Minimizing GCD sums and applications

joint work with Marc Munsch and Gérald Tenenbaum

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1. Previously in Gál sums : Large values

One traditionally defines the Gál sum

$$S(\mathcal{M}) := \sum_{m,n \in \mathcal{M}} \frac{(m,n)}{\sqrt{mn}},$$

where (m,n) denotes the greatest common divisor of m and n.

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Key point : no bound on the size of $m \in \mathcal{M}$, only bound on the size of $|\mathcal{M}|$

Improving Bondarenko and Seip ('15, '17), Tenenbaum and dlB proved when N tends to infinity,

$$\max_{|\mathcal{M}|=N} \frac{S(\mathcal{M})}{|\mathcal{M}|} = \mathcal{L}(N)^{2\sqrt{2}+o(1)},$$

with

$$\mathcal{L}(N) := \exp\left\{\sqrt{\frac{\log N \log_3 N}{\log_2 N}}\right\},$$

where we denote by $\log_k$ the k-th iterated logarithm. Gain : $2\sqrt{2}$.

The same estimate holds also for

$$Q(\mathcal{M}) := \sup_{\substack{\boldsymbol{c}\in\mathbb{C}^N \\ \|\boldsymbol{c}\|_2=1}} \left| \sum_{m,n\in\mathcal{M}} c_m \overline{c_n} \frac{(m,n)}{\sqrt{mn}} \right| = \mathcal{L}(N)^{2\sqrt{2}+o(1)}.$$

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First application

Let be

$$Z_\beta(T) := \max_{T^\beta \leqslant \tau \leqslant T} \left|\zeta\left(\tfrac{1}{2}+i\tau\right)\right| \qquad (0 \leqslant \beta < 1,\ T \geqslant 1)$$

Tenenbaum and dlB proved

$$Z_\beta(T) \geqslant \mathcal{L}(T)^{\sqrt{2(1-\beta)}+o(1)}.$$

Improvement of Bondarenko and Seip by a $\sqrt{2}$ extra factor

Second application

$$L(s,\chi) := \sum_{n\geqslant 1} \frac{\chi(n)}{n^s} \qquad (\chi \neq \chi_0,\ \Re e\,(s) > 0).$$

When q is prime and tends to ∞, Tenenbaum and dlB obtained

$$\max_{\substack{\chi \bmod q \\ \chi\neq\chi_0 \\ \chi(-1)=1}} \left|L(\tfrac{1}{2},\chi)\right| \geqslant \mathfrak{L}(q)^{1+o(1)} = \exp\left\{(1+o(1))\sqrt{\frac{\log q \log_3 q}{\log_2 q}}\right\}.$$

To compare with Hough's theorems ('16), a $\sqrt{\log_3 q}$ extra factor

Third application

Let be

$$S(x,\chi) := \sum_{n\leqslant x} \chi(n), \qquad \Delta(x,q) := \max_{\substack{\chi\neq\chi_0 \\ \chi \bmod q}} \left|S(x,\chi)\right|,$$

When $\mathrm{e}^{(\log q)^{1/2+\varepsilon}} \leqslant x \leqslant q/\mathrm{e}^{(1+\varepsilon)\omega(q)}$, Tenenbaum and dlB had

$$\Delta(x,q) \gg \sqrt{x}\mathcal{L}(3q/x)^{\sqrt{2}+o(1)} \qquad (q\to\infty).$$

Improvement of Hough by an extra factor $\sqrt{\log_3(3q/x)}$.

Valid not only for q prime.

