The Hilbert function of the Ratliff–Rush filtration

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Dedicated to Prof. Wolmer Vasconcelos on the occasion of his 65th birthday

Abstract

The Ratliff–Rush filtration has been shown to be a very useful tool for studying numerical invariants of the associated graded ring $G := \bigoplus_{t \geq 0} (I^t/I^{t+1})$ of a local ring $(A, m)$ with respect to the classical $I$-adic filtration. The advantage of this approach is that the associated graded ring $\tilde{G}$ of $A$ with respect to the Ratliff–Rush filtration has positive depth, but unfortunately $\tilde{G}$ is not necessarily a standard graded algebra.

In this paper, we study some numerical invariants of $\tilde{G}$ when $I$ is an $m$-primary ideal of a local Cohen–Macaulay ring and, as consequence, we prove an upper bound on the first coefficient of the Hilbert polynomial of $G$ which extends the already known bounds.

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0. Introduction

The notion of Ratliff–Rush closure

$$\tilde{I} := \bigcup_{n \geq 1} (I^{n+1} : I^n)$$
of an ideal $I$ in a Noetherian local ring $A$ has been introduced in [12] where the authors show that, if $I$ contains a regular element, then $I$ is a reduction of $\tilde{I}$ and, even more, $(\tilde{I})^n = I^n$ for all large $n$, $\tilde{I}$ being the largest ideal with this property. More generally it has also been proved in [12] that

$$\tilde{I} \supseteq \tilde{I}^2 \supseteq \cdots \supseteq \tilde{I}^i \supseteq \tilde{I}^{i+1} \supseteq \cdots \supseteq \tilde{I}^n = I^n$$

for all large $n$.

Since it is clear that $\tilde{I}^i \tilde{I}^j \subseteq \tilde{I}^{i+j}$ for every $i$ and $j$, the collection of ideals $\{\tilde{I}^n\}_{n \in \mathbb{N}}$ is a filtration of $A$ which is called the Ratliff–Rush filtration induced by $I$ and which is a Noetherian filtration.

The Ratliff–Rush filtration has been shown to be a very useful tool for studying numerical invariants of the associated graded ring $G := \bigoplus_{t \geq 0}(I^t/I^{t+1})$ of $A$ with respect to the classical $I$-adic filtration (see [3,4,7,8,10,11,13,15,16]).

For example, for all not negative integer $n$ the degree $n$ component of the zeroth local cohomology module of $G$ with respect to the ideal $G_+ = \bigoplus_{t \geq 1}(I^t/I^{t+1})$ can be written as

$$[H^0_{G_+}(G)]_n = (\tilde{I}^{n+1} \cap I^n)/I^{n+1}.$$  

Hence $G$ has positive depth if and only if $\tilde{I}^n = I^n$ for all $n \geq 0$.

Since $\tilde{I}^p \supseteq \tilde{I}^{p+1}$, we can consider the abelian group

$$\tilde{G} := \bigoplus_{p \geq 0}(\tilde{I}^p/I^{p+1})$$

which has a natural structure of graded algebra over its degree zero part, the local ring $\tilde{G}_0 = A/\tilde{I}$, with multiplication induced by the multiplication map $\tilde{I}^p \times \tilde{I}^q \to \tilde{I}^{p+q}$.

The ring $\tilde{G}$ is called the associated graded ring of $A$ with respect to the Ratliff–Rush filtration induced by $I$. If $I$ is $m$-primary, then $\tilde{G}_0$ is an Artinian local ring and we can consider the Hilbert function of $\tilde{G}$ which is by definition

$$\tilde{H}_t(t) := \lambda_{\tilde{A}/\tilde{I}}(\tilde{G}_t) = \lambda(\tilde{I}^t/I^{t+1}),$$

where we simply write $\lambda(M)$ for the length of the $A$-module $M$. This function gives useful information on some numerical invariants related to the classical Hilbert function of $I$. The advantage is that $\tilde{G}$ has positive depth, but unfortunately $\tilde{G}$ is not a standard graded algebra because we do not necessarily have $\tilde{G}_{t+1} = \tilde{G}_t \tilde{G}_t$. Hence, the classical tools used for the computation of the Hilbert function in the standard case, are no more available here. However, if $I$ is an $m$-primary ideal of a one-dimensional Cohen–Macaulay local ring $(A, m)$, we can prove in Theorem 2.1 that the Hilbert function of $\tilde{G}$ is strictly increasing up to reach the multiplicity $e$ of $I$, the same behaviour which the Hilbert function of $G$ has in the case $G$ is Cohen–Macaulay. By using this result and as a particular case of a more precise bound, we prove in Corollary 2.3 that for every $t \geq 0$

$$\tilde{H}_t(t) \geq \min(e, t + \lambda(A/\tilde{I})).$$
This inequality should be compared with the inequality

\[ H_R(t) \geq \min(e, t + 1) \]

which holds for a given one-dimensional standard graded algebra \( R \) over an Artinian local ring \( R_0 \) and where the Hilbert function of \( R \) is defined as \( H_R(t) := \lambda_{R_0}(R_t) \).

If \( R_0 \) is a field, this last result can be found in [9] or can be achieved as a consequence of the classical Macaulay’s theorem, while, in the case \( R_0 \) is Artinian, it follows from an extension of Macaulay’s theorem due to Blancafort (see [2, Corollary 2.11]).

