Central values of additive twists of L functions via continued fractions

Sary Drappeau
joint with Sandro Bettin (Genova)

Univ. Aix-Marseille

July 15, 2019

Central values of L-functions, non-vanishing

Some reasons for studying central values of L-functions:

- Lindelöf hypothesis: $|\zeta(1/2+it)| \ll 1+|t|^{\varepsilon}$? (…, Kolesnik, Huxley, Bourgain 2015, $t^{13/87+\varepsilon}$).
- Chowla conjecture: is $L(\chi, 1/2) \neq 0$ for χ primitive? quadratic? Results on average over χ (Balasubramanian-Murty, Iwaniec-Sarnak, Soundararajan, …)
- Birch, Swinnerton-Dyer conjecture: $E/\mathbb{Q}$ elliptic curve. Count points mod p, and build $L(E,s)$. Then $L(E,1/2)$ should vanish at order given by the rank of E.

Central values of L-functions, non-vanishing

Some reasons for studying central values of L-functions:

- Lindelöf hypothesis: $|\zeta(1/2+it)| \ll 1+|t|^{\varepsilon}$? (…, Kolesnik, Huxley, Bourgain 2015, $t^{13/87+\varepsilon}$).
- Chowla conjecture: is $L(\chi, 1/2) \neq 0$ for χ primitive? quadratic? Results on average over χ (Balasubramanian-Murty, Iwaniec-Sarnak, Soundararajan, …)
- Birch, Swinnerton-Dyer conjecture: $E/\mathbb{Q}$ elliptic curve. Count points mod p, and build $L(E,s)$. Then $L(E,1/2)$ should vanish at order given by the rank of E.
  Mazur-Rubin, Stein: fix $E/\mathbb{Q}$. How large does $\mathrm{rank}(E/K)$ get as K varies among abelian extensions of $\mathbb{Q}$?

Central values of L-functions, distribution

We wish to understand these values. What is their size as complex numbers?

- Selberg: $\left(\frac{\log \zeta(1/2+it)}{\sqrt{\log \log T}}\right)_{t \in [T,2T]}$ converges to a Gaussian, meaning $\forall R \subset \mathbb{C}$ rectangle, as $T \to \infty$,

$$\mathbb{P}_{t \in [T,2T]}\left(\frac{\log \zeta(1/2+it)}{\sqrt{\log \log T}} \in R\right) \to \mathbb{P}(\mathcal{N}_{\mathbb{C}}(0,1) \in R).$$

Not much is yet proved in other families. Conjectures of Keating-Snaith. Radziwiłł-Soundararajan '17: one-sided bounds.

Central values of L-functions, distribution

We wish to understand these values. What is their size as complex numbers?

- Selberg: $\left(\frac{\log \zeta(1/2+it)}{\sqrt{\log\log T}}\right)_{t\in[T,2T]}$ converges to a Gaussian, meaning $\forall R \subset \mathbb{C}$ rectangle, as $T \to \infty$,

$$\mathbb{P}_{t\in[T,2T]}\left(\frac{\log \zeta(1/2+it)}{\sqrt{\log\log T}} \in R\right) \to \mathbb{P}(\mathcal{N}_{\mathbb{C}}(0,1) \in R).$$

  Not much is yet proved in other families. Conjectures of Keating-Snaith. Radziwiłł-Soundararajan '17: one-sided bounds.
- Distribution happens in the log-scale, because of multiplicativity:

$$\log \zeta(1/2+it) \approx \sum_{p \ll t^{O(1)}} \frac{p^{-it}}{\sqrt{p}} + [\text{zeroes}].$$

  Sum of terms behaving independently.

Additive twists - cuspidal case

For f a holomorphic eigen-cusp form, $f(z) = \sum_{n\geq 1} a_f(n)\mathrm{e}(nz)$.
Define the twisted L-function

$$L_f(s,x) := \sum_{n\geq 1} \frac{a_f(n)\mathrm{e}(nx)}{n^s} \qquad (\Re(s) > 1/2)$$

analytically continued to $\mathbb{C}$. The value $L_f(1/2,x)$ is one incarnation of modular symbols (useful *e.g.* to compute with modular forms).

