Moments of cubic twists and averages of cubic Gauss sums over function fields.

Joint work with A. Florea and M. Lalin

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Cubic characters over $\mathbb{Q}$

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$$\chi_p(a) = \left(\frac{a}{p}\right)_3 \equiv a^{(p-1)/3} \mod p.$$

The cubic characters $\chi_p, \overline{\chi_p}$ take values in $\mu_3 \subseteq \mathbb{C}^*$ by fixing an isomorphism between the cube roots of unity in $(\mathbb{Z}/p\mathbb{Z})^*$ with μ_3.

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Extending by multiplicativity, we get the cubic characters over $\mathbb{Q}$.

Cubic characters over $\mathbb{Q}$

Let $a(n)$ is the number of primitive characters of conductor n co-prime to 3. Then,

$$G(s) = \sum_{d=1}^{\infty} \frac{a(n)}{n^s} = \prod_{p \equiv 1 \pmod 3} \left(1 + \frac{2}{p^s}\right),$$

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and

$$\#\mathcal{C}(X) \sim c_3 X$$

where

$$c_3 = \frac{11\sqrt{3}}{20\pi} \prod_{p \equiv 1 \pmod 3} \frac{(p+2)(p-1)}{p(p+1)}.$$

Cubic Dirichlet twists

For a primitive cubic character χ of conductor h, let

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$$

be the Dirichlet L-function. It has analytic continuation for all $s \in \mathbb{C}$, and functional equation

$$\Lambda(s,\chi) = \frac{G(\chi)}{\sqrt{h}} \Lambda(1-s,\overline{\chi}),$$

where

$$\Lambda(s,\chi) = \left(\frac{\pi}{h}\right)^{-s/2} \Gamma(s/2) L(s,\chi)$$

and $G(\chi)$ is the Gauss sum

$$G(\chi) = \sum_{a \,(\mathrm{mod}\, h)} \chi(a) e\left(\frac{a}{h}\right), \qquad e(x) = e^{2\pi i x}.$$

Moments of cubic Dirichlet twists

Let k, ℓ be non-negative integers. We define

$$\left\langle L(1/2,\chi)^k \overline{L(1/2,\chi)}^{\ell} \right\rangle_X := \frac{1}{\#\mathcal{C}(X)} \sum_{\chi \in \mathcal{C}(X)} L(1/2,\chi)^k \overline{L(1/2,\chi)}^{\ell}.$$

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It follows from the conjectures of Keating and Snaith that

$$\left\langle L(1/2,\chi)^k \overline{L(1/2,\chi)}^\ell \right\rangle_X \sim \frac{g_{k,\ell} a_{k,\ell}}{k!\ell!} (\log X)^{k\ell},$$

$$\text{where} \quad g_{k,\ell} = k!\ell! \, \frac{\prod_{h=1}^{k} h! \prod_{h=1}^{\ell} h!}{\prod_{h=1}^{k+\ell-1} h!}.$$

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The family of cubic Dirichlet twists is a unitary family.

Moments of cubic Dirichlet twists

Using the "recipe" of Conrey, Farmer, Keating, Rubinstein and Snaith, one can obtain the more precise conjecture

$$\left\langle L(1/2,\chi)^k \overline{L(1/2,\chi)}^\ell \right\rangle_X \sim \frac{g_{k,\ell} a_{k,\ell}}{k!\ell!} P_{k,\ell}(\log X),$$

where $P_{k,\ell}(X)$ is a monic polynomial of degree $k\ell$, and the arithmetic factor $a_{k,\ell}$ depends on the family.

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See "Conjectures for moments of cubic twists of elliptic curves L-functions" by David, Lalin and Nam, where we also give a formula for the coefficient of $(\log X)^{k\ell-1}$.

First moment of cubic Dirichlet twists

Theorem

(Baier and Young, 2010)

Let $w : (0, \infty) \to \mathbb{R}$ *be a smooth compactly supported function. Then*

$$\sum_{\chi \in \mathcal{C}(X)} L(1/2, \chi) w\left(\frac{q}{X}\right) = c\hat{w}(0)X + \left(X^{37/38+\varepsilon}\right).$$

for some explicit constant $c > 0$.

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for some explicit constant $c > 0$.

This is the case $k = 1$, $\ell = 0$, and $k\ell = 0$, which gives a polynomial of degree 0 when dividing by $\#\mathcal{C}(X)$.