2. Small Gál sums

We define

$$\mathcal{T}(\boldsymbol{c};N) := \sum_{m,n\leqslant N} \frac{(m,n)}{\sqrt{mn}} c_m c_n, \quad \mathcal{T}_N := N \inf_{\substack{\boldsymbol{c}\in(\mathbb{R}^+)^N\\ \|\boldsymbol{c}\|_1=1}} \mathcal{T}(\boldsymbol{c};N),$$

$$\mathcal{V}(\boldsymbol{c};N) := \sum_{m,n\leqslant N} \frac{(m,n)}{m+n} c_m c_n, \quad \mathcal{V}_N := N \inf_{\substack{\boldsymbol{c}\in(\mathbb{R}^+)^N\\ \|\boldsymbol{c}\|_1=1}} \mathcal{V}(\boldsymbol{c};N),$$

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Theorem 1 (BMT ’19). *Let be* $\eta := 0.16656\ldots < 1/6$. *There exists* $c > 0$ *such that*

$$(\log N)^\eta \ll \mathcal{V}_N \leqslant \tfrac{1}{2}\mathcal{T}_N \ll (\log N)^\eta L(N)^c$$

with $L(N) := \mathrm{e}^{\sqrt{\log_2 N}}$.

Application : Improvement of Burgess' bound

Let

$$S(M,N;\chi) := \sum_{M<n\leqslant M+N} \chi(n),$$

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Burgess proved the following inequality

$$(*) \qquad S(M,N;\chi) \ll N^{1-1/r} p^{(r+1)/4r^2} (\log p)^{b} \qquad (r \geqslant 1)$$

with $b = 1$. It is non trivial for $N > p^{1/4+\varepsilon}$.

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Recently, Kerr, Shparlinski and Yau proved (*) for $b = \frac{1}{4r} + o(1)$.

Theorem 2 (BMT'19). *For* $r \geqslant 1$, $p \leqslant \frac{1}{2}N$

$$S(M,N;\chi) \ll N^{1-1/r} p^{(r+1)/4r^2} \max_{1\leqslant x\leqslant p} \mathcal{T}_x^{1/2r}.$$

Hence we have (*) for $b = \frac{\eta}{2r} + o(1)$.

Let us consider the weighted version of the multiplicative energy

$$\mathcal{E}(\boldsymbol{c};N) := \sum_{1\leqslant n\leqslant N^2}\Bigg(\sum_{\substack{dt=n\\ d,t\leqslant N}} c_d c_t\Bigg)^2 = \sum_{\substack{1\leqslant d_1,t_1,d_2,t_2\leqslant N\\ d_1t_1=d_2t_2}} c_{d_1}c_{t_1}c_{d_2}c_{t_2}$$

and define

$$\mathcal{E}_N := \inf_{\substack{\boldsymbol{c}\in(\mathbb{R}^+)^N\\ \|\boldsymbol{c}\|_1=1}} N^2\mathcal{E}(\boldsymbol{c};N).$$

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Let $\delta := 1 - (1 + \log_2 2)/\log 2 \approx 0.08607$. Appears in table multiplication problem (Hall, Tenenbaum '88 and Ford '06)

$$H(N) := \left|\{n \leqslant N^2 \quad \exists a, b \leqslant N \quad n = ab\}\right| \asymp \frac{N^2}{(\log N)^{\delta} (\log_2 N)^{3/2}}.$$

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Theorem 3 (BMT'19). *For $N\geqslant 3$ and suitable constant c, we have*

$$(\log N)^{\delta}(\log_2 N)^{3/2} \ll \mathcal{E}_N \ll (\log N)^{\delta}L(N)^c.$$

First application : Non vanishing of theta functions

Balasubramanian and Murty '92 proved that a positive proportion of characters verify $L(\frac{1}{2}, \chi) \neq 0$. We consider

$$\vartheta(x;\chi) = \sum_{n\geqslant 1} \chi(n) \mathrm{e}^{-\pi n^2 x/p} \quad (\chi \in X_p^+ = \{\chi \bmod p : \chi \neq \chi_0,\ \chi(-1) = 1\}).$$

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$$\tau(\overline{\chi})\vartheta(x;\chi) = (q/x)^{1/2}\vartheta(1/x;\overline{\chi})$$

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Let $M_0(p)=\{\chi \bmod p:\quad \chi\neq\chi_0,\quad \chi(-1)=1,\quad \vartheta(1;\chi)\neq 0\}.$

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Let $M_0(p)=\{\chi \bmod p\ :\quad \chi\neq\chi_0,\quad \chi(-1)=1,\quad \vartheta(1;\chi)\neq 0\}$.
Louboutin conjectured $M_0(p)=\frac{1}{2}(p-1)$. Checked for $3\leqslant p\leqslant 10^6$ by Molin.
Louboutin and Munsch '13 showed that $M_0(p)\gg p/\log p$.