Additive twists - cuspidal case

For f a holomorphic eigen-cusp form, $f(z) = \sum_{n \geq 1} a_f(n)\mathrm{e}(nz)$.
Define the twisted L-function

$$L_f(s,x) := \sum_{n \geq 1} \frac{a_f(n)\mathrm{e}(nx)}{n^s} \qquad (\Re(s) > 1/2)$$

analytically continued to $\mathbb{C}$. The value $L_f(1/2,x)$ is one incarnation of modular symbols (useful *e.g.* to compute with modular forms).

Conjecture (Mazur-Rubin, Stein 2015)

The values $L_f(1/2,x)$ *become Gaussian distributed: for some* $\sigma_{f,q} > 0$*, as* $q \to \infty$*, when* x *is picked at random among rationals in* $(0,1]$ *with denominator* $= q$*,*

$$\mathbb{P}\Big(\frac{L_f(1/2,x)}{\sigma_{f,q}\sqrt{\log q}} \in R\Big) \to \mathbb{P}(\mathcal{N}_{\mathbb{C}}(0,1) \in R)$$

where $R \subset \mathbb{C}$ *is any fixed rectangle.*

First and second moment is known
(Blomer-Fouvry-Kowalski-Michel-Milićević-Sawin)

Additive twists - cuspidal case

$$L_f(1/2, x) := \sum_{n\geq 1} \frac{a_f(n)\mathrm{e}(nx)}{n^{1/2}} \qquad (\Re(s) > 0).$$

What about on average over q?

$$\Omega_Q := \{x \in \mathbb{Q} \in (0,1], \mathsf{denom}(x) \leq Q\}, \qquad \mathbb{E}_Q(f(x)) = \frac{1}{|\Omega_Q|} \sum_{x\in\Omega_Q} f(x).$$

Additive twists - cuspidal case

$$L_f(1/2, x) := \sum_{n \geq 1} \frac{a_f(n)\mathrm{e}(nx)}{n^{1/2}} \qquad (\Re(s) > 0).$$

What about on average over q?

$$\Omega_Q := \{x \in \mathbb{Q} \in (0,1], \mathrm{denom}(x) \leq Q\}, \qquad \mathbb{E}_Q(f(x)) = \frac{1}{|\Omega_Q|} \sum_{x \in \Omega_Q} f(x).$$

Is it true that for any rectangle $R \subset \mathbb{C}$, as $Q \to \infty$,

$$\mathbb{P}_Q\Big(\frac{L_f(1/2, x)}{\sqrt{\sigma_f \log Q}} \in R\Big) \to \mathbb{P}(\mathcal{N}_{\mathbb{C}}(0,1) \in R)?$$

Additive twists - cuspidal case

$$L_f(1/2, x) := \sum_{n \geq 1} \frac{a_f(n)\mathrm{e}(nx)}{n^{1/2}} \qquad (\Re(s) > 0).$$

What about on average over q?

$$\Omega_Q := \{x \in \mathbb{Q} \in (0,1], \mathrm{denom}(x) \leq Q\}, \qquad \mathbb{E}_Q(f(x)) = \frac{1}{|\Omega_Q|} \sum_{x \in \Omega_Q} f(x).$$

Is it true that for any rectangle $R \subset \mathbb{C}$, as $Q \to \infty$,

$$\mathbb{P}_Q\Big(\frac{L_f(1/2, x)}{\sqrt{\sigma_f \log Q}} \in R\Big) \to \mathbb{P}\big(\mathcal{N}_{\mathbb{C}}(0,1) \in R\big)?$$

Theorem (Petridis-Risager '17, Nordentoft)

Yes, in general, by automorphic methods (twisted Eisenstein series, Goldfeld '97)

Additive twists - cuspidal case

$$L_f(1/2, x) := \sum_{n\geq 1} \frac{a_f(n)\mathrm{e}(nx)}{n^{1/2}} \qquad (\Re(s) > 0).$$

What about on average over q?

$$\Omega_Q := \{x \in \mathbb{Q} \in (0,1], \mathrm{denom}(x) \leq Q\}, \qquad \mathbb{E}_Q(f(x)) = \frac{1}{|\Omega_Q|}\sum_{x\in\Omega_Q} f(x).$$

Is it true that for any rectangle $R \subset \mathbb{C}$, as $Q \to \infty$,

$$\mathbb{P}_Q\Big(\frac{L_f(1/2,x)}{\sqrt{\sigma_f \log Q}} \in R\Big) \to \mathbb{P}(\mathcal{N}_{\mathbb{C}}(0,1) \in R)?$$

Theorem (Petridis-Risager '17, Nordentoft)

Yes, in general, by automorphic methods (twisted Eisenstein series, Goldfeld '97)

Theorem (Lee-Sun, Bettin-D.)