Moments of quadratic Dirichlet twists

For the case of **quadratic** characters over $\mathbb{Q}$, the first moment was computed by Jutila, the second and third moments by Soundararajan, and the fourth moment by Shen under GRH (July 2, 2019).

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Over function fields, the first fourth moments were computed by Florea, and the results over function fields improve the number fields results (better error terms, a secondary term for the first moment, and more terms of the polynomial $P_4(\log X)$ for the fourth moment.

Cubic characters over $\mathbb{Q}(\sqrt{-3})$

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$$\chi_\pi(a) = \left(\frac{a}{\pi}\right)_3 \equiv a^{(N(\pi)-1)/3} \mod \pi.$$

There are 2 primitive cubic characters of conductor π, χ_π and $\overline{\chi}_\pi = \chi_\pi^2 = \chi_{\pi^2}$, and this choice is canonical.

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Extending by multiplicativity, we get all character modulo $d \in \mathcal{O}_K$.

Moments of cubic Dirichlet twists over $\mathbb{Q}(\sqrt{-3})$

Let $a_K(d)$ be the number of primitive characters of conductor d defined over K. Then,

$$G_K(s) = \sum_{\substack{d \in \mathcal{O}_K \\ d \equiv 1 \pmod 3}} \frac{a_K(d)}{N(d)^s} = \prod_{\pi \equiv 1 \pmod 3} \left(1 + \frac{2}{N(\pi)^s}\right),$$

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$$\#\mathcal{C}_K(X) \sim a_1 X \log X + a_0 X.$$

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Theorem

(Luo, 2004) As $X \to \infty$ and for some explicit $A > 0$,

$$\sum_{\substack{c \in \mathcal{O}_K \\ c \text{ square-free} \\ c \equiv 1 \pmod 9}} L(1/2, \chi_c) e\left(-N(c)/X\right) = AX + O\left(X^{21/22+\varepsilon}\right).$$

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This give a polynomial of degree zero when dividing by $\#\mathcal{L}(X) \sim cX$, which is density zero sub-family of $\mathcal{C}_K(X)$.

Number fields and function fields

Let q power of a prime, $\mathbb{F}_q$ finite field with q elements.

Number Fields		Function Fields
$\mathbb{Q}$	$\leftrightarrow$	$\mathbb{F}_q(T)$
$\mathbb{Z}$	$\leftrightarrow$	$\mathbb{F}_q[T]$
p positive prime	$\leftrightarrow$	$P(T)$ monic irreducible polynomial
$\lvert n\rvert = \lvert\mathbb{Z}/n\mathbb{Z}\rvert = n \in \mathbb{Z}_{\geq 0}$	$\leftrightarrow$	$\lvert F(T)\rvert = \lvert\mathbb{F}_q[T]/(F(T))\rvert = q^{\deg F}$
$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$	$\leftrightarrow$	$\zeta_q(s) = \sum_{\substack{F \in \mathbb{F}_q[T] \\ F \text{ monic}}} \frac{1}{\lvert F\rvert^s} = (1 - qq^{-s})^{-1}$
Riemann Hypothesis ???	$\leftrightarrow$	Riemann Hypothesis !!!

Moments of cubic Dirichlet twists over function fields

Let $\mathbb{F}_q$ be a finite field with q elements, and let χ be a primitive cubic Dirichlet character over $\mathbb{F}_q[T]$ with conductor h. Let $L(s, \chi)$ be the L-function

$$L(s, \chi) = \sum_{f \in \mathbb{F}_q[T]} \frac{\chi(f)}{q^{s \deg(f)}}$$

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$$L(s,\chi) = \sum_{f\in\mathbb{F}_q[T]} \frac{\chi(f)}{q^{s\deg(f)}} = \sum_{d=0}^{\deg(h)-1} \frac{1}{q^{ds}} \sum_{\substack{f\in\mathbb{F}_q[T] \\ \deg(F)=d}} \chi(f)$$

by the orthogonality relations.

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by the orthogonality relations.

We also use the notation

$$\mathcal{L}(u,\chi) = \sum_{f \in \mathbb{F}_q[T]} \chi(f) u^{\deg(f)},$$

such that $\mathcal{L}(q^{-s}, \chi) = L(s, \chi)$.

