Probabilistic models for primes and large gaps

William Banks Kevin Ford Terence Tao

July, 2019

Large gaps between primes

Def: $G(x) = \max_{p_n \leqslant x}(p_n - p_{n-1})$, p_n is the n^{th} prime.

$$2, 3, 5, 7, \ldots, 109, 113, 127, 131, \ldots, 9547, 9551, 9587, 9601, \ldots$$

Upper bound: $G(x) = O(x^{0.525})$ (Baker-Harman-Pintz, 2001).
Improve to $O(x^{1/2} \log x)$ on Riemann Hypothesis (Cramér, 1920).

Lower bound: $G(x) \gg (\log x)\frac{\log_2 x \log_4 x}{\log_3 x}$

(F,Green,Konyagin,Maynard,Tao,2018)

$$\log_2 x = \log\log x, \quad \log_3 x = \log\log\log x, \ldots$$

Conjectures on large prime gaps

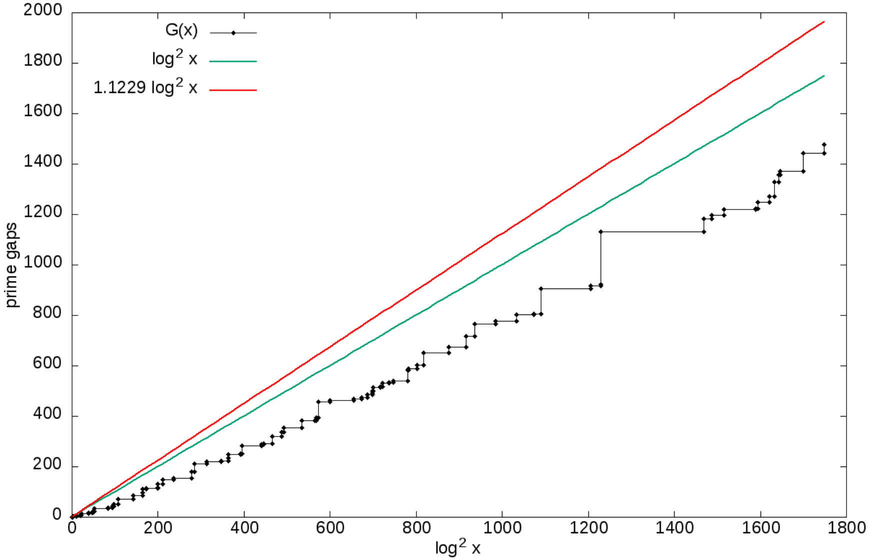
Cramér (1936): $\limsup_{x\to\infty} \frac{G(x)}{\log^2 x} = 1.$

Shanks (1964): $G(x) \sim \log^2 x.$

Granville (1995): $\limsup_{x\to\infty} \frac{G(x)}{\log^2 x} \geqslant 2e^{-\gamma} = 1.1229\ldots$

Computations: $\sup_{x\leqslant 10^{18}} \frac{G(x)}{\log^2 x} \approx 0.92.$

Computational evidence, up to 10^{18}



Cramér’s model of large prime gaps

Random set $\mathcal{C} = \{C_1, C_2, \ldots\} \subset \mathbb{N}$, choose $n \geqslant 3$ to be in $\mathcal{C}$ with probability

$$\frac{1}{\log n},$$

the $1/\log n$ matches the density of primes near n.

Cramér's model of large prime gaps

Random set $\mathcal{C} = \{C_1, C_2, \ldots\} \subset \mathbb{N}$, choose $n \geqslant 3$ to be in $\mathcal{C}$ with probability

$$\frac{1}{\log n},$$

the $1/\log n$ matches the density of primes near n.

Theorem. (Cramér 1936)

With probability 1,

$$\limsup_{m\to\infty} \frac{C_{m+1} - C_m}{\log^2 C_m} = 1.$$

Cramér: "*for the ordinary sequence of prime numbers p_n, some similar relation may hold*".

Cramér model and large gaps

$$\text{a.s.} \quad \limsup_{m\to\infty} \frac{C_{m+1} - C_m}{\log^2 C_m} = 1.$$

Proof:

$$\mathbb{P}(n+1, \dots, n+k \notin \mathcal{C}) \sim \left(1 - \frac{1}{\log n}\right)^k \sim e^{-k/\log n}.$$

$k > (1+\varepsilon)\log^2 n$, this is $\ll n^{-1-\varepsilon}$. Sum converges
$k < (1-\varepsilon)\log^2 n$, this is $\gg n^{-1+\varepsilon}$. Sum diverges.
Finish with Borel-Cantelli.

