33 and all that

Andrew Booker
University of Bristol

July 2019

Three cubes and a sum

$$\begin{aligned}(-2\,736\,111\,468\,807\,040)^3 + (-8\,778\,405\,442\,862\,239)^3 + 8\,866\,128\,975\,287\,528^3 \\ = \quad -20483367622797158223817952754905569383153664000 \\ -676467453392982277424361019810585360331722557919 \\ \underline{+696950821015779435648178972565490929714876221952} \\ 33\end{aligned}$$

Keeping secrets is hard. . .

At 9:05am GMT on February 27th, a computer in Bristol found the solution to $x^3 + y^3 + z^3 = 33$ shown on the previous slide.

I told several colleagues about it later that day.

Eleven days later, one of them sent me this:

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Dan Fretwell <daniel.fretwell@bristol.ac.uk> Mar 10, 2019, 8:39 AM Reply
to Andrew

Hi Andy,

Just found this online:

https://gilkalai.wordpress.com/2019/03/09/8866128975287528%C2%B3-8778405442862239%C2%B3-2736111468807040%C2%B3/

Is this the same solution as the one you found?

Uh oh.

It got worse from there. . .

Matt Parker
@standupmaths

Follow

#mathsnews: 33 is the sum of three cubes!

(8866128975287528)^3 +
(-8778405442862239)^3 +
(-2736111468807040)^3 = 33
pub.ist.ac.at/%257Etbrownin/

Breakthrough by Tim Browning. Watch when he explained the then-unsolved problem on @numberphile a few years ago:

The Uncracked Problem with 33 - Numberphile
Update March 2019: 8866128975287528^3+(-8778405442862239)^3+(-2736111468807040)^3 33 is the lowest unsolved problem in the world of "summing t...
youtube.com

9:16 am - 9 Mar 2019

239 Retweets **853** Likes

12 239 853

Browning! 😠

I protested:

Andrew Booker <andrew.booker@bristol.ac.uk> Mar 10, 2019, 11:17 AM Reply
to Tim

Dude, what have you done? It's all over the internet that you found the solution to this, e.g.
https://en.wikipedia.org/wiki/Sums_of_three_cubes

(Yes, there was already a Wikipedia article.)

Tim professed his innocence. Eventually we worked it out:

Tim Browning <timdanielbrowning@gmail.com> Mar 10, 2019, 6:17 PM Reply
to Andrew

It looks like it wasn't Brady but my stupid placeholder website: https://pub.ist.ac.at/~tbrownin/
I was just putting up a test page while I got my website ready...

This was Tim's web page at the time:

https://pub.ist.ac.at/~tbrownin/

(8866128975287528)^3+(-8778405442862239)^3+(-2736111468807040)^3

It turns out that this is a good marketing strategy.

NewScientist

Mathematician cracks centuries-old problem about the number 33

Quanta magazine

Sum-of-Three-Cubes Problem Solved for 'Stubborn' Number 33

Newsweek

MATHEMATICIAN SOLVES 64-YEAR-OLD 'DIOPHANTINE PUZZLE'

朝日新聞 DIGITAL

数学者悩ませ64年、難問ついに解けた カギはスパコン

University of BRISTOL

Bristol mathematician cracks Diophantine puzzle

Mirror **Mathematician solves puzzle that's baffled experts for 64 YEARS**

FiveThirtyEight **Significant Digits For Wednesday, March 27, 2019**

3 16-digit integers

For decades, mathematicians have pondered, as mathematicians are wont to do, a pressing question: Can the number 33 be expressed as the sum of three cubes? And now, at long last, to the fanfare of mathematical trumpets and the serenading of mathematical angels, Andrew Booker, a mathematician at the University of Bristol, has provided an answer. Yes it can, in the form of three 16-digits integers: $(8{,}866{,}128{,}975{,}287{,}528)^3 + (-8{,}778{,}405{,}442{,}862{,}239)^3 + (-2{,}736{,}111{,}468{,}807{,}040)^3 = 33$. I'll never forget where I was when I heard the news. [Quanta Magazine]

Meanwhile, on mathoverflow

What are some noteworthy “mic-drop” moments in math?

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Oftentimes in math the **manner** in which a solution to a problem is announced becomes a significant chapter/part of the lore associated with the problem, almost being remembered more than the manner in which the problem was solved. I think that most mathematicians as a whole, even upon solving major open problems, are an extremely humble lot. But as an outsider I appreciate the understated manner in which some results are dropped.