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The first moment is

$$\frac{1}{\#\mathcal{C}_q(d)} \sum_{\chi \in \mathcal{C}_q(d)} L(1/2, \chi) = \frac{1}{\#\mathcal{C}_q(d)} \sum_{\chi \in \mathcal{C}_q(d)} \mathcal{L}(q^{-1/2}, \chi).$$

Again, the sum is real even if $L(1/2, \chi) \in \mathbb{C}$.

Kummer case: $q \equiv 1 \mod 3$

Kummer case: Then $\mu_3 \subseteq \mathbb{F}_q^*$.

There are 2 characters modulo P for each irreducible polynomial $P \in \mathbb{F}_q[T]$ given by the cubic residue symbol

$$\chi_P(a) \equiv a^{(q^{\deg(P)}-1)/3} \pmod{P}, \qquad a \in \mathbb{F}_q[T],\ (a, P) = 1,$$

and

$$\#\mathcal{C}_q(d) \sim a_1 q^d d + a_0 q^d.$$

Theorem

(D, Florea, Lalin, 2019)

$$\sum_{\substack{\chi\in\mathcal{C}_q(d)\\ \chi|_{\mathbb{F}_q^*}=\chi_3}} L(1/2,\chi) = \sum_{\substack{d_1+d_2=d\\ d_1+2d_2\equiv 1 \pmod 3}} \sum_{\substack{\deg F_1=d_1,\deg F_2=d_2\\ F_1,F_2 \textit{ square-free}\\ (F_1,F_2)=1}} L(1/2,\chi_{F_1}\overline{\chi}_{F_2})$$

$$= c_{1,K}\, dq^d + c_{0,K}\, q^d + O\left(q^{d\left(\frac{1+\sqrt{7}}{4}+\varepsilon\right)}\right).$$

where $\frac{1+\sqrt{7}}{4} \approx 0.9114378\cdots > 0.875 = \frac{7}{8}$.

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Since χ restricts to a fixed non-trivial character of $\mathbb{F}_q^*$, the characters are odd, and the finite field L-function is the function field L-function. For odd characters, $d = g + 1$.

Non-Kummer case: $q \equiv 2 \mod 3$

Non-Kummer case: Then $\mu_3 \not\subseteq \mathbb{F}_q^*$.

There are 2 characters modulo P for each irreducible polynomial $P \in \mathbb{F}_q[T]$ of even degree given by the cubic residue symbol

$$\chi_P(a) \equiv a^{(q^{\deg(P)}-1)/3} \pmod{P}, \quad a \in \mathbb{F}_q[T], (a, P) = 1.$$

from the work of Bary-Soroker and Meisner, and

$$\#\mathcal{C}_q(d) \sim a_0 q^d.$$

Families of non-Kummer cubic twists

Lemma

Suppose $q \equiv 2 \pmod 3$. Then, for d even,

$$\sum_{\chi \in \mathcal{C}_q(d)} L(1/2, \chi) = \sum_{d_1+d_2=d} \sum_{\substack{F_1, F_2 \in \mathbb{F}_q[T] \\ F_1, F_2 \text{ square-free, coprime} \\ P \mid \deg(F_i) \Rightarrow \deg(P) \text{ even} \\ \deg(F_1)=d_1, \deg(F_2)=d_2}} L(1/2, \chi_{F_1} \overline{\chi_{F_2}})$$

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$$= \sum_{\substack{F \in \mathbb{F}_{q^2}[T] \text{ square-free} \\ P \mid F \Rightarrow P \notin \mathbb{F}_q[T] \\ \deg(F)=d/2}} L(1/2, \chi_F)$$

Key: We count the primitive cubic characters over $\mathbb{F}_{q^2}[T]$ which restrict to a primitive cubic characters over $\mathbb{F}_q[T]$.