Our approach also gives a bound on the regularity \( s \) of \( G \) in terms of the invariants of \( I \). More precisely we prove in Theorem 2.4 and 2.5 that

\[ s \leq e - \max(v(I), v(\tilde{I})) + 1, \]

where \( v(J) \) denotes the minimal number of generators of \( J \).

We remark that in most of the cases \( v(\tilde{I}) \geq v(I) \), but Example 3.6. in [14] shows that \( v(I) - v(\tilde{I}) \) can be positive and large as you want even in a regular local ring.

In the last section, as a simple numerical consequence of the described properties of the Hilbert functions of \( G \), we recover and extend in Theorem 3.1 a remarkable result proved by Elias [5].

For a Cohen–Macaulay one-dimensional local ring \((A, m)\) one has for every \( n \geq 0 \)

\[ \lambda(A/m^{n+1}) = \sum_{t=0}^{n} H_G(t) = e(n + 1) - e_1, \]

where \( e \) is the multiplicity of \( A \) and \( e_1 \) is an integer which is called the first Hilbert coefficient of \( A \).

In the quoted paper, by using deep methods related to the strict transform of the blowing up of \( A \), Elias proved that

\[ e_1 \leq \left( \frac{e}{2} \right) - \left( \frac{v - 1}{2} \right) \]

where \( v \) is the embedding dimension of \( A \). This bound is sharp and it can be used to give all the possible Hilbert–Samuel polynomials for the class of one-dimensional Cohen–Macaulay local rings with multiplicity \( e \) and embedding dimension \( v \).

In Theorem 3.2, as a consequence of a more general result, we improve the upper bound for \( e_1 \) proved by Elias by showing that for an \( m \)-primary ideal of a Cohen–Macaulay local ring \((A, m)\) one has

\[ e_1 \leq \left( \frac{e}{2} \right) - \left( \frac{v(I) - d}{2} \right) - \lambda(A/I) + 1. \]

This result can be used to give strict constraints on the Hilbert function of an \( m \)-primary ideal in a Cohen–Macaulay local ring, for example it says that the Hilbert series

\[ P_m(z) = \frac{1 + 3z - z^2 + z^3 + z^4}{1 - z} \]

is not admissible since \( e = 5 \), \( v = 4 \) and \( e_1 = 8 \).
The paper ends with a short proof (see Proposition 3.3) that, in the case $I$ is the maximal ideal of $A$, if $e_l$ reaches its maximal value, then $A$ has a specified Hilbert function, a result which was the main theorem in [6].

1. Preliminaries

Let $(A, m)$ be a local ring of dimension $d$ and $I$ an $m$-primary ideal in $A$. Let us recall a construction due to Ratliff and Rush (see [12]). For every $n \geq 0$ we have a chain of ideals

$$I^n \subseteq I^{n+1}; I \subseteq I^{n+2}; I^2 \subseteq \ldots \subseteq I^{n+k}; I^k \subseteq \ldots.$$ 

This chain stabilizes at an ideal which we will denote by

$$\widetilde{n} := \bigcup_{k \geq 1} (I^{n+k}; I^k).$$

Hence there is a positive integer $t$, depending on $n$, such that $\widetilde{n} = I^{n+k}; I^k$ for every $k \geq t$.

It is clear that we have $\widetilde{0} = A$ and for every non-negative integers $i$ and $j$

$$I^i \subseteq \widetilde{i}, \quad \widetilde{i} \subseteq \widetilde{i} + j, \quad I^{i+1} \subseteq \widetilde{i}.$$ 

We will denote by $\widetilde{G} := \bigoplus_{i \geq 0} (\widetilde{i}/\widetilde{i}^{i+1})$ the associated graded ring of $A$ with respect to the Ratliff–Rush filtration and by

$$\widetilde{H}(t) := \lambda_{\widetilde{G}}(\widetilde{G}) = \lambda(\widetilde{i}/\widetilde{i}^{i+1})$$

its Hilbert function. This is the Hilbert function we refer to in the title.

Superficial elements play an important role in this paper. We recall that an element $x$ in $I$ is called superficial for $I$ if $d \geq 1$ and there exists an integer $c > 0$ such that

$$(I^n : x) \cap I^c = I^{n-1}$$

for every $n > c$.

It is well known that if the residue field is infinite, superficial elements always exist. Further, if $A$ has positive depth, every superficial element for $I$ is also a regular element in $A$.

If $x$ is superficial for $I$ and a non-zero divisor, it is an easy consequence of the Artin Rees lemma that for every integer $j \geq 0$ we have $I^j : x = I^{j-1}$. From this we easily get $I^i = \widetilde{i}$, for $i \geq 0$.

Finally, for every $n \geq 0$, we have

$$\widetilde{n}^{n+1}; x = \widetilde{n},$$

which implies that $G$ has positive depth.