The very recent example that inspired this question:

- Andrew Booker's recent solution to $a^3 + b^3 + c^3 = 33$ with $(a, b, c) \in \mathbb{Z}^3$ as

  $$(a, b, c) = (8866128975287528, -8778405442862239, -2736111468807040)$$

  was publicized on Tim Browning's homepage. However the homepage has merely a single, austere line, and does not even indicate that this is/was a semi-famous open problem. Nor was there any indication that the cubes actually sum to 33, apparently leaving it as an exercise for the reader.

Other examples that come to mind include:

- In 1976 after Appel and Hakken had proved the Four Color Theorem, Appel wrote on the University of Illinois' math department blackboard "Modulo careful checking, it appears that four colors suffice." The statement "Four Colors Suffice" was used as the stamp for the University of Illinois at least around 1976.
- In 1697 Newton famously offered an "anonymous solution" to the Royal Society to the Brachistochrone problem that took him a mere evening/sleepless night to resolve. I think the story is noteworthy also because Johanne Bernoulli is said "recognized the lion by his paw."
- As close to a literal "mic-drop" as I can think of, after noting in his 1993 lectures that Fermat's

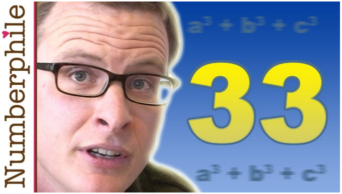
A number which will live in infamy

Bjorn Poonen, *Undecidability in Number Theory*,
AMS Notices, March 2008:

"Does the equation $x^3 + y^3 + z^3 = 29$ have a solution in integers?
Yes: $(3, 1, 1)$, for instance.
How about the equation $x^3 + y^3 + z^3 = 30$?
Again yes, although this was not known until 1999: the smallest solution is $(-283059965, -2218888517, 2220422932)$.
And how about $x^3 + y^3 + z^3 = 33$?
This is an unsolved problem."

YouTube

Numberphile



History

Ryley (1825): $x = \left(\frac{27x^3 - y^9}{3y^2(9x^2+3xy^3+y^6)}\right)^3 + \left(\frac{-27x^3+9xy^6+y^9}{3y^2(9x^2+3xy^3+y^6)}\right)^3 + \left(\frac{3xy(3x+y^3)}{9x^2+3xy^3+y^6}\right)^3$

Mordell (1953): $x^3 + y^3 + z^3 = 3$ other than $(1,1,1)$, $(4,4,-5)$?

Miller and Woolett (1955): Searched for solutions to $x^3 + y^3 + z^3 = k$ for $0 < k \le 100$ using the EDSAC at Cambridge

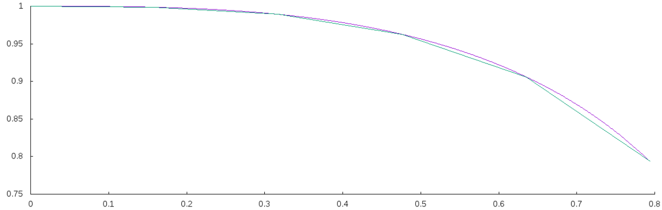
Gardiner, Lazarus, and Stein (1964): Found one more $k \le 100$

Heath-Brown (1992): Conjectured solutions exist $\forall k \not\equiv \pm 4 \pmod 9$

Heath-Brown, Lionen, and te Riele (1993)
Conn and Vaserstein (1994)
Koyama (1994), (1995)
Bremner (1995)
Koyama, Tsuruoka, and Sekigawa (1997)
Elkies (2000)
Bernstein (2001)
Beck, Pine, Tarrant, and Yarbrough Jensen (2007)
Elsenhans and Jahnel (2009)
Huisman (2016): Found all solutions for $k < 1000$ with $\max\{|x|, |y|, |z|\} \le 10^{15}$

Elkies' algorithm

Elkies (1996) described an algorithm to find all $(x, y, z) \in \mathbb{Z}^3$ with $\max\{|x|, |y|, |z|\} \le B$ and $|x^3 + y^3 + z^3| \le B$ in time $O(B \log^c B)$.
His observation is that we can rewrite $x^3 + y^3 + z^3 = k$ as $\left(-\frac{x}{z}\right)^3 + \left(-\frac{y}{z}\right)^3 = 1 - \frac{k}{z^3}$, so $\left(-\frac{x}{z}, -\frac{y}{z}\right)$ is a rational point "near" the Fermat cubic $X^3 + Y^3 = 1$ (within distance $O(B^{-2})$).
To find these points, he breaks $[0, 1/\sqrt[3]{2}]$ into $\asymp B$ subintervals of size $\asymp \frac{1}{B}$ and computes linear approximations to the curve on each.



