

# *Umbral Calculus and Approximation Operators*

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## Three unrelated topics?

$N$  black beads,  $N - M$  white beads, sampling with replacement + addition of  $c$  beads of the same colour.

$X_n$  is number of black beads after  $n$  drawings (Markov-Pólya distribution)

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**I** WLLN (Weak Law of Large Numbers) for  $X_n$ :  $\frac{X_n}{n} \xrightarrow{P} ?$

**II** Convergence of Stancu operator:  $f \in C[0, 1]$ ,

$$\left\| \frac{1}{1[n, -a]} \sum_{k=0}^n \binom{n}{k} x^{[k, -a]} (1-x)^{[n-k, -a]} f\left(\frac{k}{n}\right) - f \right\|_{\infty} \rightarrow ?$$

where  $x^{[k, -a]} = x(x+a)\dots(x+(k-1)a)$

$$\text{III } \sum_{k=0}^n \left( k - n \frac{\alpha}{\alpha + \beta} \right)^2 \frac{\binom{\alpha+k-1}{k} \binom{\beta+n-k-1}{n-k}}{\binom{\alpha+\beta+n-1}{n}} \rightarrow ?$$

## *Outline*

- explain relation between WLLN and Popoviciu approximation operators
- use Umbral Calculus to compute moments of probability distributions on  $0, 1, \dots, n$
- extensions to other types of approximation operators

## Weierstrass approximation

$f : [0, 1] \rightarrow \mathbb{R}$  continuous,  $\mathcal{P} = \mathbb{R}[x]$

$\exists (p_n)_{n \in \mathbb{N}} \in \mathcal{P} : p_n(x) \rightarrow f(x)$  uniformly in  $x$

Probabilistic proof (Bernstein, ...):  $X_n \sim \text{Bin}(n, x)$

$$p_n(x) := \mathbb{E}f\left(\frac{X_n}{n}\right) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

Chebyshev inequality:

$$\mathbb{P}(|X - \mathbb{E}(X)| > \varepsilon) \leq \frac{\text{Var}(X)}{\varepsilon^2}$$

This implies  $\mathbb{E}f\left(\frac{X_n}{n}\right) \rightarrow f(x)$  (variant of WLLN)

Convergence is uniform in  $x \in [0, 1]$  since

$$\text{Var}\left(\frac{X_n}{n}\right) = \frac{x(1-x)}{n} \leq \frac{1}{4n}$$

# Popoviciu operators

$$L_n^{D,I}(f) := \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

is positive, linear operator on  $C[0, 1]$ .

Generalization: Popoviciu (1931) + papers by Manole, Moldovan, Lupaş, Sablonnière, Stancu, ...

$$L_n^{Q,I}(f) := \frac{1}{q_n(1)} \sum_{k=0}^n q_k(x) q_{n-k}(1-x) f\left(\frac{k}{n}\right)$$

where  $q_n(x) \geq 0$  for all  $n$  and

$$\forall n, x, y : q_n(x+y) = \sum_{k=0}^n q_k(x) q_{n-k}(y)$$

$$\mathbb{P}(X_n = k) = \frac{q_k(x) q_{n-k}(1-x)}{q_n(1)}, k = 0, \dots, n$$

$Q : \mathcal{P} \rightarrow \mathcal{P}$  linear operator with  $Qq_k := q_{k-1}$ .

$$L_n^{Q,I}(1) = 1; L_n^{Q,I}(x) = \mathbb{E}\left(\frac{X_n}{n}\right); L_n^{Q,I}(x^2) = \mathbb{E}\left(\frac{X_n^2}{n^2}\right)$$

## Examples

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$$q_k(x) = \frac{x^k}{k!} \quad (Q = D)$$

$$\mathbb{P}(X_n^{\alpha,\beta} = k) = \binom{n}{k} \left(\frac{\alpha}{\alpha + \beta}\right)^k \left(\frac{\beta}{\alpha + \beta}\right)^{n-k}.$$

$$X_n^{\alpha,\beta} \sim \text{Bin}\left(n, \frac{\alpha}{\alpha + \beta}\right).$$

Corresponding approximation operator: Bernstein operator

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$$q_k(x) = \frac{x(x+a)\dots(x+(k-1)a)}{k!}$$

(shifted factorials  $Q = \frac{1}{a}(I - E^{-a})$ )

$X_n^{\alpha,\beta}$  has Markov-Pólya distribution.