If $G := \bigoplus_{i \geq 0} (I^i/I^{i+1})$ is the associated graded ring of $A$ with respect to the $I$-adic filtration, we have $\widetilde{G} = G_i$ for $i \geq 0$. We recall that $G$ is a standard graded algebra which has not necessarily positive depth, while $\widetilde{G}$ is not a standard graded algebra, but depth $\widetilde{G} > 0$ by (1).
In this paper, we study some properties of $\tilde{H}_I(t)$ and we show how these properties give information on the Hilbert function $H_I(t)$ of $I$ which, as usual, is defined as

$$H_I(t) = H_G(t) = \hat{\lambda}_{A/I}(I^t/I^{t+1}) = \hat{\lambda}(I^t/I^{t+1}).$$

The generating function of the numerical function $H_I(t)$ is the power series

$$P_I(z) = \sum_{t \geq 0} H_I(t) z^t.$$

This series is called the Hilbert series of $I$. It is well known that this series is rational and that, even more, there exists a polynomial $h_I(z)$ with integer coefficients such that $h_I(1) \neq 0$ and

$$P_I(z) = \frac{h_I(z)}{(1-z)^d}.$$

For every $i \geq 0$, the integers

$$e_i(I) := \frac{h_I^{(i)}(1)}{i!}$$

are called the Hilbert coefficients of $I$. The integer $e_0(I) = h_I(1)$ is the multiplicity of $I$ and it is simply denoted by $e(I)$.

It is well known that the polynomial

$$p_I(X) := \sum_{i=0}^d (-1)^i e_i(I) \binom{X + d - i}{d - i}$$

has the property that for every $n \geq 0$

$$p_I(n) = \hat{\lambda}(A/I^{n+1}) = \sum_{j=0}^n H_I(j).$$

Since we have $I^{n+1} = \tilde{I}^{n+1}$ for every $n$ big enough, we also get for every $n \geq 0$

$$p_I(n) = \hat{\lambda}(A/\tilde{I}^{n+1}) = \sum_{j=0}^n \tilde{H}_I(j).$$

A well-known property we will use in the paper is the following: if $x_1, \ldots, x_r$ is a superficial sequence for $I$ (which means $x_1$ is superficial for $I$ and $\overline{x_i}$ is superficial for $I/(x_1, \ldots, x_{i-1})$ for every $2 \leq i \leq r$) and we put $\overline{I} := I/(x_1, \ldots, x_r)$, then, for $i=0, \ldots, d-r$, we have $e_i(I) = e_i(\overline{I})$. Hence, for example, if $d = 1$ and $x$ is a superficial element in $I$, then $e_0(I) = e_0(I/xA) = \hat{\lambda}(A/xA)$.

When the ring $A$ has dimension one, we have nice properties of the above-defined integers. Hence, from now on, we are assuming that $(A, m)$ is a Cohen–Macaulay local ring of dimension $d = 1$ and we will simply write $e$ and $e_1$ for the Hilbert coefficients $e_0(I)$ and $e_1(I)$, respectively.
Further, we let $x$ be a superficial element of the $m$-primary ideal $I$ and we recall that, since $A$ is Cohen–Macaulay, $x$ is regular on $A$ and $\tilde{G}$ as well.

We consider for every $i \geq 0$ the following diagram:

$$
\begin{align*}
A & \supseteq \tilde{I}^{i+1} \supseteq I^{i+1} \\
xA & \supseteq x\tilde{I}^i \supseteq xI^i.
\end{align*}
$$

Accordingly, we set

$$
\rho_i := \lambda(\tilde{I}^{i+1}/x\tilde{I}^i), \quad v_i := \lambda(I^{i+1}/xI^i)
$$

and then from the diagram we get

$$
e = \lambda(A/xA) = H_I(i) + v_i = \tilde{H}_I(i) + \rho_i.
$$

Hence $H_I(i) = e$ if and only if $v_i = 0$, that is $I^{i+1} = xI^i$, and similarly $\tilde{H}_I(i) = e$ if and only if $\rho_i = 0$, that is $I^{i+1} = x\tilde{I}^i$.

Let $s$ be the integer defined by

$$
\begin{align*}
v_j & > 0 \quad \text{if } i \leq s - 1, \\
v_i & = 0 \quad \text{if } i \geq s
\end{align*}
$$

so that $s$ is exactly the reduction number of $I$. It is well known that $s \leq e - 1$ (see for example [17, Remark 6.16]).

We have $I^{i+1} = xI^i$ for every $i \geq s$, from which we easily get by induction on $t \geq 0$ and for every $p \geq s$,

$$
I^{i+p} = x^tI^p.
$$

Let $j$ be an integer, $j \geq s$, and let $t$ be a positive integer such that $\tilde{I}^j = I^{j+t}; I^t$; we have

$$
\tilde{I}^j = I^{j+t}; I^t \subseteq I^{j+s}; I^t; x^t = x^{j+t-s}I^t; x^t = x^{j-s}I^s \subseteq I^j,
$$

so that, for every $j \geq s$,

$$
\tilde{I}^j = I^j, \quad \tilde{H}_I(j) = H_I(j) = e, \quad v_j = \rho_j = 0.
$$

Since for $n \gg 0$

$$
p_I(n) = e(n + 1) - e_1 = \sum_{i=0}^{n} H_I(i) = \sum_{i=0}^{n} \tilde{H}_I(i),
$$

by (2) we get

$$
e_1 = \sum_{i=0}^{n} v_i = \sum_{i=0}^{s-1} v_i
$$
and, similarly,
\[ e_1 = \sum_{i=0}^{s-1} p_i. \]  

(5)

We want now to describe the components of the Ratliff–Rush filtration in the one-dimensional case.