Cramér's model defect: global distribution

Theorem. (Cramér 1936 "Probabilistic RH"))

With probability 1, $\pi_{\mathcal{C}}(x) := \#\{n \leqslant x : n \in \mathcal{C}\} = \mathrm{li}(x) + O(x^{1/2+\varepsilon})$.

Theorem. (Pintz)

$$\mathbb{E}(\pi_{\mathcal{C}}(x) - \mathrm{li}(x))^2 \sim \frac{x}{\log x},$$

Theorem. (Cramér 1920)

On R.H.,

$$\frac{1}{x}\int_x^{2x} |\pi(t) - \mathrm{li}(t)|^2 \, dt \ll \frac{x}{\log^2 x}$$

Cramér model defect: short intervals

Theorem. (Cramér model in short intervals)

With prob. 1, $\quad \pi_{\mathcal{C}}(x+y) - \pi_{\mathcal{C}}(x) \sim \dfrac{y}{\log x} \quad (y/\log^2 x \to \infty)$

Theorem (Selberg). Let $\frac{y}{\log^2 x} \to \infty$. On RH, for almost all x,

$$\pi(x+y) - \pi(x) \sim \frac{y}{\log x}.$$

Cramér model defect: short intervals

Theorem. (Cramér model in short intervals)

$$\text{With prob. } 1, \quad \pi_{\mathcal{C}}(x+y) - \pi_{\mathcal{C}}(x) \sim \frac{y}{\log x} \quad (y/\log^2 x \to \infty)$$

Theorem (Selberg). Let $\frac{y}{\log^2 x} \to \infty$. On RH, for almost all x,

$$\pi(x+y) - \pi(x) \sim \frac{y}{\log x}.$$

Theorem. (Maier 1985)

$$\forall M > 1, \quad \limsup_{x\to\infty} \frac{\pi(x+\log^M x) - \pi(x)}{\log^{M-1} x} > 1$$

$$\text{and} \quad \liminf_{x\to\infty} \frac{\pi(x+\log^M x) - \pi(x)}{\log^{M-1} x} < 1.$$

Cramér's model defect: k-correlations

Theorem. (k-correlations in Cramér's model)

Let $\mathcal{H}$ be a finite set of integers. With probability 1,

$$\#\{n \leqslant x : n + h \in \mathcal{C} \ \forall h \in \mathcal{H}\} \sim \frac{x}{(\log x)^{|\mathcal{H}|}}.$$

Cramér's model defect: k-correlations

Theorem. (k-correlations in Cramér's model)

Let $\mathcal{H}$ be a finite set of integers. With probability 1,

$$\#\{n \leqslant x : n + h \in \mathcal{C} \ \forall h \in \mathcal{H}\} \sim \frac{x}{(\log x)^{|\mathcal{H}|}}.$$

This fails for primes, e.g. $\mathcal{H} = \{0, 1\}$, because the primes are *biased*

For each prime p, all but one prime in $\in \{1, .., p-1\}$ modulo p;
But $\mathcal{C}$ is equidistributed in $\{0, 1, \ldots, p-1\}$ mod p.

Even for sets $\mathcal{H}$ where we expect many prime patterns, e.g. $\mathcal{H} = \{0, 2\}$ (twin primes), Cramér's model gives the wrong prediction.

Hardy-Littlewood conjectures for primes

Cramér: $\#\{n \leqslant x : n+h \in \mathcal{C}\ \forall h \in \mathcal{H}\} \sim \frac{x}{(\log x)^{|\mathcal{H}|}}$.

Prime k-tuples Conjecture (Hardy-Littlewood, 1922)

$$\#\{n \leqslant x : n+h \text{ prime } \forall h \in \mathcal{H}\} \sim \mathfrak{S}(\mathcal{H}) \frac{x}{(\log x)^{|\mathcal{H}|}} \quad (x \to \infty),$$