If $(X, Y) = \left(\frac{x}{z}, \frac{y}{z}\right)$ is a point of height $O(B)$ within distance $O(B^{-2})$ of one of the line segments, then (x, y, z) lies in a certain parallelopiped of side lengths $O(1)$, $O(B^{-1})$, and $O(B)$.
Finally, apply LLL to find the integer points.

A little algebra

Suppose that $x^3+y^3+z^3=k$, with $|x|\ge|y|\ge|z|$. Then

$$k-z^3=x^3+y^3=(x+y)(x^2-xy+y^2).$$

Writing $d=|x+y|=|x|+y\operatorname{sgn}x$, we have

$$\frac{|k-z^3|}{d}=x^2-xy+y^2=3x^2-3d|x|+d^2,$$

so that

$$\{x,y\}=\left\{\frac{1}{2}\operatorname{sgn}(k-z^3)\left(d\pm\sqrt{\frac{4|k-z^3|-d^3}{3d}}\right)\right\}.$$

Given a candidate value of z, we can try all $d>0$ dividing $|k-z^3|$. This finds all solutions to $x^3+y^3+z^3=k$ with $\min\{|x|,|y|,|z|\}\le B$ in (heuristic) time $O(B^{1+\varepsilon})$.

A better algorithm

Factoring might be subexponential, but it's expensive in practice.

So instead of running through z and solving for $d \mid (k - z^3)$, it's better to run through d and solve for z satisfying $z^3 \equiv k \pmod{d}$. With the Chinese remainder theorem and Hensel's lemma, this can be reduced to finding solutions to $z^3 \equiv k \pmod{p}$ for primes $p \mid d$.

In the particular case $k \equiv 3\epsilon \pmod{9}$ for $\epsilon \in \{\pm 1\}$, we have $x \equiv y \equiv z \equiv \epsilon \pmod{3}$, and it follows that $\operatorname{sgn} z = \epsilon \left(\frac{d}{3}\right)$. That leads to the following system:

$$\frac{d}{\sqrt[3]{2} - 1} < |z| \le B, \quad \operatorname{sgn} z = \epsilon \left(\frac{d}{3}\right), \quad z^3 \equiv k \pmod{d},$$
$$3d \left(4\epsilon \left(\frac{d}{3}\right)(z^3 - k) - d^3\right) = \square.$$

Also, some congruence constraints come for free, e.g. $z \equiv \frac{4}{3}k(2 - d^2) + 9(k + d) \pmod{18}$.

Complexity analysis

Even with the noted optimizations, there are $\gg B \log B$ candidate pairs (d, z) satisfying the first line of the system.

To get better than $O(B \log B)$ running time, we use a time-space tradeoff: If $\Delta = 3d\left(4\epsilon\left(\frac{d}{3}\right)(z^3 - k) - d^3\right)$ is a square then $\left(\frac{\Delta}{p}\right) \in \{0, 1\}$ for any odd prime p. Setting $M = \prod_{5 \le p \le P} p$ for some auxiliary parameter P, we can restrict to the residue classes of $z \pmod{M}$ satisfying this criterion for all $p \mid M$. This comes with $O(M)$ setup cost, but typically reduces the number of z by a factor of $2^{-\omega(M)}$.

Optimally choosing $P \asymp \log\log B \log\log\log B$, we get a total (heuristic) running time of $O\big(B \log\log B \log\log\log B\big)$.

There are many practical issues: 64-bit arithmetic, Montgomery multiplication, fast cube roots mod p, fast sieving for primes, ...

What's next?

The only remaining $k \leq 100$ with no local obstructions and no known solutions is. . . 42. I searched for solutions with $\min\{|x|, |y|, |z|\} \leq 10^{16}$ without success.

Mordell's question about solutions for $k = 3$ remains open.

When I shared the news with Heath-Brown on Feb 27th, he asked "What about $x^3 + y^3 + 2z^3$?"

Drew Sutherland and I are working on these, with help from our friends at charityengine.

Oh, by the way:

$795 = (-14\,219\,049\,725\,358\,227)^3 + 14\,197\,965\,759\,741\,571^3 + 2\,337\,348\,783\,323\,923^3.$