Corresponding approximation operator: Stancu operator

## Characterization of $X_n$

$$\forall n, x, y : q_n(x + y) = \sum_{k=0}^n q_k(x) q_{n-k}(y)$$

implies

$$\sum_{n=0}^{\infty} q_n(x) z^n = e^{xg(z)}$$

Define independent random variables  $X, Y$ :

$$P(X = k) = \theta^k q_k(\alpha) e^{-\alpha g(\theta)}$$

$$P(Y = h) = \theta^h q_h(\beta) e^{-\beta g(\theta)}.$$

Then

$$P(X = k | X + Y = n) = \frac{q_k(\alpha) q_{n-k}(\beta)}{q_n(\alpha + \beta)}.$$

## Popoviciu-Sheffer operators

Crăciun (2001) studied generalization of Popoviciu operators:

$$L_n^{Q,S}(f) := \frac{1}{s_n(1)} \sum_{k=0}^n q_k(x) s_{n-k}(1-x) f\left(\frac{k}{n}\right)$$

where  $q_k(x) \geq 0$ ,  $s_k(x) \geq 0$

$$s_n(x+y) = \sum_{k=0}^n q_k(x) s_{n-k}(y)$$

$$\text{cf. } q_n(x+y) = \sum_{k=0}^n q_k(x) q_{n-k}(y)$$

$Q : \mathcal{P} \rightarrow \mathcal{P}$  linear operator with  $Qq_k := q_{k-1}$ .

$S : \mathcal{P} \rightarrow \mathcal{P}$  invertible linear operator with  $Ss_k := q_k$ .

$(s_k)_{k \in \mathbb{N}}$  are called Sheffer sequences

## Examples (continued)

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$$q_k(x) = \frac{x(x+ak)^{k-1}}{k!}$$

(Abel polynomials with delta operator  $DE^{-a}$ )

$X_n^{\alpha,\beta}$  has quasi-binomial distribution.

Corresponding approximation operator: "second operator" introduced by Cheney and Sharma (1964)

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$$s_k(x) = \frac{(x+ka)^k}{k!}$$

(Sheffer sequence for  $Q = DE^{-a}$  and  $S = E^a Q' = I - aD$ )

Corresponding approximation operator: "first operator" introduced by Cheney and Sharma (1964)

## Korovkin theorems

Necessary conditions for  $(L_n f)(x) \rightarrow f(x)$  uniformly in  $x$ :

$$\|L_n x - x\|_\infty \rightarrow 0 \text{ and } \|L_n x^2 - x^2\|_\infty \rightarrow 0.$$

Special case of one of the Korovkin theorems in approximation theory (cf. version of WLLN presented earlier).

**Main problem** : compute  $L_n^{Q,S} x^\ell = \mathbb{E} \left( \frac{X_n^\ell}{n^\ell} \right)$

$$\mathbb{P}(X_n = k) = \frac{q_k(\alpha) s_{n-k}(\beta)}{s_n(\alpha + \beta)}, \quad k = 0, \dots, n$$

**Approach (first  $S = I$ )** : define

$$V_\ell q_n = \sum_{k=0}^n k^{[\ell]} q_k(\alpha) q_{n-k} \Rightarrow \frac{V_\ell q_n(\beta)}{q_n(\alpha + \beta)} = \mathbb{E} \left( \frac{X_n^{[\ell]}}{n^\ell} \right)$$

$k^{[\ell]} = k(k-1)\dots(k-\ell+1)$  (factorial moments)

# Expansions: Umbral Calculus

Use "Taylor" expansion for  $V_\ell$  in terms of  $Q$

$$q_n(x+y) = \sum_{k=0}^n q_k(x) q_{n-k}(y)$$

$Q : \mathcal{P} \rightarrow \mathcal{P}$  linear operator with  $Qq_n := q_{n-1}$ .

$$\sum_{n=0}^{\infty} q_n(x) z^n = e^x g(z)$$

## Operator Expansion Theorem

$$T = \sum_{k=0}^{\infty} [T q_k]_{x=0} Q^k$$

Apply to  $T = D$ :

$$D = \sum_{k=0}^{\infty} (Dq_k)(0) Q^k = g(Q) \Leftrightarrow Q = g^{-1}(D)$$

Apply to  $T = E^\alpha$  (shift-operator):

$$E^\alpha = \sum_{k=0}^{\infty} (E^\alpha q_k)(0) Q^k = e^{\alpha g(Q)} = e^{\alpha D}$$

*Moments:  $S = I, s_n = q_n$*

Apply expansion formulas to  $V_\ell$  (cf. Manole 1987):

$$\begin{aligned}
 V_\ell &= \sum_{k=0}^{\infty} (V_\ell q_k)(0) Q^k \\
 &= \sum_{k=0}^{\infty} k^{[\ell]} q_k(\alpha) Q^k \\
 &= \sum_{k=\ell}^{\infty} k^{[\ell]} q_k(\alpha) Q^k \\
 &= Q^\ell \sum_{k=0}^{\infty} k^{[\ell]} q_k(\alpha) Q^k \\
 &= Q^\ell \frac{d^\ell}{dQ^\ell} \left( \sum_{k=0}^{\infty} q_k(\alpha) Q^k \right) \\
 &= Q^\ell \frac{d^\ell}{dQ^\ell} e^{\alpha g(Q)}
 \end{aligned}$$

