Software for Lattice point Enumeration and application to multiway contingency tables

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Outline of this talk

- Setting problem
- Applications
- Some definitions and theorems
- Barvinok's cone decomposition
- Implementation of `LattE`
- Some new results

Getting started...

HOW MANY WAYS are there?

?	?	?	?	?	338106
?	?	?	?	?	574203
?	?	?	?	?	678876
?	?	?	?	?	1213008
20202	142746	410755	1007773	1222717	

What we want to solve

Let $P = \{x \in \Re^d | Ax = a,\ Bx \leq b\}$, where A, B are integral matrices and a, b are integral vectors.

Problem: want to compute the multivariate generating function

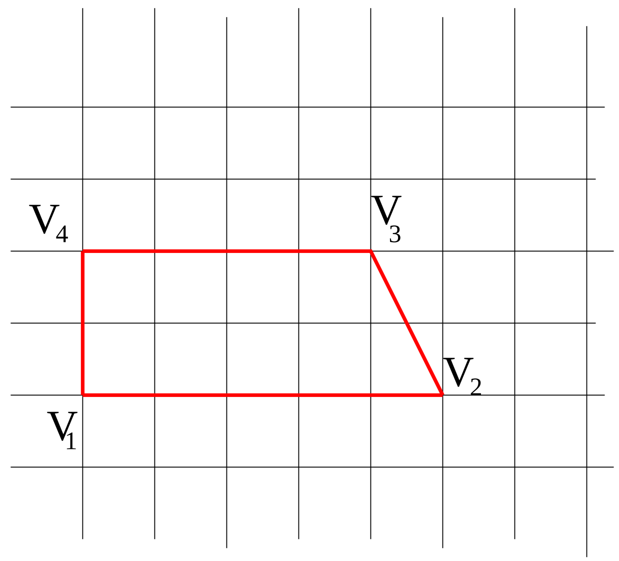
$$f(P) = \sum_{\alpha \in P \cap \mathbb{Z}^d} z^\alpha,$$

where $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_d^{\alpha_d}$.

This is an infinite formal power series if P is not bounded, but if P is a polytope it is a polynomial.

Example for $f(P)$

Let $V_1 = (0, 0)$, $V_2 = (5, 0)$, $V_3 = (4, 2)$, and $V_4 = (0, 2)$.



Each vertex is represented by the following monomials:

For $V_1 = (0,0)$, $z^{V_1} = z_1^0 z_2^0 = 1$.

For $V_2 = (5,0)$, $z^{V_2} = z_1^5 z_2^0 = z_1^5$.

For $V_3 = (4,2)$, $z^{V_3} = z_1^4 z_2^2$.

For $V_4 = (0,2)$, $z^{V_4} = z_1^0 z_2^2 = z_2^2$.

By this way, we have $f(P)$ as the following:

$f(P) = {z_1}^5 + {z_1}^4 z_2 + {z_1}^4 + {z_1}^4 {z_2}^2 + z_2 {z_1}^3 + {z_1}^3 + {z_1}^3 {z_2}^2 + z_2 {z_1}^2 + {z_1}^2 + {z_1}^2 {z_2}^2 + z_1 z_2 + z_1 + z_1 {z_2}^2 + {z_2}^2 + z_2 + 1.$

LattE

- **INPUT:** Integral matrices A, B and integral vectors a, b for the polytope P.
- **OUTPUT**: $f(P)$ written as a sum of rational functions.
- **Application**:

(A) Table counting problem,

(B) Table existence problem,

(C) To compute explicit exact formulas in terms of the given marginals.

Answer to the puzzle

?	?	?	?	?	338106
?	?	?	?	?	574203
?	?	?	?	?	678876
?	?	?	?	?	1213008
20202	142746	410755	1007773	1222717	

316052820930116909459822049052149787748004963058022997262397.

Also, we can obtain the formulas.