where

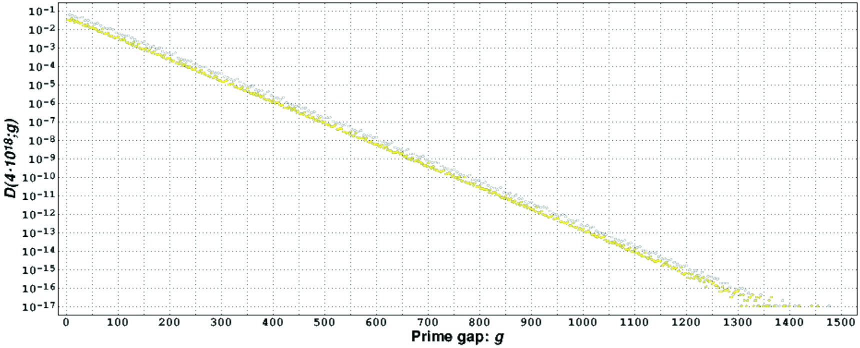
$$\mathfrak{S}(\mathcal{H}) := \prod_p \left(1 - \frac{|\mathcal{H} \text{ mod } p|}{p}\right)\left(1 - \frac{1}{p}\right)^{-|\mathcal{H}|}.$$

The factor $\mathfrak{S}(\mathcal{H})$ captures the bias of real primes;
For each p, $\mathcal{H}$ must avoid the forbidden residue class $0 \mod p$.
$\mathcal{H}$ is admissible if $|\mathcal{H} \mod p| < p$ for all p.

Cramér model defect: gaps

Theorem: With probability 1,

$$\frac{\#\{C_n \leqslant N : C_{n+1} - C_n = k\}}{\#\{C_n \leqslant N\}} \sim \frac{e^{-k/\log N}}{\log N} \quad (N \to \infty)$$



Actual prime gap statistics, $p_n < 4 \cdot 10^{18}$

Granville's refinement of Cramér's model

$$T = o(\log x) \qquad Q = \prod_{p \leqslant T} p = x^{o(1)}$$

Real primes live in $\mathcal{U}_T := \{n \in \mathbb{Z} : \gcd(n, Q) = 1\}$, the integers not divisible by any prime $p \leqslant T$. The set $\mathcal{U}_T$ has density $\theta = \prod_{p \leqslant T}(1 - 1/p)$.

Granville's random model:

For $x < n \leqslant 2x$, choose n in $\mathcal{G}$ with probability

$$\begin{cases} 0 & \text{if } \gcd(n, Q) > 1 \text{ (i.e.,} n \notin \mathcal{U}_T) \\ \frac{1/\theta}{\log n} & \text{if } \gcd(n, Q) = 1 \text{ (i.e.,} n \in \mathcal{U}_T). \end{cases}$$

k-correlations. For all $\mathcal{H}$, with probability 1 we have

$$\#\{n \leqslant x : n + h \in \mathcal{G} \; \forall h \in \mathcal{H}\} \sim \mathfrak{S}(\mathcal{H}) \frac{x}{(\log x)^{|\mathcal{H}|}}, x \to \infty.$$

Granville's refinement of Cramér's model, II

$\mathcal{U}_T := \{n \in \mathbb{Z} : (n, Q) = 1\}$, (integers with no prime factor $\leqslant T$)

Theorem. (Granville 1995)

Write $\mathcal{G} = \{G_1, G_2, \ldots\}$. With probability 1,

$$\limsup_{n\to\infty} \frac{G_{n+1} - G_n}{\log^2 G_n} \geqslant 2e^{-\gamma} = 1.1229\ldots$$

Idea: with $y = c\log^2 x$, $T = y^{1/2+o(1)}$,

$$\#\left([Qm, Qm + y] \cap \mathcal{U}_T\right) = \#\left([0, y] \cap \mathcal{U}_T\right) \sim \frac{y}{\log y}.$$

By contrast, for a *typical* $a \in \mathbb{Z}$,

$$\#\left([a, a + y] \cap \mathcal{U}_T\right) \sim y \prod_{p\leqslant T}\left(1 - \frac{1}{p}\right) \sim 2e^{-\gamma}\frac{y}{\log y} \qquad \text{(Mertens)}$$

Minor flaw in Granville's model

Hardy-Littlewood statistics:

$$\#\{n \leqslant x : n+h \in \mathcal{G}\ \forall h \in \mathcal{H}\} = \mathfrak{S}(\mathcal{H}) \int_2^x \frac{dt}{(\log t)^{|\mathcal{H}|}} + E_{\mathcal{G}}(x;\mathcal{H}),$$

where

$$E_{\mathcal{G}}(x;\mathcal{H}) = \Omega(x/(\log x)^{|\mathcal{H}|+1}).$$

Conjecture

For any admissible $\mathcal{H}$, we have

$$\#\{n \leqslant x : n+h \text{ prime } \ \forall h \in \mathcal{H}\} = \mathfrak{S}(\mathcal{H}) \int_2^x \frac{dt}{(\log t)^{|\mathcal{H}|}} + O(x^{1/2+\varepsilon}).$$

Much numerical evidence for this, especially for
$\mathcal{H} = \{0,2\}, \{0,2,6\}, \{0,4,6\}, \{0,2,6,8\}$.