Let \( t \geq 0 \) and \( j \) and \( p \) integers such that \( 0 \leq j \leq s \leq p \); if \( ax^t \in I^{j+t} \), then, by (4),
\[ ax^t I^{p-j} \subseteq I^{t+p} = x^t I^p \]

so that \( a \in I^p : I^{p-j} \). This proves that
\[ I^{j+t} : x^t \subseteq I^p : I^{p-j} \]  

(6)

for every \( t \geq 0 \) and \( 0 \leq j \leq s \leq p \).

**Proposition 1.1.** Let \( (A, \mathfrak{m}) \) be a Cohen–Macaulay local ring of dimension one and let \( I \) be an \( \mathfrak{m} \)-primary ideal in \( A \) with reduction number \( s \). Let \( p \geq s \) be an integer, then for every \( j \geq 0 \) we have
\[ \tilde{I}^j = \begin{cases} I^p : x^{p-j} = I^p : I^{p-j} & \text{if } j \leq s, \\ I^j & \text{if } j \geq s. \end{cases} \]

**Proof.** We have already seen that \( \tilde{I}^j = I^j \) if \( j \geq s \).

Now, let \( t \) be a positive integer such that \( \tilde{I}^j = I^{j+t} : I^t \).

If \( j \leq s \) we can use (6) and (4) to get
\[
\tilde{I}^j = I^{j+t} : I^t \subseteq I^{j+t} : x^t \subseteq I^p : I^{p-j} \subseteq I^p : x^{p-j} \subseteq I^{p+s} : x^{p-j} I^s 
\]
\[ = I^{p+s} : I^{p+s-j} \subseteq \tilde{I}^j. \]

The conclusion follows. \( \square \)

2. The Hilbert function of \( \tilde{G} \)

In this section \( (A, \mathfrak{m}) \) is a local Cohen–Macaulay ring of dimension one, \( I \) an ideal which is primary for \( \mathfrak{m} \), \( x \) a superficial element in \( I \) and \( s \) the reduction number of \( I \).

We will simply write \( H(t) \) and \( \tilde{H}(t) \) instead of \( H_1(t) \) and \( \tilde{H}_1(t) \) for the Hilbert function of \( G \) and \( \tilde{G} \), respectively, and \( e \) for the multiplicity \( e(I) \) of \( I \).

Since by (1) we have \( I^{j+1} : x = \tilde{I}^j \), for every \( t \geq 0 \) the multiplication by \( x \) gives an injective map
\[ 0 \rightarrow \tilde{G}_j \xrightarrow{x} \tilde{G}_{j+1}. \]
whose cokernel is
\[ \tilde{G}_{t+1}/x\tilde{G}_t = \tilde{I}^{t+1}/(x\tilde{I}^{t+2}). \]

Since we have
\[ x\tilde{I}^{t+2} + \tilde{I}^{t+2} \subseteq \tilde{I}^{t+1} + \tilde{I}^{t+2} \subseteq \tilde{I}^{t+1}, \]
if we let
\[ b_t := \tilde{\lambda}(I\tilde{I}^t + \tilde{I}^{t+2}/x\tilde{I}^t + \tilde{I}^{t+2}) \]
and
\[ c_t := \tilde{\lambda}(I\tilde{I}^{t+1}/I\tilde{I}^t + \tilde{I}^{t+2}) \]
for every \( t \geq 0 \) we get
\[ \tilde{H}(t+1) = \tilde{H}(t) + c_t + b_t. \tag{7} \]

Further, since for every \( t \geq s \) we have \( \tilde{H}(t) = e \), it is clear that \( c_t = b_t = 0 \) for every \( t \geq s \).

The next result is the main theorem of this section. We recall that if \( R \) is a one-dimensional Cohen–Macaulay standard graded algebra over a field, its Hilbert function is strictly increasing until it reaches the multiplicity at which it stabilizes. We prove that the same property holds for the Cohen–Macaulay graded algebra \( \tilde{G} \), even if \( \tilde{G} \) is an algebra over an Artinian local ring and it is not standard.

**Theorem 2.1.** Let \((A, m)\) be a Cohen–Macaulay local ring of dimension one, let \( I \) be an \( m \)-primary ideal in \( A \) and let \( t \geq 0 \) be an integer. The following conditions are equivalent:

(a) \( \tilde{H}(t+1) = \tilde{H}(t) \).
(b) \( b_2 = 0 \).
(c) \( \tilde{H}(t) = e \).
(d) \( \tilde{H}(n) = e \) for every \( n \geq t \).

**Proof.** It is clear by (7) that (a) implies (b). Let us prove that (b) implies (c). If \( t \geq s \), then \( \tilde{H}(t) = H(t) = e \). So let \( t + 1 \leq s \). By assumption we have
\[ I\tilde{I}^{t+1} \subseteq x\tilde{I}^{t} + \tilde{I}^{t+2} \]
and we claim that
\[ I^s = x^{s-t}\tilde{I}^t. \]

We have
\[ x^{s-t}\tilde{I}^t \subseteq \tilde{I}^s = I^s, \]
on the other hand
\[ I^s = I^{s-t-1}I^{t+1} \subseteq I^{s-t-1}I\tilde{I}^t \subseteq I^{s-t-1}(x\tilde{I}^t + \tilde{I}^{t+2}) \subseteq xI^{s-t-1}\tilde{I}^t + \tilde{I}^{s+1} = xI^{s-t-1}\tilde{I}^t + I^{s+1}. \]
If \( s = t + 1 \) we are done by Nakayama. Otherwise \( s > t + 1 \) and we have
\[
x I^{s-t-1} \tilde{I} + I^{s+1} = x I^{s-t-2} \tilde{I} \tilde{I} + I^{s+1} \subseteq x I^{s-t-2} (x \tilde{I} + \tilde{I} + I^{s+1}) + I^{s+1} \subseteq x^2 I^{s-t-2} \tilde{I} \tilde{I} + I^{s+1} \subseteq \cdots \subseteq x^{s-t} \tilde{I} \tilde{I} + I^{s+1}.
\]
The claim follows again by Nakayama.