Yes, by dynamical systems methods,

Additive twists - cuspidal case

$$L_f(1/2,x) := \sum_{n\geq 1} \frac{a_f(n)\mathrm{e}(nx)}{n^{1/2}} \qquad (\Re(s)>0).$$

What about on average over q?

$$\Omega_Q := \{x \in \mathbb{Q} \in (0,1], \mathrm{denom}(x) \leq Q\}, \qquad \mathbb{E}_Q(f(x)) = \frac{1}{|\Omega_Q|}\sum_{x\in\Omega_Q} f(x).$$

Is it true that for any rectangle $R \subset \mathbb{C}$, as $Q \to \infty$,

$$\mathbb{P}_Q\Big(\frac{L_f(1/2,x)}{\sqrt{\sigma_f \log Q}} \in R\Big) \to \mathbb{P}(\mathcal{N}_{\mathbb{C}}(0,1) \in R)?$$

Theorem (Petridis-Risager '17, Nordentoft)

Yes, in general, by automorphic methods (twisted Eisenstein series, Goldfeld '97)

Theorem (Lee-Sun, Bettin-D.)

Yes, by dynamical systems methods, if f has weight 2,

Additive twists - cuspidal case

$$L_f(1/2, x) := \sum_{n\geq 1} \frac{a_f(n)\mathrm{e}(nx)}{n^{1/2}} \qquad (\Re(s) > 0).$$

What about on average over q?

$$\Omega_Q := \{x \in \mathbb{Q} \in (0,1], \mathsf{denom}(x) \leq Q\}, \qquad \mathbb{E}_Q(f(x)) = \frac{1}{|\Omega_Q|} \sum_{x\in\Omega_Q} f(x).$$

Is it true that for any rectangle $R \subset \mathbb{C}$, as $Q \to \infty$,

$$\mathbb{P}_Q\Big(\frac{L_f(1/2, x)}{\sqrt{\sigma_f \log Q}} \in R\Big) \to \mathbb{P}(\mathcal{N}_{\mathbb{C}}(0,1) \in R)?$$

Theorem (Petridis-Risager '17, Nordentoft)

Yes, in general, by automorphic methods (twisted Eisenstein series, Goldfeld '97)

Theorem (Lee-Sun, Bettin-D.)

Yes, by dynamical systems methods, if f has weight 2, or if f has level 1.

Additive twists - Estermann function

Non-cuspidal analogue: for $\Re(s) > 1$, τ divisor function, let

$$D(s,x) := \sum_{n \geq 1} \frac{\tau(n)\mathrm{e}(nx)}{n^s}.$$

Additive twists - Estermann function

Non-cuspidal analogue: for $\Re(s) > 1$, τ divisor function, let

$$D(s,x) := \sum_{n\geq 1} \frac{\tau(n)\mathrm{e}(nx)}{n^s}.$$

Meromorphically continued to $\mathbb{C}$ if $x \in \mathbb{Q}$. The value $D(1/2,x)$ is linked (via orthogonality) to twisted moments of Dirichlet L-functions.

Additive twists - Estermann function

Non-cuspidal analogue: for $\Re(s) > 1$, τ divisor function, let

$$D(s,x) := \sum_{n\geq 1} \frac{\tau(n)\mathrm{e}(nx)}{n^s}.$$

Meromorphically continued to $\mathbb{C}$ if $x \in \mathbb{Q}$. The value $D(1/2, x)$ is linked (via orthogonality) to twisted moments of Dirichlet L-functions.

Theorem (Bettin-D.)

For all rectangle $R \subset \mathbb{C}$, as $Q \to \infty$,

$$\mathbb{P}_Q\Big(\frac{D(1/2,x)}{\sqrt{\sigma(\log Q)(\log\log Q)^3}} \in R\Big) \to \mathbb{P}(\mathcal{N}_{\mathbb{C}}(0,1) \in R).$$

All moments are known by Bettin '18 (with single average!), but don't tell about the limit law, because of few bad terms, *e.g.* $D(1/2, 1/q) \asymp q^{1/2} \log q$.