A new "random sieve" model of primes

Random set $\mathcal{B} \subset \mathbb{N}$:

- For prime p, take a random residue class $a_p \in \{0, \dots, p-1\}$, uniform probability, independent for different p;
- Let $\mathcal{S}_z = \{n \in \mathbb{Z} : n \not\equiv a_p \pmod{p}, p \leqslant z\}$, **random sieved set** with

$$\text{density}(\mathcal{S}_z) = \theta_z = \prod_{p \leqslant z}(1 - 1/p) \sim \frac{e^{-\gamma}}{\log z}.$$

- Take $z = z(n) \sim n^{1/e^{\gamma}} = n^{0.56\dots}$ so that $\theta_{z(n)} \sim \frac{1}{\log n}$, density of primes.
- Define $\mathcal{B} = \{n \in \mathbb{N} : n \notin \mathcal{S}_{z(n)}\}$.

A new "random sieve" model of primes

Random set $\mathcal{B} \subset \mathbb{N}$:

- For prime p, take a random residue class $a_p \in \{0, \ldots, p-1\}$, uniform probability, independent for different p;
- Let $\mathcal{S}_z = \{n \in \mathbb{Z} : n \not\equiv a_p \pmod{p}, p \leqslant z\}$, **random sieved set** with

$$\text{density}(\mathcal{S}_z) = \theta_z = \prod_{p \leqslant z}(1 - 1/p) \sim \frac{e^{-\gamma}}{\log z}.$$

- Take $z = z(n) \sim n^{1/e^{\gamma}} = n^{0.56\ldots}$ so that $\theta_{z(n)} \sim \frac{1}{\log n}$, density of primes.
- Define $\mathcal{B} = \{n \in \mathbb{N} : n \notin \mathcal{S}_{z(n)}\}$.

Global density: $\mathbb{P}(n \in \mathcal{B}) = \mathbb{P}(n \notin \mathcal{S}_{z(n)}) \sim \frac{1}{\log n}$. Matches primes.
Difficulty: $n_1 \in \mathcal{B}$, $n_2 \in \mathcal{B}$ not independent.

We conjecture that the primes and $\mathcal{B}$ share similar *local statistics*.

New model $\mathcal{B}$ and Hardy-Littlewood conjectures

Strong Hardy-Littlewood conjecture (standard version)

$$\#\{n \leqslant x : n+h \text{ prime } \forall h \in \mathcal{H}\} = \mathfrak{S}(\mathcal{H}) \int_2^x \frac{dt}{(\log t)^{|\mathcal{H}|}} + O(x^{1/2+\varepsilon}).$$

$$\mathfrak{S}(\mathcal{H}) \approx \underbrace{\prod_{p \leqslant z(t)} \left(1 - \frac{|\mathcal{H} \bmod p|}{p}\right)}_{=\mathbb{P}(\mathcal{H} \subset \mathcal{S}_{z(t)})} \underbrace{\prod_{p \leqslant z(t)} \left(1 - \frac{1}{p}\right)^{-|\mathcal{H}|}}_{\approx (\log t)^{|\mathcal{H}|}}.$$

New model $\mathcal{B}$ and Hardy-Littlewood conjectures

Strong Hardy-Littlewood conjecture (standard version)

$$\#\{n \leqslant x : n+h \text{ prime } \forall h \in \mathcal{H}\} = \mathfrak{S}(\mathcal{H}) \int_2^x \frac{dt}{(\log t)^{|\mathcal{H}|}} + O(x^{1/2+\varepsilon}).$$

$$\mathfrak{S}(\mathcal{H}) \approx \underbrace{\prod_{p \leqslant z(t)} \left(1 - \frac{|\mathcal{H} \bmod p|}{p}\right)}_{=\mathbb{P}(\mathcal{H} \subset \mathcal{S}_{z(t)})} \underbrace{\prod_{p \leqslant z(t)} \left(1 - \frac{1}{p}\right)^{-|\mathcal{H}|}}_{\approx (\log t)^{|\mathcal{H}|}}.$$