Approximate functional equation

Let χ be a primitive cubic character of modulus h. Then

$$\mathcal{L}(u,\chi) = \sum_{f \in \mathbb{F}_q[T]} \chi(f) u^{\deg(f)} = \sum_{\substack{f \in \mathbb{F}_q[T] \\ \deg(f) \leq \deg(h)-1}} \chi(f) u^{\deg(f)}.$$

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Proposition (Approximate Functional Equation)

Let χ be a primitive **even** *cubic character of modulus h. Then,*

$$\mathcal{L}\left(\frac{1}{\sqrt{q}},\chi\right) = \sum_{\substack{f\in\mathbb{F}_q[T]\\ \deg(f)\leq A}} \frac{\chi(f)}{q^{\deg(f)/2}} + \omega(\chi) \sum_{\substack{f\in\mathbb{F}_q[T]\\ \deg(f)\leq\deg(h)-1-A}} \frac{\overline{\chi(f)}}{q^{\deg(f)/2}}$$

as principal sum + dual sum.

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$$+ \frac{1}{1-\sqrt{q}} \sum_{\substack{f \in \mathbb{F}_q[T] \\ \deg(f) = A+1}} \frac{\chi(f)}{q^{\deg(f)/2}} + \frac{\omega(\chi)}{1-\sqrt{q}} \sum_{\substack{f \in \mathbb{F}_q[T] \\ \deg(F) \le \deg(h)-A}} \frac{\overline{\chi(f)}}{q^{\deg(f)/2}}.$$

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Approximate functional equation

The sign of the functional equation relating $\mathcal{L}(u,\chi)$ to $\mathcal{L}(1/qu,\overline{\chi})$ is given by

$$\omega(\chi) = q^{-\deg(h)/2} G(\chi),$$

where $G(\chi)$ is the cubic Gauss sum of the primitive character χ, i.e.

$$G(\chi) = \sum_{a \pmod h} \chi(a)\, e_q\left(\frac{a}{h}\right) \quad \in \mathbb{C}^*$$

of size $|G(\chi)| = q^{\deg(h)/2}$.

Gauss sums

The exponential function e_q was introduced by Carlitz and Hayes. For any $a \in \mathbb{F}_q((1/T))$, we define

$$e_q(a) = \exp\left(\frac{2\pi i \operatorname{tr}_{\mathbb{F}_q/\mathbb{F}_p}(a_1)}{p}\right) \in \mu_p \subseteq \mathbb{C}^*,$$

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We then have that

- $e_q(a+b) = e_q(a)e_q(b)$
- $e_q(a) = 1$ for $a \in \mathbb{F}_q[T]$
- $e_q(a/h) = e_q(b/h)$ for $a, b, h \in \mathbb{F}_q[T]$ with $a \equiv b \pmod{h}$.

Generalized cubic Gauss sums

Recall that we are considering the characters over $\mathbb{F}_{q^2}[T]$, sieving out those which are not defined over $\mathbb{F}_q[T]$, and those which are not primitive.
Notice that $q^2 \equiv 1 \pmod 3$.

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Let $q \equiv 1 \mod 3$. Let χ_F be the cubic residue symbol defined above for any $F \in \mathbb{F}_q[T]$. This is a character of modulus F, but not necessarily primitive. We define the generalized cubic Gauss sum by

$$G_q(V, F) = \sum_{u \pmod F} \chi_F(u)\, e_q\left(\frac{uV}{F}\right).$$

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If $(a, F) = 1$, we have

$$G_q(aV, F) = \overline{\chi_F}(a) G_q(V, F).$$

Generalized cubic Gauss sums

Suppose that $q \equiv 1 \pmod 3$.

(i) If $(F_1, F_2) = 1$, then

$$\begin{aligned} G_q(V, F_1F_2) &= \chi_{F_1}(F_2)^2 G_q(V, F_1) G_q(V, F_2) \\ &= G_q(VF_2, F_1) G_q(V, F_2). \end{aligned}$$

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$$\begin{aligned} G_q(V, F_1F_2) &= \chi_{F_1}(F_2)^2 G_q(V, F_1) G_q(V, F_2) \\ &= G_q(VF_2, F_1) G_q(V, F_2). \end{aligned}$$

(ii) If $V = V_1 P^\alpha$ where $P \nmid V_1$, then

$$G_q(V, P^i) = \begin{cases} 0 & \text{if } i \le \alpha \text{ and } i \not\equiv 0 \pmod 3, \\ \phi(P^i) & \text{if } i \le \alpha \text{ and } i \equiv 0 \pmod 3, \\ -|P|_q^{i-1} & \text{if } i = \alpha + 1 \text{ and } i \equiv 0 \pmod 3, \\ \epsilon(\chi_{P^i})\omega(\chi_{P^i})\chi_{P^i}(V_1^{-1})|P|_q^{i-\frac{1}{2}} & \\ \qquad \text{if } i = \alpha + 1 \text{ and } i \not\equiv 0 \pmod 3, \\ 0 & \text{if } i \ge \alpha + 2, \end{cases}$$