From the claim we get
\[
I^{s+1} = x I^s \subseteq x^{s-t} \tilde{I} \tilde{I} + I^{s+1} = I^{s+1},
\]
hence \( I^{s+1} = x^{s-t} \tilde{I} \tilde{I} + I^{s+1} \), and we finally get
\[
e = \lambda(I^s/I^{s+1}) = \lambda(x^{s-t} \tilde{I} / x^{s-t} \tilde{I} + I^{s+1}) = \lambda(\tilde{I} / \tilde{I} + I^{s+1}) = \tilde{H}(t).
\]

Let us finally prove that (c) implies (d). If \( n \geq t \), we have
\[
e \geq e - \rho_n = \tilde{H}(n) - \tilde{H}(t) = e
\]
and the conclusion follows. \( \square \)

As an easy consequence of this result, we have the following crucial corollary.

**Corollary 2.2.** Let \( j \) be a non-negative integer; then for every \( n \geq j \) we have
\[
\tilde{H}(n) \geq \min \left( e, \tilde{H}(j) + n - j + \sum_{i=0}^{n-1} c_i \right).
\]

**Proof.** If \( j = n \) there is nothing to prove. So let \( n > j \) and consider the sequence
\[
\tilde{H}(j) \leq \tilde{H}(j + 1) \leq \cdots \leq \tilde{H}(n).
\]
If for some \( j \leq i \leq n - 1 \) we have \( \tilde{H}(i) = \tilde{H}(i + 1) \), then \( e = \tilde{H}(i) \leq \tilde{H}(n) \) and the conclusion follows. Otherwise \( b_j, \ldots, b_{n-1} > 0 \) and we have
\[
\tilde{H}(n) = \tilde{H}(j) + \sum_{i=j}^{n-1} (c_i + b_i) \geq \tilde{H}(j) + \sum_{i=j}^{n-1} c_i + n - j,
\]
as wanted. \( \square \)

We can get free of the nasty term involving the \( c_i \)'s in the above inequality by proving the following corollary. We will use throughout the notation
\[
\lambda := \lambda(A/\tilde{I}) = \tilde{H}(0).
\]

**Corollary 2.3.** For every \( n \geq 0 \) we have
\[
\tilde{H}(n) \geq \min(e, n + \lambda).
\]
Proof. We have $\tilde{H}(0) = \lambda$ so that by the above corollary we get

$$\tilde{H}(n) \geq \min \left( e, \tilde{H}(0) + n + \sum_{i=0}^{n-1} c_i \right) \geq \min(e, n + \lambda). \quad \Box$$

The next result of this main section gives an upper bound for the reduction number of the Ratliff–Rush filtration.

In the rest of the paper we let

$$\lambda := \lambda(I + \tilde{I}/\tilde{I}^2)$$

so that $c_0 + \lambda = \lambda(\tilde{I}/I + \tilde{I}^2) + \lambda(I + \tilde{I}^2/\tilde{I}^2) = \lambda(\tilde{I}/\tilde{I}^2) = \tilde{H}(1)$.

We also denote by $g$ the integer

$$g := \sum_{i \geq 0} c_i + \sigma = \lambda(\tilde{I}/\tilde{I}^2) + \sum_{i \geq 1} c_i.$$

**Theorem 2.4.** Let $(A, m)$ be a Cohen–Macaulay local ring of dimension one and let $I$ be an $m$-primary ideal in $A$. We have $e - g + 1 \geq 1$ and

$$\tilde{H}(e - g + 1) = e.$$

Proof. Since $c_j \neq 0$ for $j \geq 0$, we can consider the least integer $t \geq 0$ such that $c_j = 0$ for every $j > t$.

If $t = 0$, then $g = \sigma$ and, in this case, $e \geq \tilde{H}(1) = c_0 + \sigma = \sigma$, so that $e - \sigma \geq 0$ and $e - g + 1 = e - \sigma + 1 \geq 1$. By Corollary 2.2 we get

$$\tilde{H}(e - g + 1) = \tilde{H}(e - \sigma + 1) \geq \min \left( e, \tilde{H}(1) + e - \sigma + 1 - 1 + \sum_{i=1}^{e-g} c_i \right) = e.$$

If instead $t \geq 1$, then $g = \sum_{i=0}^{t-1} c_i + \sigma$ with $c_{t-1}, b_{t-1} > 0$. Since $b_{t-1} > 0$, we have $\tilde{H}(t - 1) < e$, hence, if $t \geq 2$, we can apply Corollary 2.2 with $j = 1$, $n = t - 1$ to get

$$\tilde{H}(t) = \tilde{H}(t - 1) + b_{t-1} + c_{t-1} \geq \tilde{H}(1) + t - 1 - 1 + \sum_{i=1}^{t-2} c_i + b_{t-1} + c_{t-1}$$

$$= c_0 + \sigma + t - 2 + \sum_{i=1}^{t-2} c_i + b_{t-1} + c_{t-1} \geq g + t - 1.$$