Symmetries

Abbreviate $L_f(x) := L_f(1/2, x)$, $L_\tau(x) := D(1/2, x)$.

Claim (Bettin '17)

Both functions above satisfy symmetries of the following kind

$$L(1 + x) = L(x), \qquad L(x) = L(1/x) + \phi_*(x)$$

where ϕ_f and ϕ_τ are analytically nice, meaning that they can be continued to $\mathbb{R}$, with some regularity.

Symmetries

Abbreviate $L_f(x) := L_f(1/2, x)$, $L_\tau(x) := D(1/2, x)$.

Claim (Bettin '17)

Both functions above satisfy symmetries of the following kind

$$L(1+x) = L(x), \qquad L(x) = L(1/x) + \phi_*(x)$$

where ϕ_f and ϕ_τ are analytically nice, meaning that they can be continued to $\mathbb{R}$, with some regularity.

This is what Zagier calls "quantum modular forms" (some exotic examples came from quantum algebra).

Symmetries

Abbreviate $L_f(x) := L_f(1/2, x)$, $L_\tau(x) := D(1/2, x)$.

Claim (Bettin '17)

Both functions above satisfy symmetries of the following kind

$$L(1 + x) = L(x), \qquad L(x) = L(1/x) + \phi_*(x)$$

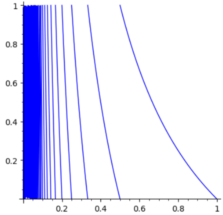
where ϕ_f and ϕ_τ are analytically nice, meaning that they can be continued to $\mathbb{R}$, with some regularity.

This is what Zagier calls "quantum modular forms" (some exotic examples came from quantum algebra).

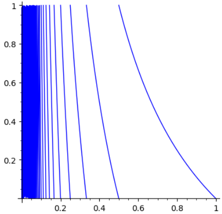
The symmetries above are all one needs to get a limit law.

Heuristics and continued fractions

Let $T(x) = \{1/x\}$ be the Gauss map.



Heuristics and continued fractions



Let $T(x) = \{1/x\}$ be the Gauss map.

$$\begin{aligned} L(x) &= L(T(x)) + \phi(x) \\ &= L(T^2(x)) + \phi(x) + \phi(T(x)) \\ &= \cdots \\ &= L(0) + S_\phi(x), \end{aligned}$$

where

$$S_\phi(x) := \phi(x) + \phi(T(x)) + \cdots + \phi(T^{N-1}(x)),$$

and $N = N(x)$ is minimal with $T^N(x) = 0$ ($N(a/q) \ll \log q$).

Heuristics and continued fractions

$$S_\phi(x) := \phi(x) + \phi(T(x)) + \cdots + \phi(T^{N-1}(x))$$

where $N = N(x)$ minimal with $T^N(x) = 0$ ($N(a/q) \ll \log q$).

Heuristics and continued fractions

$$S_\phi(x) := \phi(x) + \phi(T(x)) + \cdots + \phi(T^{N-1}(x))$$

where $N = N(x)$ minimal with $T^N(x) = 0$ ($N(a/q) \ll \log q$).
The map T is ergodic, exponentially mixing: terms far apart in the sum behave independently.

Heuristics and continued fractions

$$S_\phi(x) := \phi(x) + \phi(T(x)) + \cdots + \phi(T^{N-1}(x))$$

where $N = N(x)$ minimal with $T^N(x) = 0$ ($N(a/q) \ll \log q$).
The map T is ergodic, exponentially mixing: terms far apart in the sum behave independently.
Pick $x \in \Omega_Q$ randomly, then we expect

$$S_\phi(x) \approx \phi(X_1) + \cdots + \phi(X_N)$$

Heuristics and continued fractions

$$S_\phi(x) := \phi(x) + \phi(T(x)) + \cdots + \phi(T^{N-1}(x))$$

where $N = N(x)$ minimal with $T^N(x) = 0$ ($N(a/q) \ll \log q$).
The map T is ergodic, exponentially mixing: terms far apart in the sum behave independently.
Pick $x \in \Omega_Q$ randomly, then we expect

$$S_\phi(x) \approx \phi(X_1) + \cdots + \phi(X_N)$$

- X_j iid according to $\frac{\mathrm{d}x}{(1+x)\log 2}$ (Gauss, Khintchine, Wirsing, Kuzmin...)