Theorem 4 (BMT'19). *With* $\delta:=1-(1+\log_2 2)/\log 2\approx 0.08607$, *we have*

$$M_0(p)\gg\frac{p}{\mathcal{E}_{\lfloor\sqrt{q/3}\rfloor}}\gg\frac{p}{(\log p)^{\delta}L(p)^{c}}.$$

Second application : Lower bounds for low moments of character sums

Recently, Harper '17 announced

$$\frac{1}{p-2}\sum_{\chi\neq\chi_0}\Big|\sum_{n\leqslant N}\chi(n)\Big| \ll \frac{\sqrt{N}}{\min\left(\log_2 L, \log_3 6p\right)^{1/4}}$$

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where $L := \min\{N, p/N\}$. *More than squareroot cancellation !*

Theorem 5 (BMT'19). *Let* $r \in]0, 4/3[$ *be fixed. For sufficiently large* p *and all* $N \in [1, \sqrt{p}[$, *we have*

$$\frac{1}{p-2}\sum_{\chi\neq\chi_0}\Big|S(N;\chi)\Big|^r \gg \frac{N^{r/2}}{\mathcal{E}_N^{1-r/2}}.$$

In particular, for a suitable constant c,

$$\frac{1}{p-2}\sum_{\chi\neq\chi_0}\Big|S(N;\chi)\Big| \gg \sqrt{\frac{N}{\mathcal{E}_N}} \gg \frac{\sqrt{N}}{(\log N)^{\delta/2}L(N)^c}.$$

Note $\frac{1}{2}\delta \approx 0,04303$.

Same result when $T \geqslant 1$, $1 \leqslant N \leqslant \sqrt{T}$ for

$$\frac{1}{T}\int_0^T \Big|\sum_{n\leqslant N} n^{it}\Big|^r \mathrm{d}t \gg \frac{N^{r/2}}{\mathcal{E}_N^{1-r/2}}.$$

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3. Proofs for $\mathcal{E}(\boldsymbol{c}\,;\mathrm{N})$

The lower bound immediately follows from the Cauchy-Schwarz inequality. Indeed, defining $r(n) := \sum_{\substack{dt=n\\ d,t\leqslant N}} c_d c_t$, we have

$$\|\boldsymbol{c}\|_1^4 = \Big(\sum_{n\leqslant N^2} r(n)\Big)^2 \leqslant H(N)\sum_{n\leqslant N^2} r(n)^2 = H(N)\mathcal{E}(\boldsymbol{c}; N)$$

where $H(N) := \big|\{n \leqslant N^2 : \quad \exists a, b \leqslant N \quad n = ab\}\big|$.
Following Ford'06, we have

$$H(N) \ll N^2/\{(\log N)^{\delta}(\log_2 N)^{3/2}\}.$$

To establish the upper bound, select $m \mapsto c_m$ as the indicator function of the set of those integers $m \in \left]\frac{1}{2}N, N\right]$ satisfying $\Omega(m) = \left\lfloor \frac{1}{\log 4} \log_2 N \right\rfloor$

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$$\Omega(m; t) \leqslant \frac{1}{\log 4} \log_2(3t) + C\sqrt{\log_2 N} \qquad (1 \leqslant t \leqslant N).$$

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We have

$$\sum_{n \leqslant N^2} r(n) = \|\boldsymbol{c}\|_1^2 \gg \frac{N^2}{(\log N)^{\delta} \log_2 N}.$$

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We get

$$\frac{N^2}{\|\boldsymbol{c}\|_1^4} \mathcal{E}(\boldsymbol{c}; N) = \frac{N^2}{\|\boldsymbol{c}\|_1^4} \sum_{n \leqslant N^2} r(n)^2 \ll \frac{N^4 L(N)^c}{\|\boldsymbol{c}\|_1^4 (\log N)^{\delta}}.$$

The last upper bound was proved in the book Divisors '88 by Hall and Tenenbaum.

