Oil fields and Hilbert schemes

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In the realm of polynomial algebra two main ingredients need manipulation and implementation, discrete and continuous data. In particular, a polynomial over the reals or the complex numbers is built on top of a discrete object, the support, and a continuous object, the set of its coefficients. The support is very well understood and has a strong foot inside classical algebra. On the other hand, if the coefficients are not exact, the very notion of a polynomial, and all the classically derived algebraic structures, such as ideals, free resolutions, Hilbert functions, tend to be blurred.

An easy example is the following; consider three distinct non-aligned points in the affine plane over the reals. First of all, it is not clear what is the meaning of *being non-aligned*; a better description might be *being far from aligned*. Now consider the vanishing ideal; it is generated by three quadratic polynomials. However, if we change some of the coefficients of these polynomials by a small amount, almost surely we get the unit ideal, since the first two conics still intersect in four points, but the third will almost certainly miss all of them.

Based on these observations, a new fields of investigation is emerging. We have named it Approximate Commutative Algebra (ApCoA), see

`http://www.ricam.oeaw.ac.at/specsem/srs/groeb/schedule_B1.html`
`http://cocoa.dima.unige.it/conference/apcoa2008/.`

Approximate coefficients and/or continuous families of points may encode experimental data like measures of physical quantities in an **oil field**, see

`http://staff.fim.uni-passau.de/algebraic-oil/en/index.html.`

If we want to use algebraic methods with the goal of building up polynomial models, we face the difficulty of doing good multivariate interpolation (see [1], [2], [3]). To this end, Gröbner bases are not well-suited because of the rigid structure imposed by term orderings. Other objects behave better, are called border bases. They have emerged as good candidates to complement, and in many cases substitute for, Gröbner bases (see [11], [13], [14], [16], [19]). But possibly the most important break through is a recent discovery of a link between border bases and Hilbert schemes. It may provide a solid mathematical foundation for the new emerging field (see [6], [7], [8], [9], [10], [15], [17], [18]).

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