Heuristics and continued fractions

$$S_\phi(x) := \phi(x) + \phi(T(x)) + \cdots + \phi(T^{N-1}(x))$$

where $N = N(x)$ minimal with $T^N(x) = 0$ ($N(a/q) \ll \log q$).
The map T is ergodic, exponentially mixing: terms far apart in the sum behave independently.
Pick $x \in \Omega_Q$ randomly, then we expect

$$S_\phi(x) \approx \phi(X_1) + \cdots + \phi(X_N)$$

- X_j iid according to $\frac{\mathrm{d}x}{(1+x)\log 2}$ (Gauss, Khintchine, Wirsing, Kuzmin...)
- N is distributed according to a normal, mean $N_\mu := \frac{12\log 2}{\pi^2}\log Q$ and variance $\asymp \log Q$ (Heilbronn, ..., Hensley 1994).

Heuristics and continued fractions

$$S_\phi(x) := \phi(x) + \phi(T(x)) + \cdots + \phi(T^{N-1}(x))$$

where $N = N(x)$ minimal with $T^N(x) = 0$ ($N(a/q) \ll \log q$).
The map T is ergodic, exponentially mixing: terms far apart in the sum behave independently.
Pick $x \in \Omega_Q$ randomly, then we expect

$$S_\phi(x) \approx \phi(X_1) + \cdots + \phi(X_N)$$

- X_j iid according to $\frac{\mathrm{d}x}{(1+x)\log 2}$ (Gauss, Khintchine, Wirsing, Kuzmin...)
- N is distributed according to a normal, mean $N_\mu := \frac{12 \log 2}{\pi^2} \log Q$ and variance $\asymp \log Q$ (Heilbronn, ..., Hensley 1994).

$$\mathbb{E}_Q(\mathrm{e}^{itS_\phi(x)}) \overset{?}{\approx} \mathbb{E}(\mathrm{e}^{it\phi(X)})^{N_\mu} = \exp\left\{N_\mu \log(1 + \mathbb{E}(\mathrm{e}^{it\phi(X)} - 1))\right\}$$

$$\approx \exp\left\{N_\mu \mathbb{E}(\mathrm{e}^{it\phi(X)} - 1)\right\}$$

Limit theorem for rational CF

$$\mathbb{E}_Q(\mathrm{e}^{itS_\phi(x)}) \overset{?}{\approx} \exp\left\{N_\mu \mathbb{E}(\mathrm{e}^{it\phi(X)} - 1)\right\}$$

Limit theorem for rational CF

$$\mathbb{E}_Q(\mathrm{e}^{itS_\phi(x)}) \overset{?}{\approx} \exp\left\{N_\mu \mathbb{E}(\mathrm{e}^{it\phi(X)} - 1)\right\}$$

Theorem (Bettin-D.)

Let $\alpha, \kappa > 0$. Suppose $\phi : [0,1] \to \mathbb{C}$ is κ-Hölder on $(\frac{1}{n+1}, \frac{1}{n})$, $\forall n \geq 1$, and suppose $\int_{[0,1]} |\phi|^\alpha < \infty$.

Limit theorem for rational CF

$$\mathbb{E}_Q(\mathrm{e}^{itS_\phi(x)}) \overset{?}{\approx} \exp\left\{ N_\mu \mathbb{E}(\mathrm{e}^{it\phi(X)} - 1) \right\}$$

Theorem (Bettin-D.)

Let $\alpha, \kappa > 0$. Suppose $\phi : [0,1] \to \mathbb{C}$ is κ-Hölder on $(\frac{1}{n+1}, \frac{1}{n})$, $\forall n \geq 1$, and suppose $\int_{[0,1]} |\phi|^\alpha < \infty$.
For some $\delta > 0$ and small $t \in \mathbb{R}$,

$$\mathbb{E}_Q(\mathrm{e}^{itS_\phi(x)}) = \exp\left\{ \tfrac{12 \log 2}{\pi^2} (\log Q) I_\phi(t) + O((t^2 + t^{2\alpha}) \log Q + Q^{-\delta}) \right\}$$

where $I_\phi(t) = \int_0^1 (\mathrm{e}^{it\phi(x)} - 1) \frac{dx}{(1+x)\log 2}$.