The inequality $\tilde{H}(t) \geq g + t - 1$ holds true also if $t = 1$ because, in that case, $g = c_0 + \sigma = \tilde{H}(1)$. Hence if $t \geq 1$, we have

$$e \geq \tilde{H}(t) \geq g + t - 1.$$
so that \( e - g + 1 \geq t \geq 1 \) and finally by Corollary 2.2 we get

\[
\tilde{H}(e - g + 1) \geq \min \left( e, \tilde{H}(t) + e - g + 1 - t + \sum_{i=t}^{e-g} c_i \right) 
\geq \min \left( e, g + t - 1 + e - g + 1 - t + \sum_{i=t}^{e-g} c_i \right) = e,
\]

as wanted. □

We have seen that the integer \( g \) plays a central role in the above theorem. Unfortunately, it looks like a mysterious invariant of the ideal \( I \) involving unaccessible integers. Nevertheless, the next theorem proves that it is bounded below by nice numerical invariants of the ideal \( I \).

We will denote by \( v(J) \) the minimal number of generators of an ideal \( J \) of a local ring \( A \).

**Theorem 2.5.** Let \((A, m)\) be a Cohen–Macaulay local ring of dimension one and let \( I \) be an \( m \)-primary ideal in \( A \). Then

\[
g \geq \max(v(\tilde{I}), v(I)).
\]

**Proof.** Recall that

\[
g = \lambda(\tilde{I}/\tilde{I}^2) + \sum_{i \geq 1} c_i = \lambda(\tilde{I}/\tilde{I}^2) + \sum_{i \geq 1} \lambda(\tilde{I}^{i+1}/\tilde{I}^i + \tilde{I}^{i+2}).
\]

We remark that for every \( i \geq 1 \), \( I \tilde{I}^i \subseteq I_m \cap \tilde{I}^{i+1} \), hence we have

\[
\lambda(\tilde{I}^{i+1}/\tilde{I}^i + \tilde{I}^{i+2}) \geq \lambda(\tilde{I}^{i+1}/I_m \cap \tilde{I}^{i+1} + \tilde{I}^{i+2}) = \lambda(I^{i+1} + I_m/\tilde{I}^{i+2} + I_m).
\]

We know that there exists an integer \( N \) such that for every \( j > N \) we have \( \tilde{I}^j = I^j \). Hence for \( i \geq 0 \) we have \( \tilde{I}^{i+2} = I^{i+2} = I \tilde{I}^{i+1} \subseteq I_m \) and so it is easy to see that

\[
g \geq \lambda(\tilde{I}/\tilde{I}^2) + \lambda(\tilde{I}^2 + I_m/I_m).
\]

Now

\[
\lambda(\tilde{I}/\tilde{I}^2) + \lambda(\tilde{I}^2 + I_m/I_m) \geq \lambda(\tilde{I}/\tilde{I}^2 + I_m) + \lambda(\tilde{I}^2 + I_m/I_m) \geq v(\tilde{I}).
\]

On the other hand \( \lambda(\tilde{I}/\tilde{I}^2) \geq \lambda(I + \tilde{I}^2/I^2) \geq \lambda(I + \tilde{I}^2/I^2 + I_m) \), hence

\[
g \geq \lambda(I + \tilde{I}^2/I_m) \geq v(I)
\]

as desired. □

By analogy with the classical case, let \( s \) be the least integer \( t \) such that \( \tilde{H}(t) = e \). Since \( e = \tilde{H}(j) + \rho_j \), it is clear that \( s \) is also the least integer \( t \) such that \( \rho_t = 0 \). Since \( H(s) = \tilde{H}(s) = e \) we have \( s \leq s \).
We will denote by
\[ v := \max(v(\tilde{I}), v(I)). \]
By the above theorems we have
\[ \tilde{s} \leq e - g + 1 \leq e - v + 1. \]

We end this section by proving a far reaching property of the powers of an \( m \)-primary ideal \( I \) in a one-dimensional Cohen–Macaulay local ring.

**Proposition 2.6.** With the above notation, we have \( I^s \subseteq x^s A : x^{e-g+1} \).

**Proof.** We know that \( \tilde{H}(e - g + 1) = e \). This implies \( \rho_j = 0 \) for every \( j \geq e - g + 1 \) so that \( I^s \subseteq x^s A : x^{e-g+1} \).

Hence, using (4), we get
\[ x^{e-g+1} I^s = I^{s+e-g+1} = I^{g+1} A \subseteq x^{g+1} A. \]

The first assertion follows.

As for the second, we have \( I^{e-1} = x^{e-1-s} I^s \subseteq x^{e-1-s} (x^s A : x^{e-g+1}) \).

Now, if \( s \geq e - g + 1 \), then we get \( I^{e-1} \subseteq x^{e-1-s} x^{e-g+1} A = x^{g+1} A \) if \( s \leq e - g + 1 \), then \( e - 1 - s \geq g - 2 \) so that \( x^{e-1-s} A \subseteq x^{g+1} A \). This proves the second assertion. □

3. The bound for \( e_1 \)

In this section, we use the result on the Hilbert function of \( \tilde{G} \) to get an upper bound for the Hilbert coefficient \( e_1 \) of \( I \).

