own behaviour.

2.3. *Normative and descriptive aspects*

In situations where mathematics is incorporated, there is often overlap of so-called *normative* aspects (mathematics used for building the object) and *descriptive* aspects (mathematics for describing the object). For example, if we graph Italian railway fares in relation to mileage (see Fig. 2), the discovery that this step function can be approximated with a continuous piecewise linear function not only leads to concise modelling but also reveals the "logic" behind fare definition.

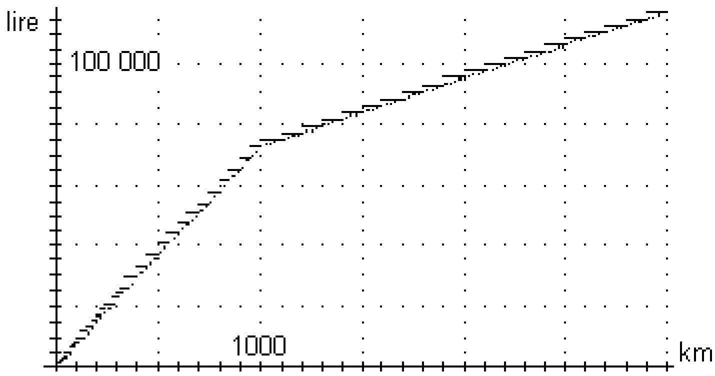

Figure 2

In money and calendar fields this overlapping is evident. It is present in the drawing field too. But this twofold aspect is especially present in the *automation* field of experience.

In our projects, *automation* (from programmed machines to programmable ones) is first outlined in the pocket-calculator field of experience and in activities within other contexts involving automated systems. Later on, it is featured in other learning activities both as a direct context and incidentally. The interaction between automation and *mathematical* fields of experience is *special* and *multifaceted*: • it provides tools (which have now become basic) for the study of mathematics; • it is an out-of-school context that draws heavily on mathematics in a meaningful way; • the use and mastery of automatic processing systems calls for special mathematical languages and models, and so on (cf. Blum & Niss, 1991).

In learning activities, the external component of this field of experience often overlaps (in a different way by comparison with previous fields) with students' and teacher's internal ones: suggestions and elaborations made by students and teacher can develop integrated with information, forms of knowledge, etc. that are incorporated into computer files and software; the pupils can work autonomously, conduct self-evaluation and work at their own pace on the basis of the computer's response to their behaviour; etc.. In particular, in order to carry out the didactical potentialities of computer's use, links with *out-of-school* modes of using (in work or play) must be maintained: the natural combination of reflective and experiential aspects the computer allows, the shift in the boundary between hard and soft skills, etc.: see 1.4.

The relation between normative and descriptive aspects in this field of experience provides a context for learning activities of various levels: • interpreting *unexpected outcomes* (seeking to understand the underlying mathematics and then mathematically modelling the behaviour of a calculator or computer program that gives an unexpected value for a numerical term, or the behaviour of Derive when it makes a mistake in solving or graphically representing an equation, etc.); • learning to use *paint* or *draw* applications (What is a figure within these applications? When ruler and compass are abandoned in favour of computer graphics, what are the new basic "geometric" entities?); • exploring/describing the working logic of *black boxes*; • exploring/studying *artificial languages* to communicate commands and procedures to computer; • pointing out the *role of mathematical modelling* in creating automated systems, simulations, etc.; up to • focusing on formal and historically determined nature of mathematical *proofs* (which cannot be identified with absolute ideas of reasoning, truth, etc. but depends on *a* definition of *rule* concept; cf. Davis, 1978).

2.4. *Some difficulties inherent in situated teaching*

Two of the most common objections to situated teaching of a discipline are that (1) reference to real contexts involves the use of models from *other disciplines* and the consequent presence of interferences of goals, conceptions and languages which are difficult to manage, and (2) reference to particularsituations of use may inhibit the shift to abstract reasoning, misdirect it or hinder the *transfer* of abstract reasoning to other contexts. These objections in some cases are a priori, in others correspond to teachers' real difficulties.

2.4.1. *The predominance of mathematical aspects*

When presented with concepts inside a modelling activity, pupils may face *additional difficulties* in relation to the knowledge of the context. At the same time, teachers may need extra knowledge and skills in order to fully exploit the learning potential that the context offer (see 2.2). It is not easy to draw a line between mathematical and non-mathematical work, nor to strike the right balance (between artistic and geometrical aspects in drawing field, biological and statistical ones in the bodily development field, physical and geometrical ones in the shadow field, etc.). There is a risk of concentrating too much on the introduction of mathematical models and *neglecting* the specificity of the viewpoints, models, languages, etc. of *other disciplines* (or knowledge domains), running into possible cultural errors. It is important to allow for mental interference with concept construction in other disciplines and with the pupils' related conceptions and misconceptions.