Limit theorem for rational CF

$$\mathbb{E}_Q(\mathrm{e}^{itS_\phi(x)}) \overset{?}{\approx} \exp\left\{N_\mu \mathbb{E}(\mathrm{e}^{it\phi(X)} - 1)\right\}$$

Theorem (Bettin-D.)

Let $\alpha, \kappa > 0$. Suppose $\phi : [0,1] \to \mathbb{C}$ is κ-Hölder on $(\frac{1}{n+1}, \frac{1}{n})$, $\forall n \geq 1$, and suppose $\int_{[0,1]} |\phi|^\alpha < \infty$.
For some $\delta > 0$ and small $t \in \mathbb{R}$,

$$\mathbb{E}_Q(\mathrm{e}^{itS_\phi(x)}) = \exp\left\{\frac{12\log 2}{\pi^2}(\log Q) I_\phi(t) + O((t^2 + t^{2\alpha})\log Q + Q^{-\delta})\right\}$$

where $I_\phi(t) = \int_0^1 (\mathrm{e}^{it\phi(x)} - 1)\frac{\mathrm{d}x}{(1+x)\log 2}$. Moreover, if $\alpha > 1$,

$$\mathbb{E}_Q(\mathrm{e}^{itS_\phi(x)}) = \exp\left\{\frac{12\log 2}{\pi^2}(\log Q)(I_\phi(t) + C_\phi t^2) + O((t^3 + t^{1+\alpha})\log Q + Q^{-\delta})\right\}$$

Previous work by Vallée '02 and Baladi-Vallée '05
($\phi(x) = f(\lfloor 1/x \rfloor) \ll |\log 1/x|$, Gaussian).
In the continuous case: many works (..., Aaronson-Denker).
Limit law is not necessarily Gaussian: stable law (Levy, Cauchy, ...)

Applications to additive twists (cusp case)

Case when f is a cuspidal eigen-cusp form.

$$L_f(x) := \sum_{n \geq 1} \frac{a_f(n)\mathrm{e}(nx)}{n^{1/2}}.$$

Applications to additive twists (cusp case)

Case when f is a cuspidal eigen-cusp form.

$$L_f(x) := \sum_{n \geq 1} \frac{a_f(n)\mathrm{e}(nx)}{n^{1/2}}.$$

$$L_f(x) = L_f(1/x) + \phi(x),$$

Here ϕ is $(1-\varepsilon)$-Hölder on $\mathbb{R}$ and bounded.

Applications to additive twists (cusp case)

Case when f is a cuspidal eigen-cusp form.

$$L_f(x) := \sum_{n \geq 1} \frac{a_f(n)\mathrm{e}(nx)}{n^{1/2}}.$$

$$L_f(x) = L_f(1/x) + \phi(x),$$

Here ϕ is $(1-\varepsilon)$-Hölder on $\mathbb{R}$ and bounded.

$$\begin{aligned} I_\phi(t) + C_\phi t^2 &= \int_0^1 (\mathrm{e}^{it\phi(x)} - 1)\,\mathrm{d}\mu(x) + C_\phi t^2 \\ &= i\mu t - \tfrac{1}{2}\sigma^2 t^2 + O(t^3). \end{aligned}$$

In fact $\mu = 0$ and σ is related to the Petersson norm of f (not seen from dynamics!).

Applications to additive twists (cusp case)

Case when f is a cuspidal eigen-cusp form.

$$L_f(x) := \sum_{n \geq 1} \frac{a_f(n)\mathrm{e}(nx)}{n^{1/2}}.$$

$$L_f(x) = L_f(1/x) + \phi(x),$$

Here ϕ is $(1-\varepsilon)$-Hölder on $\mathbb{R}$ and bounded.

$$\begin{aligned} I_\phi(t) + C_\phi t^2 &= \int_0^1 (\mathrm{e}^{it\phi(x)} - 1)\,\mathrm{d}\mu(x) + C_\phi t^2 \\ &= i\mu t - \tfrac{1}{2}\sigma^2 t^2 + O(t^3). \end{aligned}$$

In fact $\mu = 0$ and σ is related to the Petersson norm of f (not seen from dynamics!).
This implies the Gaussian behaviour with variance $\sigma^2 \log Q$.