We recall that we have defined for every \( t \geq 0 \) the integers
\[ c_t := \lambda(I^{t+1}/I^t I^{t+2}). \]
Further we set
\[ \sigma := \lambda(I + I^2/\tilde{I}^2), \quad g := \sum_{i \geq 0} c_i + \sigma, \quad \lambda := \lambda(A/\tilde{I}) = \tilde{H}(0) \]
and \( \tilde{s} \) the least integer \( t \) such that \( \tilde{H}(t) = e \).

At the end of the last section, in Theorem 2.5, we proved that \( g \geq v \). We remark now that the integer \( \tilde{s} \) can be zero, but, if this is the case, then \( \lambda(A/I) \geq \lambda(A/\tilde{I}) = e \), hence \( H(t) = e \) for every \( t \geq 0 \) and \( e_1 = 0 \). Thus, we will tacitly assume in the rest of this section that \( \tilde{s} \geq 1 \).

A final remark on the integer \( g \) is needed. Namely we claim that \( g \geq 2 \), unless \( I = m \) and \( A \) is regular. In fact \( g \geq c_0 + \sigma = \tilde{H}(1) = \lambda + c_0 + b_0 \), hence \( g \geq 1 \) and if \( g = 1 \) then \( \lambda = 1 \) and \( b_0 = 0 \). This implies \( 1 = \lambda = e \) so that \( H(0) \geq \lambda = e = 1 \), which implies \( H(0) = 1 \). Hence \( I = m \) and \( A \) is regular.
Theorem 3.1. Let \((A, m)\) be a Cohen–Macaulay local ring of dimension one and let \(I\) be an \(m\)-primary ideal in \(A\). Then
\[
e_1 \leq \left( \frac{e}{2} \right) - \left( \frac{g - 1}{2} \right) - \tilde{s}(\lambda - 1).
\]

Proof. We have by (5) and (2)
\[
e_1 = \sum_{j=0}^{\tilde{s}-1} \rho_j = \sum_{j=0}^{\tilde{s}-1} \rho_j = e\tilde{s} - \sum_{j=0}^{\tilde{s}-1} \tilde{H}(j).
\]
Since \(0 \leq j \leq \tilde{s} - 1\), we have \(\tilde{H}(j) < e\) so that, by Corollary 2.3, we get
\[
\tilde{H}(j) \geq j + \lambda.
\]
Hence
\[
e_1 = e\tilde{s} - \sum_{j=0}^{\tilde{s}-1} \tilde{H}(j) \leq e\tilde{s} - (1 + 2 + \cdots + \tilde{s} - 1) - \tilde{s}\lambda
\]
\[
= e\tilde{s} - \left( \frac{\tilde{s} + 1}{2} \right) - \tilde{s}(\lambda - 1).
\]
By Theorem 2.4 we have \(\tilde{s} \leq e - g + 1\). Because \(g \geq 2\) we also have
\[
\tilde{s} \leq e + g - 2.
\]
An easy computation shows that
\[
\left( \frac{e}{2} \right) - \left( \frac{g - 1}{2} \right) - e\tilde{s} + \left( \frac{\tilde{s} + 1}{2} \right) = \frac{(e - \tilde{s} - g + 1)(e - \tilde{s} + g - 2)}{2} \geq 0,
\]
hence
\[
e_1 \leq e\tilde{s} - \left( \frac{\tilde{s} + 1}{2} \right) - \tilde{s}(\lambda - 1) \leq \left( \frac{e}{2} \right) - \left( \frac{g - 1}{2} \right) - \tilde{s}(\lambda - 1). \quad \Box
\]
Since by Theorem 2.5 we have \(g \geq v\), we can give a weaker bound for \(e_1\) which, however, uses more accessible invariants.

For every primary ideal \(I\) of the one-dimensional Cohen–Macaulay local ring \(A\), we have
\[
e_1 \leq \left( \frac{e}{2} \right) - \left( \frac{v - 1}{2} \right) - \tilde{s}(\lambda - 1) \leq \left( \frac{e}{2} \right) - \left( \frac{v - 1}{2} \right) - \lambda + 1.
\]
We would like to extend the inequality
\[
e_1 \leq \left( \frac{e}{2} \right) - \left( \frac{v - 1}{2} \right) - \lambda + 1
\]
to the higher-dimensional case. Unfortunately, the integer \( v(\tilde{I}) \) does not behave well under reduction modulo a superficial sequence. For this reason we will extend to higher dimension the inequality

\[
e_1 \leq \left( \frac{e}{2} \right) - \left( \frac{v(I) - 1}{2} \right) - \lambda + 1
\]

which holds in the one-dimensional case by the above remark.

**Theorem 3.2.** Let \((A, \mathfrak{m})\) be a Cohen–Macaulay local ring of dimension \( d \) and let \( I \) be an \( \mathfrak{m} \)-primary ideal in \( A \). Then

\[
e_1 \leq \left( \frac{e}{2} \right) - \left( \frac{v(I) - d}{2} \right) - \lambda(A/I) + 1.
\]

**Proof.** In the one-dimensional case the result follows by (8). Let \( d \geq 2 \), let \( x_1, \ldots, x_{d-1} \) be a superficial sequence in \( I \) and denote \( J := I/(x_1, \ldots, x_{d-1}) \) in the one-dimensional Cohen–Macaulay local ring \( A/(x_1, \ldots, x_{d-1}) \). The multiplicity and \( e_1 \) do not change from \( I \) to \( J \) so that, by induction, we have

\[
e_1 \leq \left( \frac{e}{2} \right) - \left( \frac{v(J) - 1}{2} \right) - \lambda(A/I) + 1.
\]

The conclusion follows because we clearly have

\[v(J) \geq v(I) - (d - 1)\].