Main term of the principal sum

When f is a cube

$$\chi_F(f) = \begin{cases} 1 & (F, f) = 1 \\ 0 & (F, f) \neq 1 \end{cases},$$

Main term of the principal sum

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and this gives the main term of the principal sum

$$S_{\text{cube}} = \sum_{\deg(f) \leq A/3} \frac{1}{q^{3\deg(f)/2}} \sum_{\substack{F \in \mathbb{F}_q[T] \text{ square-free} \\ (F,f)=1 \\ \deg F = d \\ P|F \Rightarrow 2|\deg(P)}} 2^{\omega(F)}.$$

Main term of the principal sum

Let $$G_f(u) = \sum_F a_f(F) u^{\deg(F)} = \prod_{\substack{\deg P \text{ even} \\ P \nmid f}} \left(1 + u^{2 \deg P}\right),$$

where $a_f(F)$ is the number of primitive cubic characters of conductor F with $(F, f) = 1$.

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By Perron's formula,

$$\sum_{\substack{F \in \mathbb{F}_q[T] \text{ square-free} \\ (F,f)=1 \\ \deg F = d \\ P|F \Rightarrow 2|\deg(P)}} 2^{\omega(F)} = \sum_{\deg F = d} a_f(F) = \frac{1}{2\pi i} \oint_{|u|=q^{-2}} \frac{G_f(u)}{u^d} \frac{du}{u},$$

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and move from small circle $|u| = q^{-2}$ to large circle $|u| = q^{-1/2-\varepsilon}$.

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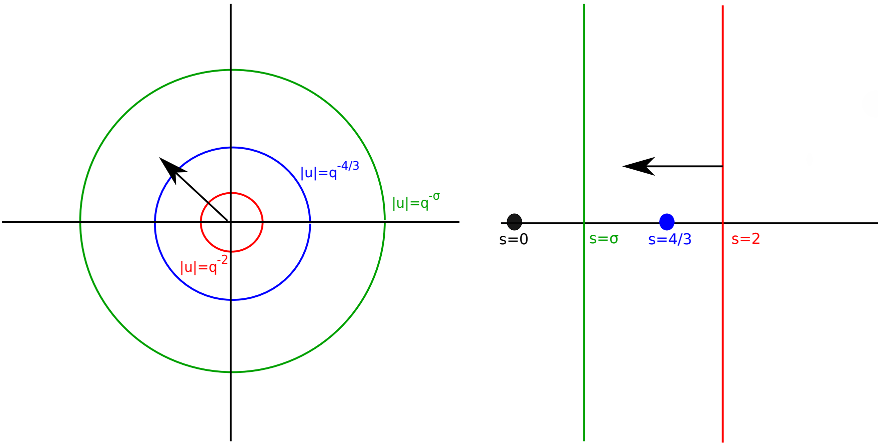
and move from small circle $|u| = q^{-2}$ to large circle $|u| = q^{-1/2-\varepsilon}$.

Since $u = q^{-s}$, it is moving from $\mathrm{Re}(s) = 2$ to $\mathrm{Re}(s) = 1/2 + \varepsilon$.

Perron's formula



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$$= \frac{1}{2\pi i}\oint_{|u|=q^{-2}} \frac{\prod_{2|\deg P, P|f}\left(1+2u^{\deg(P)}\right)^{-1} \mathcal{Z}_q(u) F(u)}{u^d}\frac{du}{u}$$

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$$= F(1/q)q^d \prod_{2|\deg P,P|f}\left(1+2q^{-\deg(P)}\right)^{-1} + O\left(q^{(1/2+\varepsilon)d}\right)$$

where $\mathcal{Z}_q(u) = (1-qu)^{-1}$ has a simple pole at $u = q^{-1}$ with residue $-1/q$, and d is even.

Main term of the principal sum

Then, the contribution due to f cubes to principal sum is asymptotic to

$$S_{\text{cube}} = F(1/q)q^d \sum_{\deg(f)\le A/3} \frac{1}{q^{3\deg(f)/2}} \prod_{2|\deg P, P|f} \left(1 + 2q^{-\deg(P)}\right)^{-1},$$

where the number of characters is asymptotic to $F(1/q)q^d$.