Applications to additive twists (Estermann case)

Case of the Estermann function.

$$L_\tau(x) := \sum_{n \geq 1} \frac{\tau(n)\mathrm{e}(nx)}{n^{1/2}}.$$

Applications to additive twists (Estermann case)

Case of the Estermann function.

$$L_\tau(x) := \sum_{n \geq 1} \frac{\tau(n)\mathrm{e}(nx)}{n^{1/2}}.$$

$$L_\tau(x) = L_\tau(1/x) + \phi(x),$$

Now ϕ is $(\frac{1}{2} - \varepsilon)$-Hölder on $\mathbb{R} \smallsetminus \mathbb{Z}$ and not bounded! By Bettin '16 : $\phi(x) \sim cx^{-1/2} \log x$ as $x \to 0$.

Applications to additive twists (Estermann case)

Case of the Estermann function.

$$L_\tau(x) := \sum_{n\geq 1} \frac{\tau(n)\mathrm{e}(nx)}{n^{1/2}}.$$

$$L_\tau(x) = L_\tau(1/x) + \phi(x),$$

Now ϕ is $(\frac{1}{2}-\varepsilon)$-Hölder on $\mathbb{R}\smallsetminus\mathbb{Z}$ and not bounded! By Bettin '16 : $\phi(x) \sim cx^{-1/2}\log x$ as $x \to 0$.

$$\begin{aligned} I_\phi(t) &= \int_0^1 (\mathrm{e}^{it\phi(x)} - 1)\,\mathrm{d}\mu(x) \\ &= i\mu t - \tfrac{1}{2}\sigma^2 t^2(\log t)^3 + o(t^2(\log t)^3) \end{aligned}$$

In fact $\mu = 0$ and $\sigma = \pi$.

Applications to additive twists (Estermann case)

Case of the Estermann function.

$$L_\tau(x) := \sum_{n\geq 1} \frac{\tau(n)\mathrm{e}(nx)}{n^{1/2}}.$$

$$L_\tau(x) = L_\tau(1/x) + \phi(x),$$

Now ϕ is $(\frac{1}{2}-\varepsilon)$-Hölder on $\mathbb{R}\smallsetminus\mathbb{Z}$ and not bounded! By Bettin '16 : $\phi(x) \sim cx^{-1/2}\log x$ as $x \to 0$.

$$\begin{aligned} I_\phi(t) &= \int_0^1 (\mathrm{e}^{it\phi(x)} - 1)\,\mathrm{d}\mu(x) \\ &= i\mu t - \tfrac{1}{2}\sigma^2 t^2(\log t)^3 + o(t^2(\log t)^3) \end{aligned}$$

In fact $\mu = 0$ and $\sigma = \pi$.
This implies the Gaussian behaviour with variance $\sigma^2 \log Q(\log\log Q)^3$.

Applications to sum of CF coefficients

The law is not in general Gaussian: stable laws.

Applications to sum of CF coefficients

The law is not in general Gaussian: stable laws.
Example: sum of continued fractions coefficients.

$$\Sigma(x) := \sum_{j=1}^{r} a_j(x) \quad \text{if} \quad x = \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}$$

Applications to sum of CF coefficients

The law is not in general Gaussian: stable laws.
Example: sum of continued fractions coefficients.

$$\Sigma(x) := \sum_{j=1}^{r} a_j(x) \quad \text{if} \quad x = \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}$$

Theorem (Bettin-D.)

As $Q \to \infty$, $\Sigma(x) = (1 + o(1))\frac{12}{\pi^2} \log Q \log\log Q$ a.s. for $x \in \Omega_Q$.
(Proof: take $\phi(x) = \lfloor 1/x \rfloor$, then $I_\phi(t) \sim ct \log t$)

Applications to sum of CF coefficients

The law is not in general Gaussian: stable laws.
Example: sum of continued fractions coefficients.

$$\Sigma(x) := \sum_{j=1}^{r} a_j(x) \quad \text{if} \quad x = \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}$$

Theorem (Bettin-D.)

As $Q \to \infty$, $\Sigma(x) = (1 + o(1))\frac{12}{\pi^2} \log Q \log\log Q$ a.s. for $x \in \Omega_Q$.