Strong Hardy-Littlewood conjecture (probabilistic version)

$$\#\{n \leqslant x : n+h \text{ prime } \forall h \in \mathcal{H}\} = \int_2^x \mathbb{P}(\mathcal{H} \subset \mathcal{S}_{z(t)})\, dt + O(x^{1/2+\varepsilon}).$$

New model and Hardy-Littlewood, II

Theorem. (BFT 2019)

Fix $\frac{1}{2} \leqslant c < 1, \varepsilon > 0.$ With probability 1,

$$\#\{n \leqslant x : n+h \in \mathcal{B}\ \forall h \in \mathcal{H}\} = \int_2^x \mathbb{P}(\mathcal{H} \subset \mathcal{S}_{z(t)})\, dt + O(x^{1/2+\delta(c)+o(1)})$$

uniformly for $\mathcal{H} \subset [0, \exp\{(\log x)^{c-\varepsilon}\}]$ and $|\mathcal{H}| \leqslant (\log x)^c$, where

$$\delta(1/2) = 0, \qquad \delta(c) < 1/2 \quad (c > 1/2).$$

Notes. Best possible when $c = 1/2$, matches strongest conjectured HL.

When $|\mathcal{H}| \geqslant \frac{\log x}{\log\log x}$, $\mathbb{P}(\mathcal{H} \subset \mathcal{S}_{z(t)})$ is very tiny ($< 1/x$), and we cannot expect a result uniformly for such $\mathcal{H}$.

New model and large gaps: Interval sieve

Interval sieve extremal bound

Define

$$W_y := \min_{(a_p)} \Big| [0, y] \cap \mathcal{S}_{(y/\log y)^{1/2}} \Big|$$

$$= \min_u \# \Big\{ n \in (u, u+y] : n \text{ has no prime factor } \leqslant \left(\frac{y}{\log y} \right)^{1/2} \Big\}.$$

Known bounds:

$$\frac{4y \log_2 y}{\log^2 y} \lesssim W_y \lesssim \frac{y}{\log y}.$$

Upper bound: $u = 0$ and $a_p = 0 \ \forall p$. **Lower bound:** Iwaniec, linear sieve.

Folklore conjecture

$W_y \sim y/\log y$ as $y \to \infty$.

New model and large gaps

Def: $G_{\mathcal{B}}(x)$ is largest gap between consec. elements of $\mathcal{B}$ that are $\leqslant x$.

$$W_y := \min \left| [0,y] \cap \mathcal{S}_{(y/\log y)^{1/2}} \right|. \qquad g(u) := \max\{y : W_y \log y \leqslant u\}.$$

Then

$$\frac{4y \log_2 y}{\log^2 y} \lesssim W_y \lesssim \frac{y}{\log y} \quad \Rightarrow \quad u \lesssim g(u) \lesssim \frac{u \log u}{4 \log_2 u}.$$

Theorem. (BFT 2019)

Let $\xi = 2e^{-\gamma}$. For all $\varepsilon > 0$, with probability 1 there is x_0 s.t.

$$g((1-\varepsilon)\xi \log^2 x) \leqslant G_{\mathcal{B}}(x) \leqslant g((1+\varepsilon)\xi \log^2 x) \quad (x > x_0).$$

Proof tools: Small sieve, large sieve, large deviation inequalities (Bennett's inequality, Azuma's martingale inequality), combinatorics, ...

New model and large gaps

Theorem. (BFT 2019)

Let $\xi = 2e^{-\gamma}$. For all $\varepsilon > 0$, with probability 1 there is x_0 s.t.

$$g((1-\varepsilon)\xi \log^2 x) \leqslant G_{\mathcal{B}}(x) \leqslant g((1+\varepsilon)\xi \log^2 x) \quad (x > x_0).$$

Conjecture. (BFT 2019)

For the largest gap $G(x)$ between primes $\leqslant x$,

$$G(x) \sim g(\xi \log^2 x) \qquad (x \to \infty).$$

New model and large gaps

Theorem. (BFT 2019)

Let $\xi = 2e^{-\gamma}$. For all $\varepsilon > 0$, with probability 1 there is x_0 s.t.