Theorem The number $T(3,s)$ of $3 \times 3 \times 3$ integer cubes with sum along rows, columns, etc adding to a constant s is:

$$\begin{cases} \frac{9}{2240}s^8 + \frac{27}{560}s^7 + \frac{87}{320}s^6 + \frac{297}{320}s^5 + \frac{1341}{640}s^4 + \frac{513}{160}s^3 + \frac{3653}{1120}s^2 + \frac{627}{280}s + 1 & \text{if } 2|s, \\ \frac{9}{2240}s^8 + \frac{27}{560}s^7 + \frac{87}{320}s^6 + \frac{297}{320}s^5 + \frac{1341}{640}s^4 + \frac{513}{160}s^3 + \frac{3653}{1120}s^2 + \frac{4071}{2240}s + \frac{47}{128} & \text{otherwise.} \end{cases}$$

Theory behind the implementation

Definition: A tangent cone K_v at a vertex v of P is a cone such that $K_v = \{u \in \Re^d | v + \delta u \in P\}$ for small $\delta > 0$.

Theorem[Brion, Lawrence]

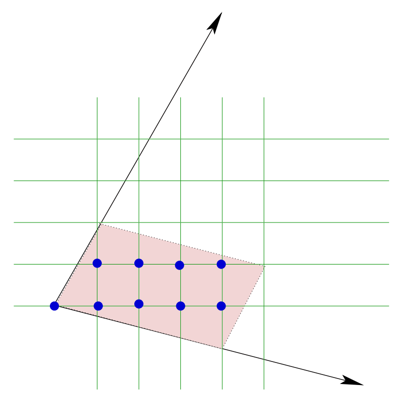
Let P be a convex polytope and let $V(P)$ be the vertex set of P. Let K_v be the tangent cone at $v \in V(P)$. Then

$$f(P) = \sum_{v \in V(P)} f(K_v).$$

For a simplicial cone $K \subset \Re^d$,

$$f(K) = \frac{\sum_{u \in \Pi \cap \mathbb{Z}^d} z^u}{(1 - z^{c_1})(1 - z^{c_2}) \dots (1 - z^{c_d})}$$

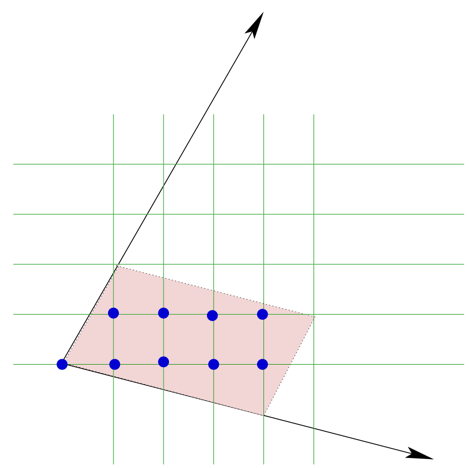
where Π is the open parallelepiped spanned by the rays of the cone K, $c_1, \dots, c_d$.



Example

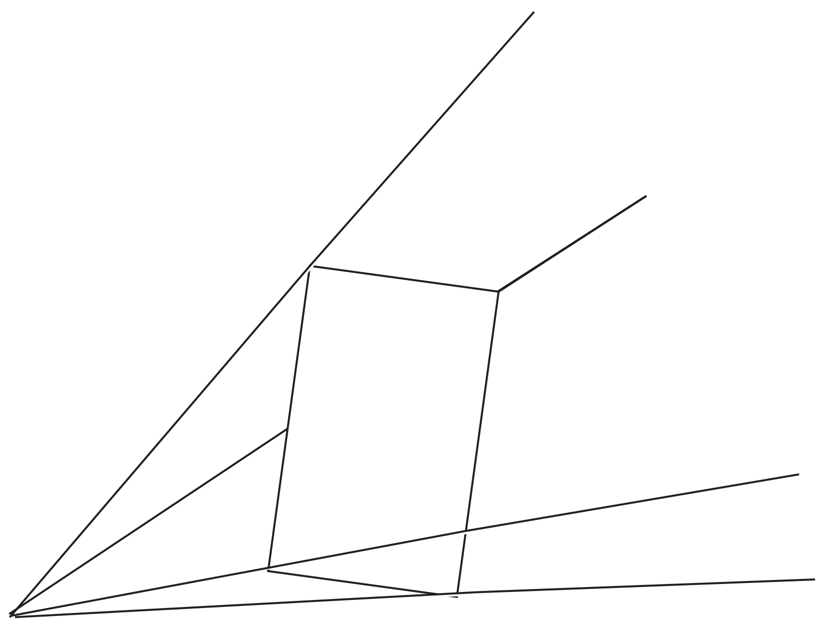
In this case, we have $d = 2$ and $c_1 = (1, 2)$, $c_2 = (5, -1)$. We have:

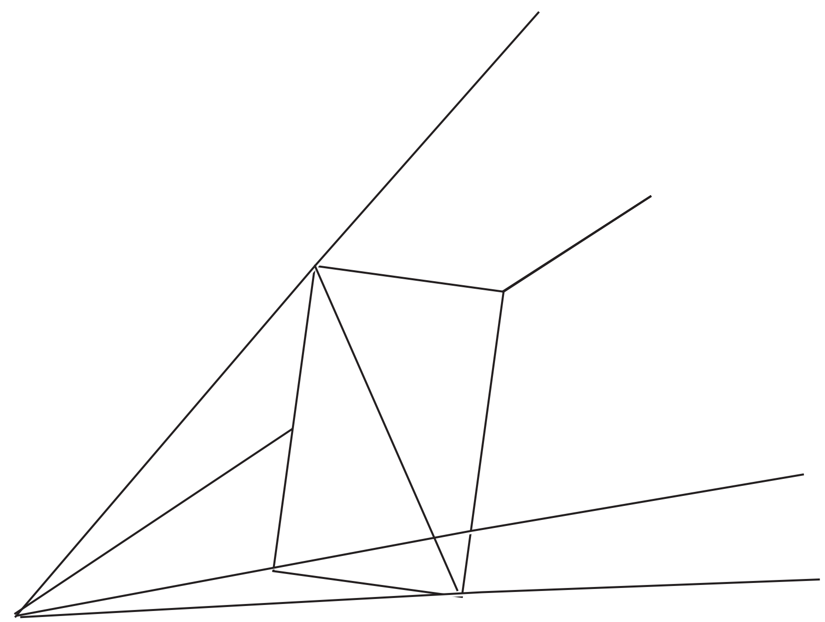
$$f(K) = \frac{z_1^4 z_2 + z_1^3 z_2 + z_1^2 z_2 + z_1 z_2 + z_1^4 + z_1^3 + z_1^2 + z_1 + 1}{(1 - z_1 z_2^2)(1 - z_1^5 z_2^{-1})}.$$



If a cone K is not simplicial....

We Triangulate a cone into simplicial cones.





Fact: If the parallelepiped Π for a simplicial cone K has one integral point u, then

$$f(K) = \frac{z^u}{\prod_{i=1}^{d}(1 - z^{c_i})}.$$

Goal: Want to decompose K into simplicial cones whose parallelepipeds Π have ONLY one integral point.