Depending on how closely the choice of ERs and the ER $\rightarrow$ EM association are linked to mastery of R, a *mathematical model* may be either *conceptual* or *formal*. Conceptual models are based on careful identification of AR factors and of the relations between them, as it occurred in the mathematical modelling of classical physics. Formal (analogical or empirical, for some authors) models are those in which we seek to define M so that there is a behaviour analogy (an "isomorphism" according to Bertalanffy, 1968) between AM and AR, omitting to study (by reasoning about R) in what sense the ERs are factors of AR, as it occurred in modelling heartbeat as relaxation-oscillations (cf. Israel, 1996).

Teaching often trivialises the modelling process, concentrating on a "formal" approach even in cases where a "conceptual" approach would be more meaningful, for example when the role of mathematics in physics is reduced to graphic representation of data or to the search for an approximation function. In other occasions, the application of certain models is extended groundlessly, for example when the sum of vectors is introduced for consecutive displacements and then also used, without explanations, for simultaneous displacements.

Intertwining and confusion with *physics* inevitably occur in (situated or not) mathematics teaching: mathematics only became a discipline in its own right in the late nineteenth century, an evolution that, emblematically, corresponds with the transition from (Euler's) concept of function as a graph of a physical phenomenon to the first mathematical definition of function, as "a set of pairs such that…".

This is a fruitful combination in educational terms but can also create serious conceptual confusion if differences are not clearly focused at some point. Consider, for example, the subtle presence of the *time* variable in many geometric activities (when dealing with movements, defining or describing curves, etc.), as well as the presence of "physical" reasoning in activities on euclidean geometry (using point as particle or centroid as barycentre, using movements without an axiomatic or analytical definition,…).

A "dual" aspect (with respect to 2.3 and this section) is that of interference with (or separation from) the ways in which mathematical concepts and techniques are *used* or *introduced* in other disciplines. This area has not been the subject of in-depth educational research yet (it is worth mentioning Pollak, 1976); we started confronting this issue at secondary school level.

2.4.2. *Transfer*

In this section we argue (with references to previous sections) that the transfer problem involves not only situated teaching but all forms of teaching which are not limited to experiential development of certain hard skills, and that teaching practice which explicitly deals with interference between different types and levels of knowledge seems to be better equipped to tackle it.

- In 1.4 (and 1.3.2) we observed *doing mathematics* (not only applying it) consists mainly in transforming problems from a language or an (internal or external) context into another. These transformations are of *various* kinds, also because the ways in which *mathematical concepts* formed are various.

On the one extreme many concepts do not emerge from generalisations but surface in response to one or just a few situations, corresponding to primordial experiences. Consider for example the structure of natural numbers (even a child can quickly grasp the abstract structure, $\bullet \rightarrow \bullet \rightarrow \dots$, drawing on the subjective sensation of time passing) and the concepts of function, order and soon. Only at a later point when the abstract structure has been perceived can analogies be drawn with other situations.

On the other extreme there are concepts which have not been conceived as refinements of intuitive ones: many concepts originated as models of situations that had already been transformed into mathematical form (the group concept, the polynomial function concept, etc.).

In between there are cases of a complex and discontinuous transition from intuitive or everyday concepts related to "common sense" (the line as a pen stroke, a piece of cord, etc.) to concepts related to a mathematical practice but not rigorously defined (the lines of descriptive geometry, or of Euclid's Elements), and to formally defined concepts (the line as an equation or system of equations, or defined vectorially, or defined implicitly by a system of axioms, etc.).

- One didactical aspect of these considerations is how to take into consideration the variety of forms of relation between mathematics and *situations of use*, at which we already hinted in 2.3 and 2.4.1.

Educating students to use concepts requires not so much tackling a lot of examples as dealing with some meaningful situations (*prototypes*: see 2.1 and 2.2) which point up the nature and the problems of the main kinds of transfer, enable students to grasp and experiment with shifts between different layers of meaning and can be internalized as emblematic *stories* (Schank, 1990): recollecting how the class dealt with a context may become a schema by which the pupil mentally arranges information, concepts, relationships between models and contexts (how to formalise, what the limits are of modelling, etc.), the strategies at play, the cognitive conflicts encountered, etc., as well as the feelings experienced. In order that this happen it is not enough to deal a lot of times with the same contexts: the apparent mastery of a kind of use can fade away shortly after, if teaching did not build *communication channels* between didactic activities and real situations of use (see 2.2) or other disciplines (see 2.4.1).

- Another aspect is the control (during the shifts) of *linguistic interference.*

Interference between sectorial languages and *natural language* underlies many pupils' difficulties (cf. Pimm, 1994). This is true particularly of scientific languages, whose ambiguous terminology evolved before the "model" idea had emerged (scientific concepts as models instead of descriptions) and specialised languages had come into common use. The same is true (though less justifiably) for scholastic jargon, like the algebraic jargon that uses "to move right", "to cancel", etc..