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where the number of characters is asymptotic to $F(1/q)q^d$.

Using Perron's formula

$$\sum_{\deg(f)\le A/3} = \frac{1}{2\pi i}\oint_{|u|=q^{-2}} \frac{G(u)}{u^{A/3}(u-1)}\frac{du}{u},$$

where

$$G(u) = \sum_{f\in\mathbb{F}_q[T]} \frac{1}{q^{3\deg(f)/2}} \prod_{2|\deg P, P|f} \left(1+2q^{-\deg(P)}\right)^{-1}.$$

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Using Perron's formula

$$\sum_{\deg(f)\le A/3} = \frac{1}{2\pi i}\oint_{|u|=q^{-2}} \frac{\mathcal{Z}_q(u/q^{3/2})H(u)}{u^{A/3}(u-1)}\frac{du}{u}.$$

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Moving the integral from $u = q^{-2}$ to $u = q^{3/2-\varepsilon}$, we encounter the poles at $u = 1 = q^0$ and $u = q^{1/2}$, and the contribution from f a cube is

$$F(1/q)H(1)q^d + q^{1/2}F(1/q)H(q^{1/2})q^{d-A/6} + ET$$

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Key point: Since $A \leq d$, to get an error term of size $q^{5d/6}$, we have to cancel the $q^{d-A/6}$ with some other contribution. We believe that there could be a secondary term of size $q^{5d/6}$.

Dual sum

After Poisson summation and various reductions, we have to study

$$S_{\text{dual}} = q^{-\frac{d}{2}} \sum_{\deg(f) \le d-A-1} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{F \in \mathbb{F}_{q^2}[T] \text{ square-free} \\ \deg(F)=d/2 \\ (F,f)=1 \\ P|F \Rightarrow P \notin \mathbb{F}_q[T]}} G_{q^2}(f,F) + OT$$

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$$= q^{-\frac{d}{2}} \sum_{\deg(f)\leq d-A-1} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{D\in\mathbb{F}_q[T] \\ \deg(D)\leq d/2 \\ (D,f)=1}} \mu_q(D) G_{q^2}(f,D) \times$$

$$\times \sum_{\substack{F\in\mathbb{F}_{q^2}[T] \\ \deg(F)=d/2-\deg(D) \\ (F,Df)=1}} G_{q^2}(fD,F) + OT.$$

Dual sum: Evaluating sums of cubic Gauss sums

To study

$$\sum_{\deg F=d} G_{q^2}(f,F) \quad \text{and} \quad \sum_{\substack{\deg F=d \\ (F,f)=1}} G_{q^2}(f,F),$$

we need the analytic properties of the generating series

$$\Psi(f,u) = \sum_{F} G_{q^2}(f,F)u^{\deg F} \quad \text{and} \quad \tilde{\Psi}(f,u) = \sum_{(F,f)=1} G_{q^2}(f,F)u^{\deg F}.$$

Recall that the Gauss sums are NOT multiplicative. By the work of Kubota and Patterson, the generating series above correspond to forms on the metaplectic group, and one can deduce a functional equation from there.

Dual sum: Evaluating sums of cubic Gauss sums

Let $q \equiv 1 \mod 3$. In the case of function fields, it follows from the work of Kazhdan and Patterson [1984], Hoffstein [1992] and Patterson [2007] that

$$\psi(f,i,u) = \sum_{\deg(F)\equiv i \pmod 3} G_q(f,F)u^{\deg(P)} = \frac{u^i P(f,i,u^3)}{1-q^4u^3}$$

where $P(f,i,x)$ is a polynomial of x-degree $\leq \lfloor (1+\deg f - i)/3 \rfloor$. It follows that

$$\Psi(f,u) = (1-u^3q^3)\sum_{i=1}^{3}\psi(f,i,u)$$

is analytic for all u, except for a simple pole at $u = q^{-4/3}$.