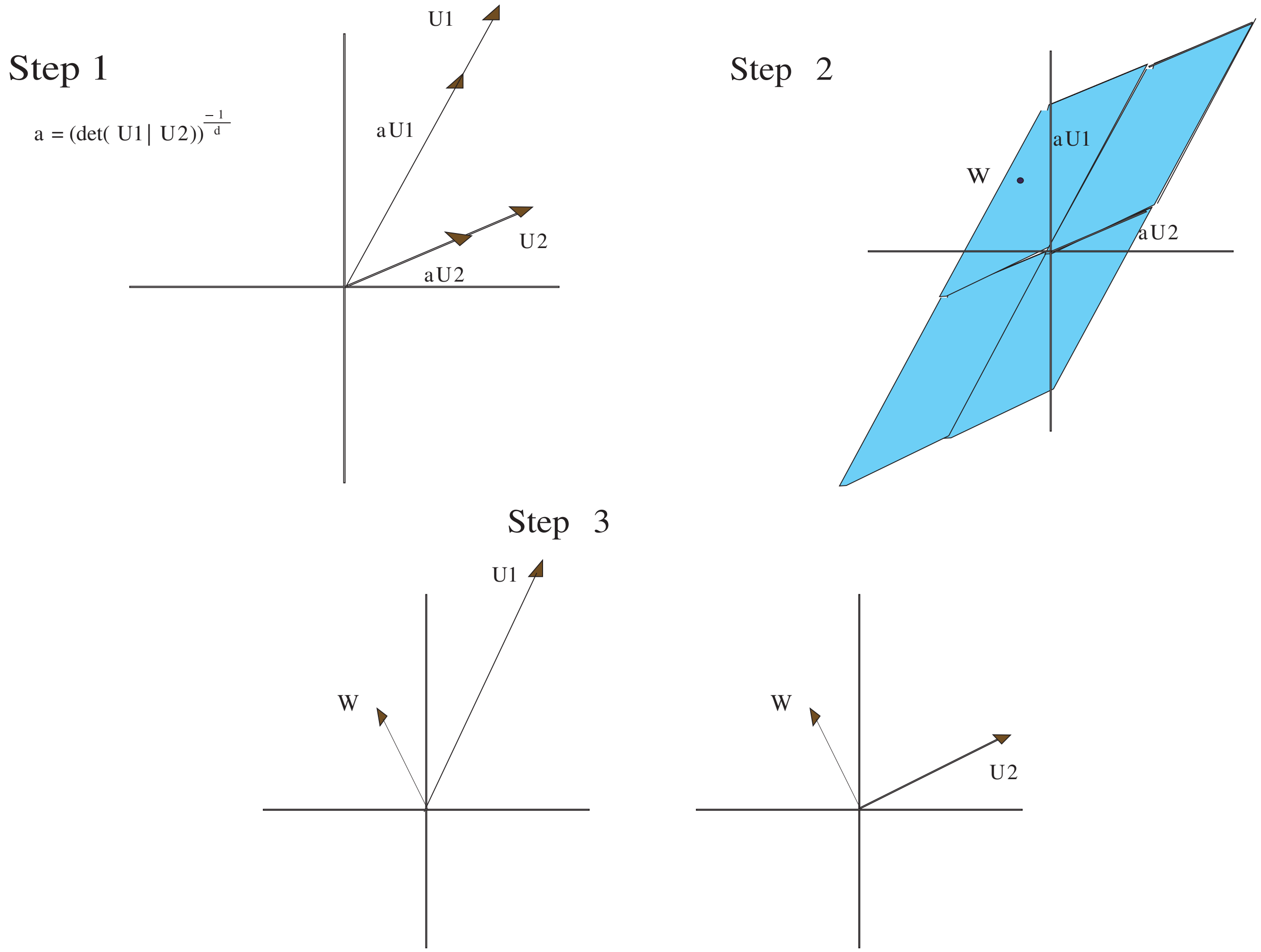
Barvinok's cone decomposition

Definition: A unimodular cone K is a simplicial cone such that the fundamental parallelepiped generated by the rays of K contains only one lattice point.

Theorem [Barvinok] Fix the dimension d. Then there exists a polynomial time algorithm which decomposes a rational polyhedral cone $K \subset \Re^d$ into unimodular cones K_i with numbers $\epsilon_i \in \{-1, 1\}$ such that

$$f(K) = \sum_{i \in I} \epsilon_i f(K_i), \ |I| < \infty.$$

We do this process until all cones become unimodular.



We have so far...

1. Find the vertices of the given polytope and their defining tangent cones.

2. Triangulate and apply Barvinok's cone decomposition to each of the cones.

3. Obtain the rational function with its sign of each cone and sum them up.

We need to care about lower dimensional cones and to use the inclusion-and-exclusion principle to get the multivariate generating function $f(P)$.

Brion's polarization trick

Lemma: Let $K \subset \Re^d$ be a cone. If K contains a straight line then $f(K) \equiv 0$.

By this lemma we can avoid using the inclusion-and-exclusion principle and to save memory and time. The trick is the following:

1. Polarize the tangent cone.

2. Decompose the polar into unimodular cones.

3. Polarize back each of the unimodular cones.

If we do this process, the multivariate function $f(K)$ for lower dimensional cones K become zero.

Outline of the implementation

1. Find the vertices of the given polytope and their defining tangent cones.

2. Compute the polar cone to each of the cones.

3. Triangulate and apply Barvinok's cone decomposition to each of the polar cones.

4. Polarize back each of the full dimensional unimodular cones.

5. Obtain the rational function with its sign of each cone and sum them up.

New results.....