When using terms that are also part of everyday language, there is a risk of retaining meanings that have no place in the discipline concerned (in front of 2x=0 a pupil can transform it into x=-2 by a movement to right, if he/she believes a movement cannot make "2" vanish). Likewise, when we reason about a concept and picture it in a less abstract *context*, we *add "data"* that may prove misleading. We saw some examples in 2.1 and 2.2; here are some more. By materializing the triangle concept we are unable to represent it in both a non-isosceles and a non-scalene manner. When thinking of a continuous function F as a graphic representation, we actually include conditions on F (finite length, countable set of slopeless points,…), etc.. Even a mathematically-versed person may experience difficulty in: dealing with the idea that A has the same number of elements as B even when $A \subsetneq B$ (if he/she interprets the sets in a finite universe); grasping complex numbers without an ordinal relationship (if he/she links the number concept to real numbers); realising that, even if $3x^2+3x$ divided by $x+1$ "equals" $3x$, $(3x^2+3x)/(x+1)$ is not equivalent to it (if he/she interprets the division as one between numbers); and so on.

- If it is true that relation of mathematics with reality is often mediated by other disciplines' models, transfer problem of mathematics involves transfer problems of these ones too; for instance, in a didactical activity about the mathematization of a physical phenomenon, it may be necessary to tackle the difficulties which follow from the fact that "force" is not an abstraction of the intuitive concept of "force" (related to "push", "effort", etc.) but can only be grasped in relation to other physics concepts. On this subject links with *educational research in other disciplines* appear weak. Problems related to modelling, misconceptions, student motivation for schooling, etc. are often treated with a separate, unilateral approach within each disciplinary sector of educational research (for instance, many issues in common to physics education emerge from reading McDermott, 1993).

- In conclusion, pupils have to be prepared to use and control the shift from contexts to abstractions and vice versa, must be confronted with the leaps and breakaways that occur (with respect to common sense, to other knowledge domains, and to their previous experience) into the development of new concepts. They must also be aware of the limits of mathematization and of the applicability of mathematical language (outside mathematics, interpreting the "angle" as an unlimited figure maybe "wrong"). They must become conscious that every modelling activity presents the risk of tunnel vision (see 1.5): confining oneself to those things that can be dealt with using the disciplinary tools one has chosen.

The *source of difficulties* does not lie in the reference to contexts but rather in disregarding them or in neglecting the consideration of their characteristics while planning and managing didactic activities.

2.5. *Other issues and open problems*

In 2.2-2.4 we analysed how different modelling activities can present different *roles of mathematics*, different forms of *interference* with other knowledge domains, and different ways of interacting of the three *components of experience fields*. In 2.1 we alluded to other issues too, which we do not go into in this paper; in particular:

- the vital role of curricular research based on contexts: where contexts are chosen and developed not just in response to exemplification established prior to the use of concepts, this research may permit a *debate* on teaching *objectives* in mathematics and their *updating*;

- the "potentialities" of referring to *historical formation of concepts* as a source of clues for identifying meaningful (today too) contexts and as a source of contexts (e.g. analysis of historical documents to explore the evolution of algebraic thought); but also its "limitations", if one overlooks the fact that the network of concepts of students and the social diffusion of mathematics differ widely from those of scholars of the past (e.g. experience with cars - knowing the speedometer, talking about acceleration, etc. - offers references for the introduction of mathematical analysis that were not available to Galileo or Newton).

Context-based teaching can modify the respective roles that the *teacher* and the *pupil* play in traditional teaching. We mention some important related problems which actually deserve more detailed examination (for some references, cf. Blum & Niss, 1991; Boero, Dapueto & Parenti, 1996; Sierpinska, 1997):

- pupil and teacher can no longer rely on their familiar reference points (syllabus schedule, identification of basic skills to be tested, etc.): it is necessary to outline a new set of *basic skills* and a new balance between experiential and reflective learning, to develop new criteria for the assessment which exploit the opportunity for *dynamic evaluation* (day-to-day, in various activities,…) that differs from pre-set and ad hoc testing, but overcome the risk of prejudicial evaluations;

- the motivations and viewpoints of *teacher-researchers* (involved in designing projects or in critical experimenting and evaluating them) differ from those of "*normal" teachers*; the non-neutral act of choosing contexts means assuming *greater educational responsibility*; managing the transition from contexts to abstraction, from individual ideas to shared knowledge, etc. and the interaction with out-of-school ways of training and communication arouse *conflicts* with the *conventional wisdom of school* education; all that makes the question of spreading situated teaching complex.

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