$$g((1-\varepsilon)\xi \log^2 x) \leqslant G_{\mathcal{B}}(x) \leqslant g((1+\varepsilon)\xi \log^2 x) \quad (x > x_0).$$

Conjecture. (BFT 2019)

For the largest gap $G(x)$ between primes $\leqslant x$,

$$G(x) \sim g(\xi \log^2 x) \qquad (x \to \infty).$$

Possible range of $g()$ implies

$$\underbrace{\xi \log^2 x}_{\text{Granville's lower bound}} \lesssim g(\xi \log^2 x) \lesssim \xi \log^2 x \, \frac{\log_2 x}{2 \log_3 x}.$$

Gallagher: HL implies Poisson gaps

Theorem. (Gallagher 1976)

Assume:

$$\#\{n \leqslant x : n + h \text{ prime } \forall h \in \mathcal{H}\} \sim \mathfrak{S}(\mathcal{H}) \int_2^x \frac{dt}{\log^{|\mathcal{H}|} t}$$

uniformly for $|\mathcal{H}| \leqslant k$ (k fixed) and $\mathcal{H} \subset [0, \log^2 x]$. Then $\pi(x + \lambda \log x) - \pi(x) \stackrel{d}{=} \text{Poisson}(\lambda)$, e.g.,

$$\#\{n \leqslant x : p_{n+1} - p_n > \lambda \log x\} \sim e^{-\lambda}\pi(x)$$

Main tool: $\displaystyle\sum_{\substack{\mathcal{H} \subset [0,y] \\ |\mathcal{H}|=k}} \mathfrak{S}(\mathcal{H}) \sim y^k/k!.$

Montgomery-Soundararajan improvement (2004). Poor uniformity in k.

Hardy-Littlewood implies large gaps

Theorem. (BFT 2019)

Let $\mathbf{1}_\mathcal{A}$ be the indicator function of $\mathcal{A}$. If $\mathcal{A} \subset \mathbb{N}$ satisfies

$$\sum_{n \leqslant x} \prod_{h \in \mathcal{H}} \mathbf{1}_\mathcal{A}(n+h) = \mathfrak{S}(\mathcal{H}) \int_2^x \frac{dt}{\log^{|\mathcal{H}|} t} + O(x^{2/3})$$
$$= \int_2^x \mathbb{P}(\mathcal{H} \subset \mathcal{S}_{z(t)})\, dt + O(x^{2/3})$$

uniformly over all tuples $\mathcal{H} \subset [0, \log^2 x]$ with $|\mathcal{H}| \leqslant \frac{\log x}{6 \log_2 x}$, then

$$G_\mathcal{A}(x) := \max\{b - a : 1 \leqslant a < b \leqslant x, (a, b] \cap \mathcal{A} = \emptyset\} \geqslant c \frac{\log^2 x}{\log_2 x}.$$

Averaged Hardy-Littlewood implies large gaps

Theorem. (BFT 2019)

Fix $0 < c < 1$. Suppose that $\mathcal{A} \subset \mathbb{N}$ satisfies the averaged Hardy-Littlewood type conjecture

$$\sum_{\substack{\mathcal{H}\subset[0,y] \\ |\mathcal{H}|=k}} \sum_{n\leqslant x} \prod_{h\in\mathcal{H}} \mathbf{1}_{\mathcal{A}}(n+h) = \sum_{\substack{\mathcal{H}\subset[0,y] \\ |\mathcal{H}|=k}} \int_2^x \frac{\mathfrak{S}_{z(t)}(\mathcal{H})}{\log^k t}\, dt + O(x^{1-c})$$

uniformly for $k \leqslant \frac{Cy}{\log x}$ and $\log x \leqslant y \leqslant (\log^2 x)\log_2 x$. Then

$$G_{\mathcal{A}}(x) \gtrsim g(c\xi \log^2 x).$$

Recall: $G_{\mathcal{B}}(x) \approx g(\xi \log^2 x)$, and $u \lesssim g(u) \ll u(\log u)^{1-o(1)}$.