164424	324745	127239
262784	601074	9369116
149654	7618489	1736281

163445	49395	403568
1151824	767866	8313284
1609500	6331023	1563901

184032	123585	269245
886393	6722333	935582
1854344	302366	9075926

Table 1: Marginals for $3 \times 3 \times 3$ example.

The number of ways for the $3 \times 3 \times 3$ tables with 2-marginals displayed above is $22498479001740171525592709675890109772\,93$. The CPU time for this example is 19.67 seconds.

Marginals for the $3 \times 3 \times 4$ example.

273510	273510	273510	191457
273510	273510	547020	191457
273510	547020	273510	191457

464967	273510	273510
547020	273510	464967
410265	601722	273510

273510	273510	273510
410265	547020	136755
547020	136755	410265
191457	191457	191457

Table 2:

We can compute this with LattE...

The number of ways for the first $3 \times 3 \times 4$ tables with 2-marginals displayed in Table 2 is

$4091700129572445106288079361219676736812805058988286839062994$.

The CPU time for this example is 44 minutes 42.22 seconds.

Knapsack/Frobenius problem

A knapsack problem is defined by a single equation and non-negativity conditions.

Mathematically we can set $P = \{x \in \Re^d | ax = b,\ x \geq 0,\ a \in \mathbb{Z}_+^d$, and $b \in \mathbb{Z}_+\}$.

Example: $x + 2y + 3z = 6$, $x \geq 0$, $y \geq 0$, and $z \geq 0$.

Definition: The Frobenius number for a knapsack problem P is the biggest integral right hand side b such that P does not contain an integral point.

We can apply to Frobenius problem.

											Frobenius #	Time (m, s)
cuww1	12223	12224	36674	61119	85569						89643481	0.55s
cuww2	12228	36679	36682	48908	61139	73365					89716838	1.78s
cuww3	12137	24269	36405	36407	48545	60683					58925134	1.27s
cuww4	13211	13212	39638	52844	66060	79268	92482				104723595	2.042s
cuww5	13429	26850	26855	40280	40281	53711	53714	67141			45094583	16.05s
prob1	25067	49300	49717	62124	87608	88025	113673	119169			33367335	47.07s
prob2	11948	23330	30635	44197	92754	123389	136951	140745			14215206	1m0.58s
prob3	39559	61679	79625	99658	133404	137071	159757	173977			58424799	1m28.3s
prob4	48709	55893	62177	65919	86271	87692	102881	109765			60575665	59.04s
prob5	28637	48198	80330	91980	102221	135518	165564	176049			62442884	1m41.78s
prob6	20601	40429	40429	45415	53725	61919	64470	69340	78539	95043	22382774	3m45.86s
prob7	18902	26720	34538	34868	49201	49531	65167	66800	84069	137179	27267751	2m57.64
prob8	17035	45529	48317	48506	86120	100178	112464	115819	125128	129688	21733990	8m29.78s
prob10	45276	70778	86911	92634	97839	125941	134269	141033	147279	153525	106925261	4m24.67

Table 3: Knapsack problems with no integral solution.

cuww1 to cuww5 are generated by Cornuéjols, Urbaniak, Weismantel, and Wolsey (1997) and prob1 to prob10 are generated by Aardal and Lenstra (2002).

We can do more...

We can compute the number of lattice points if we add 1 to the Frobenius number of each problem.

Problem	RHS	# of lattice points.
cuww1	89643482	1
cuww2	89716839	1
cuww3	58925135	2
cuww4	104723596	1
cuww5	45094584	1
prob1	33367336	859202692
prob2	14215207	2047107
prob3	58424800	35534465752
prob4	60575666	63192351
prob5	62442885	21789552314
prob6	22382775	218842
prob7	27267752	4198350819898
prob8	21733991	6743959
prob10	106925262	102401413506276371

Table 4: The number of lattice points if we add 1 to the Frobenius number.

Software

An executable will be available soon and you can download from our website:

`http://www.math.ucdavis.edu/~latte`.

If you want to try your examples, please try a website interface on the homepage above.