Proofs for $\mathcal{T}(\boldsymbol{c};\mathbf{N})$. To establish the upper bound, select $m \mapsto c_m$ as the indicator function of the set of $m \in]\frac{1}{2}N, N]$ satisfying $\Omega(m) = \lfloor \beta \log_2 N \rfloor$ satisfying $\Omega(m;t) \leqslant \beta \log_2(3t) + C\sqrt{\log_2 N} \qquad (1 \leqslant t \leqslant N)$.

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Using the convolution identity $(m,n) = \sum_{d|m,n} \varphi(d)$, we get

$$\mathcal{T}(\boldsymbol{c}; N) = \sum_{m,n \leqslant N} \frac{(m,n)}{\sqrt{mn}} c_m c_n = \sum_{d \leqslant N} \frac{\varphi(d)}{d} x_d^2,$$

with $x_d := \sum_{m \leqslant N/d} \frac{c_{md}}{\sqrt{m}}$.

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with $x_d := \sum_{m \leqslant N/d} \frac{c_{md}}{\sqrt{m}}$.
Let $\beta \in]0,1[$ be an absolue constant. For all $y,z \in]\beta,1]$ and suitable $c = c(\beta)$, we may write $x_d \leqslant L(N)^{c/2} U_d$ with

$$U_d \leqslant \begin{cases} \displaystyle\sum_{m \leqslant N/d} \frac{y^{\Omega(md)} z^{\Omega(md,d)}}{\sqrt{m}(\log N)^{\alpha \log y} (\log 2d)^{\alpha \log z}} & \text{if } d \leqslant \sqrt{N}, \\ \displaystyle\sum_{m \leqslant N/d} \frac{y^{\Omega(md)} z^{\Omega(md,N/d)}}{\sqrt{m}(\log N)^{\alpha \log y} (\log 2N/d)^{\alpha \log z}} & \text{if } \sqrt{N} < d \leqslant N. \end{cases}$$

Proof of Theorem 5 for $r = 1$. Given $\boldsymbol{c} \in (\mathbb{R}^+)^N$, we define

$$M(N;\chi) = \sum_{m \leqslant N} c_m \overline{\chi(m)}.$$

Let us put

$$\mathfrak{S}_k(N) := \frac{1}{p-1} \sum_{\chi \neq \chi_0} |S(N;\chi)|^k \quad (k > 0), \quad \mathfrak{M}_4(N) := \frac{1}{p-1} \sum_{\chi \neq \chi_0} |M(N;\chi)|^4.$$

Applying Hölder's inequality, we get

$$\|\boldsymbol{c}\|_1 \ll \frac{1}{p-1} \left| \sum_{\chi \neq \chi_0} S(N;\chi) M(N;\chi) \right| \leqslant \mathfrak{S}_1(N)^{1/2} \mathfrak{S}_2(N)^{1/4} \mathfrak{M}_4(N)^{1/4}.$$

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Orthogonality relations yield that $\mathfrak{S}_2(N) \ll N$ and $\mathfrak{M}_4(N) \ll \mathcal{E}(\boldsymbol{c}; N)$. By choosing $\boldsymbol{c}$ optimally, we deduce

$$\mathfrak{S}_1(N) \gg \frac{\|\boldsymbol{c}\|_1^2}{\mathcal{E}(\boldsymbol{c}; N)^{1/2} N^{1/2}} \gg \frac{N^{1/2}}{\mathcal{E}_N^{1/2}}.$$

Thank you for your attention !