Abstract for GROSTAT VI
Menton, IUT STID, France
February 17-20, 2003 November 22, 2002

On the conditional independence structures of Gaussian vectors*

František Matúš, Prague†

For the finite set $N = \{1, \dots, n\}$ let $\mathcal{R}$ be the family of all ordered pairs $(ij|K)$ where $K \subseteq N$ is disjoint from ij, the two-element subset $\{i, j\}$ of N. Given a symmetric matrix $A = (a_{i,j})_{i,j \in N}$ of complex numbers, $|A|$ denotes its determinant, $A_{I,J} = (a_{i,j})_{i \in I, j \in J}$, $I, J \subseteq N$, submatrices, and

$$\langle\!\langle A \rangle\!\rangle = \{(ij|K) \in \mathcal{R}\,;\ |A_{\{i\} \cup K, \{j\} \cup K}| = 0\}\ .$$

If A is the covariance matrix of a nondegenerate Gaussian vector $\xi = (\xi_i)_{i \in N}$ then it is well-known that $(ij|K) \in \langle\!\langle A \rangle\!\rangle$ if and only if ξ_i is conditionally independent (CI) of ξ_j given the subvector $\xi_K = (\xi_k)_{k \in K}$.

The aim of this contribution to discuss the problem which subsets $\mathcal{K}$ of $\mathcal{R}$ can be represented as $\mathcal{K} = \langle\!\langle A \rangle\!\rangle$ by means of a matrix A, and related problems on CI-structures of random vectors. The matrix A is assumed to be positive definite, in which case the problem concerns directly the CI in Gaussian vectors, or to belong to a class of symmetric matrices. Depending on the chosen class of matrices the problem concerns semialgebraic or, preferably, algebraic sets of matrices, in which case numerous methods of classical algebraic geometry, supported by computational procedures for Groebner bases, are at disposal [1, 5]. For example, one interesting class of matrices is defined by requiring the principal minors $|A_{I,I}|$ to be nonzero for $\emptyset \neq I \subsetneq N$. In turn, the discrete structures $\langle\!\langle A \rangle\!\rangle$ are within the class of semigraphoids, which has been a standard framework for dealing with CI constraints in statistical models, see [6, 2, 3].

Basic properties of CI in nondegenerate Gaussian vectors will be derived from a single matrix identity. For the class of matrices A that violate each of given polynomial equalities a necessary and sufficient condition for the representation $\mathcal{K} = \langle\!\langle A \rangle\!\rangle$ will be presented. The condition can be practically verified by computation of a single Groebner basis. Possible uses of this result in the problem of the representation of $\mathcal{K}$ by a positive definite matrix will be discussed.

References

[1] D. Cox, J. Little and D. O'Shea (1997) *Ideals, Varieties, and Algorithms.* Undergraduate Texts in Mathematics, Springer: New York.
[2] S.L. Lauritzen (1996) *Graphical Models.* Oxford University Press, Oxford.
[3] F. Matúš (1997) Conditional independence structures examined via minors. *Annals of Mathematics and Artificial Intelligence* **21** 99–128.
[4] F. Matúš (2002) Conditional independences in Gaussian vectors and rings of polynomials. (submitted to Springer-Proc. Workshop "Conditionals, Information, and Inference" Hagen, Germany, May 13-15, 2002)
[5] B. Mishra (1993) *Algorithmic Algebra.* Texts and monographs in Computer Science, Vol. XIV, Springer-Verlag: New York.
[6] J. Pearl (1988) *Probabilistic Reasoning in Intelligent Systems.* Morgan Kaufman: San Meteo, California.

*Based on the manuscript [4] available at `http://www.utia.cas.cz/user_data/matus/fmda.html`.

†This research was supported by the grants A 1075801 of GA AV and 201/01/1482 of GA ČR.

Keywords: Gaussian vectors, positively definite matrices, determinants, principal minors, Schur complement, conditional independence structures, semigraphoids, pseudographoids, geometric theorem proving, Groebner basis.

Permanent postal address: Institute of Information Theory and Automation, Pod vodárenskou věží 4, 182 08 Prague, Czech Republic; `matus@utia.cas.cz`.