- use  $\frac{V_\ell q_n(\beta)}{q_n(\alpha+\beta)} = \mathbb{E} \left( \frac{X_n^{[\ell]}}{n^\ell} \right)$

- use  $g'(Q) = \left( (g^{-1}(D))' \right)^{-1} = (Q')^{-1}$

## *Special case $\ell = 1$*

$$\begin{aligned}
 V_1 q_n(\beta) &= \alpha g'(Q) Q E^\alpha q_n(\beta) \\
 &= \alpha (Q')^{-1} Q E^\alpha q_n(\beta) \\
 &= \alpha E^\alpha \left( (Q')^{-1} q_{n-1} \right) (\beta) \\
 &= \alpha E^\alpha n \frac{q_n(x)}{x} \Big|_{x=\beta} \\
 &= n\alpha \frac{q_n(\alpha + \beta)}{\alpha + \beta}
 \end{aligned}$$

N.B. Rodrigues formula:  $n q_n = (x(Q')^{-1}) q_{n-1}$   
 (equivalent to Lagrange inversion)

$$\mathbb{E} \left( \frac{X_n}{n} \right) = \frac{V_1 q_n(\beta)}{q_n(\alpha + \beta)} = \frac{\alpha}{\alpha + \beta}$$

Similarly:

$$V_2 q_n(\beta) = \alpha \left[ E^\alpha \{g''(Q) + \alpha (g'(Q))^2\} q_{n-2} \right] (\beta)$$

## Moment formulas: general $S$

Same ideas as before, but now use  $S s_n = q_n$   
 + commutativity of operators

$$\begin{aligned}\mathbb{E} \left( X_n^{[\ell]} \right) &= \sum_{k=0}^n k^{[\ell]} \frac{q_k(\alpha) s_{n-k}(\beta)}{s_{n-k}(\alpha + \beta)} \\ &= \frac{V_\ell s_n(\beta)}{s_n(\alpha + \beta)} \\ &= \frac{Q^\ell \frac{d^\ell}{dQ^\ell} e^{\alpha g(Q)} s_n(\beta)}{s_n(\alpha + \beta)}\end{aligned}$$

Special case  $\ell = 1$ :

$$\begin{aligned}\mathbb{E} (X_n) &= \frac{\alpha \left[ E^\alpha (Q')^{-1} s_{n-1} \right] (\beta)}{s_n(\alpha + \beta)} \\ &= \frac{\alpha \left[ E^\alpha S^{-1} \left( \frac{nq_n(x)}{x} \right) \right]_{x=\beta}}{s_n(\alpha + \beta)}\end{aligned}$$

## Extensions: generalized differentiation

$$D_c x^k = \frac{c_k}{c_{k-1}} x^{k-1}$$

shift operator  $E_c^y = \Phi(yD_c)$  with  $\Phi(t) = \sum_{n=0}^{\infty} \frac{t^n}{c_n}$

$$E_c^y q_n(x) = \sum_{k=0}^n q_k(x) q_{n-k}(y)$$

$$\sum_{n=0}^{\infty} q_n(x) t^n = \Phi(xg(t)) \text{ and } Q = g^{-1}(D_c)$$

Choices for  $c_k$ :

- $c_k = k!$  : ordinary differentiation
- $c_k = (1-q) \dots (1-q^k) / (1-q)^k$ :  $q$ -differentiation  
 $D_q p(x) = (p(x) - p(qx)) / (x - qx)$

Special case  $q = 0$ : divided differences

## Divided differences

Case  $c_k = 1$ :  $D_c x^k = x^{k-1}$

$$E_c^y p(x) = \frac{xp(x) - yp(y)}{x - y}$$

$$\mathbb{P}(X_n^{\alpha, \beta} = k) = \frac{(\alpha - \beta) q_k(\alpha) q_{n-k}(\beta)}{\alpha q_k(\alpha) - \beta q_k(\beta)}$$

$$\mathbb{E} \left( \frac{X_n^{[\ell]}}{n^\ell} \right) = \frac{(\alpha - \beta) Q^\ell \frac{d^\ell}{dQ^\ell} \left( \frac{1}{1 - \alpha g(Q)} \right) q_n(\beta)}{\alpha q_k(\alpha) - \beta q_k(\beta)}$$

Open problems:

- Probability distributions of this form??
- Interpretation in terms of conditional distributions??

## *Conclusions*

- we have seen link between WLLN and approximation theory
- Umbral Calculus yields tools to compute moments of certain prob. distributions
- extensions need further investigation (interpretation?)
- ...

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