Dual sum: Evaluating sums of cubic Gauss sums

We are interested in the residue at $u = q^{-4/3}$ i.e. $s = 4/3$. Let

$$\rho(f, i) = \lim_{u \to q^{-4/3}} (1 - q^4 u^3) u^{-i} \psi(f, i, u) = P(f, i, q^{-4/3}).$$

Let π be a prime such that $\pi \nmid f$. Then,

- **Periodicity Theorem** $\rho(f\pi^3, i) = \rho(f, i)$.
- $\rho(f\pi^2, i) = 0$.
- $\rho(f\pi, i) = G_q(f, \pi) |\pi|_q^{-2/3} q^{8 \deg(\pi)/3} \rho(f, [i - 2\deg(\pi)]_3)$.

Computing the residues

Lemma

Let $f = f_1 f_2^2 f_3^3$ with f_1, f_2 square-free and co-prime. Then, $\rho(f,i) = 0$ if $f_2 \neq 1$, and for $f_2 = 1$,

$$\rho(f,i) = \overline{G_q(1,f_1)}|f_1|_q^{-2/3} q^{4i/3-4/3[i-2\deg(f)]_3} \rho(1,[i-2\deg(\pi)]_3).$$

Furthermore,

$$\rho(1,0) = 1, \quad \rho(1,1) = \tau(\chi_3)q, \quad \rho(1,2) = 0,$$

where $\tau(\chi_3)$ is the Gauss sum of the cubic character χ_3 of $\mathbb{F}_q^*$.

Properties of the residues

Then, we can evaluate

$$\sum_{\deg F=d} G_q(f,F) = \frac{1}{2\pi i}\int_{q^{-10}} \frac{\Psi(f,u)}{u^d}\frac{du}{u}$$

$$= -q^{\frac{4}{3}(d+1)}\mathrm{Res}_{u=q^{-4/3}}\Psi(f,u) + \int_{|u|=q^{-\sigma}} \frac{\Psi(f,u)}{u^d}\frac{du}{u}.$$

For the bounds on $\Psi(f,u)$ for $|u| = q^{-\sigma}$ for $\sigma < 4/3$, the best that we can do is apply the Maximum Modulus Principle (Phragmen-Lindelof) to get the convexity bound.

Convexity Bound

The trivial bound is given by $|G_q(f,F)| \leq q^{\deg F/2}$ when $(f,F) = 1$, and the appropriate results in the other cases.

Lemma

If $1/2 \leq \sigma \leq 3/2$, and $|u^3 - q^{-4}|, |u^3 - q^{-2}| > \delta$, then

$$\Psi(f,u) \ll_\delta |f|_q^{\frac{1}{2}\left(\frac{3}{2}-\sigma\right)+\varepsilon}.$$

Dual sum: From $\Psi(f,u)$ to $\tilde{\Psi}(f,u)$

Recall that we defined

$$\begin{aligned}\Psi(f,u) &= \sum_{F} G_{q^2}(f,F)u^{\deg F}\\ \tilde{\Psi}(f,u) &= \sum_{(F,f)=1} G_{q^2}(f,F)u^{\deg F}.\end{aligned}$$

We know from the study of metaplectic forms that $\Psi(f,u)$ is analytic for all u, except for a simple pole at $u=q^{-4/3}$. We want to deduce the analytic properties of $\tilde{\Psi}(f,u)$.

Dual sum: From $\Psi(f,u)$ to $\tilde{\Psi}(f,u)$

Let $f = f_1 f_2^2 f_3^3$ with f_1, f_2 square-free and co-prime, and let f_3^* be the product of the primes dividing f_3 but not dividing $f_1 f_2$ Then, $\tilde{\Psi}(f,u)$

$$
\begin{aligned}
= \quad & \prod_{P|f_1 f_2} \left(1 - (u^3 q^2)^{\deg(P)}\right)^{-1} \sum_{a|f_3^*} \mu(a) G_q(f_1 f_2^2, a) u^{\deg(a)} \times \\
& \times \prod_{P|a} (1 - (u^3 q^2)^{\deg(P)})^{-1} \times \\
& \times \sum_{\ell|af_1} \mu(\ell)(u^2 q)^{\deg(\ell)} \overline{G_q(1,\ell)} \chi_\ell(af_1 f_2^2/\ell) \Psi(af_1 f_2^2/\ell, u).
\end{aligned}
$$

Then, we can compute the residue of $\tilde{\Psi}(f,u)$ at $q^{-4/3}$.

Notice that contrary to $\Psi(f,u)$, $\tilde{\Psi}(f,u)$ has poles at $u = q^{-2/3}$ (of high multiplicity).