Coming back to the one-dimensional case we remark that, if \( I = \mathfrak{m} \), then (8) becomes

\[
e_1 \leq \left( \frac{e}{2} \right) - \left( \frac{v(\mathfrak{m}) - 1}{2} \right).
\]

In [6] we proved that equality holds if and only if

\[
P_A(z) = \frac{1 + (v(\mathfrak{m}) - 1)z + \sum_{j=v(\mathfrak{m})}^{e-1} z^j}{(1 - z)}.
\]

The proof there was very hard and long. We end this paper by giving a shorter proof using the methods developed in the previous sections.

**Proposition 3.3.** Let \((A, \mathfrak{m})\) be a one-dimensional Cohen–Macaulay local ring of embedding dimension \( v \). If

\[
e_1 = \left( \frac{e}{2} \right) - \left( \frac{v - 1}{2} \right)
\]

then

\[
P_A(z) = \frac{1 + (v - 1)z + \sum_{j=v}^{e-1} z^j}{(1 - z)}.
\]
Proof. It is well known (see [1]) that we always have $e \geq v$ and if $e = v$ then $e_1 = 0$. Hence we have $e \geq v + 1$. By looking at the proof of Theorem 3.1 we see immediately that

$$e_1 = \left( \frac{e}{2} \right) - \left( \frac{v - 1}{2} \right)$$

implies $v = g$ and

$$\tilde{H}(t) = \begin{cases} 
    t + 1 & \text{if } t \leq e - v, \\
    e & \text{if } t \geq e - v + 1.
\end{cases}$$

This implies that $b_i = 1$ and $c_i = 0$ for every $i = 0, \ldots, e - v - 1$. Hence we have

$$\tilde{m}_{i+1} = m\tilde{m}^i + \tilde{m}^{i+2}$$

for every $i = 0, \ldots, e - v - 1$.

Since $\tilde{H}(1) = 2$, we can find an element $y \in m$ such that $m = (x, y) + \tilde{m}^2$. Using this and (9) we easily get

$$m = (x, y) + \tilde{m}^{e-v+1}.$$  

By induction on $j$ one gets for every $j \geq 1$

$$m^j = (x, y)^j + x^{j-1}m\tilde{m}^{e-v+1}.$$  

We claim that this implies $m^j = (x, y)^j$ for every $j \geq v$. Since $j \geq v$, we have $e - v + j \geq s + 1$, hence we may apply Proposition 1.1 to get

$$\tilde{m}^{e-v+1} = \begin{cases} 
    m^{e-v+j} : x^{e-v+j-(e-v+1)} = m^{e-v+j} : x^{j-1} & \text{if } e - v + 1 \leq s, \\
    m^{e-v+1} & \text{if } e - v + 1 > s.
\end{cases}$$

It follows that

$$m^j = (x, y)^j + x^{j-1}m\tilde{m}^{e-v+1} \subseteq (x, y)^j + \tilde{m}^{e-v+j}.$$  

By Nakayama we get the claim.

Since $H(j) \geq \min(e, j + 1)$, this implies

$$H(j) = \begin{cases} 
    j + 1 & \text{if } v \leq j \leq e - 1, \\
    e & \text{if } j \geq e.
\end{cases}$$

On the other hand for every $j \geq 1$ we have

$$m^{j+1}/xm^j = \frac{(x, y)^{j+1} + x^{j}m^{e-v+1}}{x(x, y)^{j} + x^{j}m^{e-v+1}},$$

hence $m^{j+1}/xm^j$ is a cyclic module generated by $y^{j+1}$ so that

$$m^{j+1}/xm^j \simeq A/(xm^j : y^{j+1}).$$
For every $t \geq e - v + 1$, $\tilde{H}(t) = e$ so that $\rho_t = 0$; this implies
\[
\tilde{m}^t = x^{t-e+v-1}m^{e-v+1}
\]
for every $t \geq e - v + 1$. Hence for every $j \geq 1$ we have
\[
y^{j+1}m^{e-v} \subseteq m^{e-v+j+1} = x^j m^{e-v+1} \subseteq x^m^j.
\]
Thus we get a chain
\[
xA \subseteq xA + m^{e-v} \subseteq x^m^j : y^{j+1} \subseteq A.
\]
This implies for every $j \geq 1$
\[
v_j \leq e - \lambda(xA + m^{e-v}/xA) = e - \lambda(m^{e-v}/x^m^{e-v-1}) = e - \rho_{e-v-1}
\]
\[
= \tilde{H}(e - v - 1) = e - v.
\]
Finally we get
\[
e_1 = \sum_{i=0}^{e-2} v_i = \sum_{i=0}^{v-1} v_i + \sum_{i=0}^{e-2} v_i \leq e - 1 + (v - 1)(e - v) + \sum_{i=v}^{e-2} (e - i - 1)
\]
\[
= \binom{e}{2} - \binom{v-1}{2}.
\]
From this it follows that $v_i = e - v$ for $i = 1, \ldots, v - 1$ and this gives the conclusion. \qed

References