(Proof: take $\phi(x) = \lfloor 1/x \rfloor$, then $I_\phi(t) \sim ct \log t$)

This applies to a class of knot invariants, the Kashaev's invariants (Zagier's modularity conjecture '08).

Theorem (Bettin-D.)

For $x \in \mathbb{Q}$, let $J(x) := \sum_{n=0}^{\infty} \prod_{r=1}^{n} |1 - \mathrm{e}^{2\pi i r x}|^2$. Then for some $\mu > 0$,

$$\log J(x) \sim \mu \Sigma(x) \sim \mu \frac{12}{\pi^2} \log Q \log\log Q \quad \text{a.s. for } x \in \Omega_Q.$$

Another application: Dedekind sums

Define the Dedekind sums:

$$s\left(\frac{a}{q}\right) := \sum_{h=1}^{q-1} \left(\left(\frac{ha}{q}\right)\right) \left(\left(\frac{h}{q}\right)\right), \qquad ((x)) := \begin{cases} \{x\} - 1/2 & (x \notin \mathbb{Z}), \\ 0 & \text{(otherwise)}. \end{cases}$$

Another application: Dedekind sums

Define the Dedekind sums:

$$s\left(\frac{a}{q}\right) := \sum_{h=1}^{q-1} \left(\left(\frac{ha}{q}\right)\right)\left(\left(\frac{h}{q}\right)\right), \qquad ((x)) := \begin{cases} \{x\} - 1/2 & (x \notin \mathbb{Z}), \\ 0 & \text{(otherwise)}. \end{cases}$$

Theorem (Vardi '93)

As $Q \to \infty$,

$$\mathbb{P}_Q\left(\frac{s(x)}{\log Q} \leq \frac{v}{2\pi}\right) \to \frac{1}{\pi}\int_{-\infty}^{v} \frac{\mathrm{d}y}{1+y^2}$$

Achieved by Vardi '93 using trace formulas, twisted Eisenstein series. . .

Another application: Dedekind sums

Define the Dedekind sums:

$$s\left(\frac{a}{q}\right) := \sum_{h=1}^{q-1} \left(\left(\frac{ha}{q}\right)\right) \left(\left(\frac{h}{q}\right)\right), \qquad ((x)) := \begin{cases} \{x\} - 1/2 & (x \notin \mathbb{Z}), \\ 0 & \text{(otherwise)}. \end{cases}$$

Theorem (Vardi '93)

As $Q \to \infty$,

$$\mathbb{P}_Q\left(\frac{s(x)}{\log Q} \leq \frac{v}{2\pi}\right) \to \frac{1}{\pi} \int_{-\infty}^{v} \frac{\mathrm{d}y}{1+y^2}$$

Achieved by Vardi '93 using trace formulas, twisted Eisenstein series...
Or: by Dedekind '53, $s(x) = s(-1/x) + \phi(x)$ where $\phi(x) \approx 1/x$.

Glimpse of the proof

Following Vallée '02, Baladi-Vallée '05, express things in term of a transfer operator. This means replacing the map T (which has $T' > 1$) by its adjoint

$$H[f](x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^2} f\Big(\frac{1}{n+x}\Big).$$

Which has much nicer properties.

Glimpse of the proof

Following Vallée '02, Baladi-Vallée '05, express things in term of a transfer operator. This means replacing the map T (which has $T' > 1$) by its adjoint

$$H[f](x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^2} f\Big(\frac{1}{n+x}\Big).$$

Which has much nicer properties. More precisely, we need to study (perturbations of)

$$H_\tau[f](x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^{2+i\tau}} f\Big(\frac{1}{n+x}\Big).$$

Glimpse of the proof

Following Vallée '02, Baladi-Vallée '05, express things in term of a transfer operator. This means replacing the map T (which has $T' > 1$) by its adjoint

$$H[f](x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^2} f\Big(\frac{1}{n+x}\Big).$$

Which has much nicer properties. More precisely, we need to study (perturbations of)

$$H_\tau[f](x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^{2+i\tau}} f\Big(\frac{1}{n+x}\Big).$$

Methods of Dolgopyat '98. Main challenge is to adapt this when very little is known on ϕ.

Thanks for your attention!