Dual sum: Ready for Perron's formula

Recall that we want to evaluate

$$S_{\text{dual}} = q^{-\frac{d}{2}} \sum_{\deg(f)\leq d-A-1} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{F\in\mathbb{F}_{q^2}[T] \text{ square-free} \\ \deg(F)=d/2 \\ (F,f)=1 \\ P|F\Rightarrow P\notin\mathbb{F}_q[T]}} G_{q^2}(f,F) + OT$$

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$$= q^{-\frac{d}{2}} \sum_{\deg(f) \le d-A-1} \frac{1}{q^{\deg(f)/2}} \sum_{\substack{D \in \mathbb{F}_q[T] \\ \deg(D) \le d/2 \\ (D,f)=1}} \mu_q(D) G_{q^2}(f, D) \times$$

$$\times \sum_{\substack{F \in \mathbb{F}_{q^2}[T] \\ \deg(F) = d/2 - \deg(D) \\ (F, Df)=1}} G_{q^2}(fD, F) + OT.$$

Dual sum: After Perron's formula

$$
\begin{aligned}
S_{\mathrm{dual}} &= -\frac{q^{d-\frac{A}{6}} H(q^{-1/6}) \mathcal{Z}_q(q^{-1/2})}{\mathcal{Z}_{q^2}(q^{-4})} \\
&+ q^{-d} \frac{1}{2\pi i} \oint_{|u|=q^{-2\sigma}} \sum_{\deg(f) \le d-A-1} \frac{1}{q^{\deg(f)/2}} \times \\
&\quad \times \sum_{\substack{D \in \mathbb{F}_q[T] \\ \deg(D) \le d/2+1 \\ (D,f)=1}} \mu(D) G_{q^2}(f, D) \frac{\tilde{\Psi}_{q^2}(fD, u)}{u^{d/2+1-\deg(D)}} \frac{du}{u} \\
&+ \mathrm{ET}
\end{aligned}
$$

for some function $H(u)$ which is analytic at $u = q^{-1/6}$.

Cancellation between principal sum and dual sum

$$
\begin{aligned}
&S_{\mathrm{cube}} + S_{\mathrm{dual}} \\
&= F(1/q)G(1)q^{d} \\
&\quad + q^{1/2}F(1/q)G(q^{1/2})q^{d-A/6} - \frac{q^{d-A/6}H(q^{-1/6})\mathcal{Z}_q(q^{-1/2})}{\mathcal{Z}_{q^2}(q^{-4})} \\
&+q^{-d}\frac{1}{2\pi i}\oint_{|u|=q^{-2\sigma}} \sum_{\deg(f)\le d-A-1} \frac{1}{q^{\deg(f)/2}} \times \\
&\quad \times \sum_{\substack{D\in\mathbb{F}_q[T] \\ \deg(D)\le d/2+1 \\ (D,f)=1}} \mu(D)G_{q^2}(f,D)\frac{\tilde{\Psi}_{q^2}(fD,u)}{u^{d/2+1-\deg(D)}}\frac{du}{u} \\
&+\mathrm{ET}
\end{aligned}
$$

Cancellation between principal sum and dual sum

$$S_{\mathrm{cube}} + S_{\mathrm{dual}}$$
$$= F(1/q)G(1)q^d$$

$$+q^{-d}\frac{1}{2\pi i}\oint_{|u|=q^{-2\sigma}} \sum_{\deg(f)\le d-A-1} \frac{1}{q^{\deg(f)/2}} \times$$
$$\times \sum_{\substack{D\in\mathbb{F}_q[T] \\ \deg(D)\le d/2+1 \\ (D,f)=1}} \mu(D) G_{q^2}(f,D)\frac{\tilde{\Psi}_{q^2}(fD,u)}{u^{d/2+1-\deg(D)}}\,\frac{du}{u}$$

$$+\mathrm{ET}$$

Final Result

Let $q \equiv 2 \mod 3$ and let $\mathcal{C}_q(d)$ be the set of primitive cubic characters over $\mathbb{F}_q[T]$ with conductor of degree d. Then,

$$\sum_{\chi \in \mathcal{C}_q(d)} L(1/2, \chi) \quad = \quad c_3 q^d + O\left(q^{7d/8+\varepsilon}\right)$$