Large gaps from Hardy-Littlewood

Proof sketch: Weighted count of gaps of size $\geqslant y$:

$$\#\{n \leqslant x : [n, n+y] \cap \mathcal{A} = \emptyset\} = \sum_{n \leqslant x} \underbrace{\prod_{0 \leqslant h \leqslant y} (1 - \mathbf{1}_{\mathcal{A}}(n+h))}_{\text{gap detector}}$$

$$= \sum_{k=0}^{y} (-1)^k \sum_{\substack{\mathcal{H} \subset [0,y] \\ |\mathcal{H}|=k}} \underbrace{\sum_{n \leqslant x} \prod_{h \in \mathcal{H}} \mathbf{1}_{\mathcal{A}}(n+h)}_{\text{HL assumption}}$$

$$\approx \sum_{k=0}^{y} (-1)^k \sum_{\substack{\mathcal{H} \subset [0,y] \\ |\mathcal{H}|=k}} \int_2^x \mathbb{P}(\mathcal{H} \subset \mathcal{S}_{z(t)})\, dt$$

$$= \int_2^x \mathbb{E} \sum_{k=0}^{y} (-1)^k \binom{|\mathcal{S}_{z(t)} \cap [0,y]|}{k}\, dt$$

$$= \int_2^x \mathbb{P}(\mathcal{S}_{z(t)} \cap [0,y] = \emptyset)\, dt.$$

HL imlies no super-large gaps?

Does a uniform HL for $\mathcal{A}$ imply an *upper bound* on large gaps?

HL imlies no super-large gaps?

Does a uniform HL for $\mathcal{A}$ imply an *upper bound* on large gaps?

Answer: NO!

Removal of all elements of $\mathcal{A}$ in an interval $(y, y+\sqrt{y})$, for an infinite sequence of y's, does not affect the HL statistics but creates a very large gap.

Longer intervals (D. Koukoulopoulos)

Definition

$$\beta^+(u) = \limsup_{y\to\infty} \max_{(a_p)} \left(\left| [0,y] \cap S_{y^{1/u}} \right| \frac{\log(y^{1/u})}{e^{-\gamma} y} \right),$$

$$\beta^-(u) = \liminf_{y\to\infty} \min_{(a_p)} \left(\left| [0,y] \cap S_{y^{1/u}} \right| \frac{\log(y^{1/u})}{e^{-\gamma} y} \right).$$

Conjecture

$$\forall u > 2, \quad \limsup_{x\to\infty} \frac{\pi(x+\log^u x) - \pi(x)}{\log^{u-1} x} = \beta^+(u)$$

$$\text{and} \quad \liminf_{x\to\infty} \frac{\pi(x+\log^u x) - \pi(x)}{\log^{u-1} x} = \beta^-(u).$$

Linear sieve + Maier: $0 < \beta^-(u) < 1 < \beta^+(u)$ for $u > 2$.

Summary

Set	Hardy-Littlewood conjecture?	Asymptotic largest gap up to x
$\mathcal{C}$	No (singular series is missing)	$\log^2 x$
$\mathcal{G}$	Yes (with weak error term)	$\xi \log^2 x \leqslant \cdot \leqslant \frac{\xi}{2} \frac{(\log^2 x) \log_2 x}{\log_3 x}$
$\mathcal{B}$	Yes (with error $O(x^{1-c})$)	$\xi \log^2 x \leqslant \cdot \leqslant \frac{\xi}{2} \frac{(\log^2 x) \log_2 x}{\log_3 x}$
Primes	Yes (conjecturally)	$\xi \log^2 x$ (conjecturally)
$\mathcal{A}$	Assumed (error $O(x^{1-c})$)	$\gg c \frac{\log^2 x}{\log_2 x}$
$\mathcal{A}$	Assumed on avg. (error $O(x^{1-c})$)	$\geqslant g(c\xi \log^2 x)$

References

- H. Cramér, *Some theorems concerning prime numbers*, Ark. Mat. Astr. Fys. **15** (1920), 1–33.
- H. Cramér, *On the order of magnitude of the difference between consecutive prime numbers*, Acta Arith. **2** (1936), 396–403.
- Andrew Granville, *Harald Cramér and the distribution of prime numbers*, Scandanavian Actuarial J. **1** (1995), 12–28.
- H. L. Montgomery and K. Soundararajan, *Primes in short intervals*, Comm. Math. Phys. **252** (2004), 589–617.
- T. R. Nicely, *First occurrence prime gaps*, web page: `http://www.trnicely.net/gaps/gaplist.html`
- J. Pintz, *Cramér vs. Cramér. On Cramér's Probabilistic Model for primes*, Func. Approx. Comment. Math. **37** (2007), 361–376.
- T. Oliveira e Silva, *Gaps between consecutive primes* (web page), `http://sweet.ua.pt/tos/gaps.html`