# A new method to compute the Fourier coefficients of Langlands Eisenstein series on GL(n)

*(joint work with S.D. Miller and M. Woodbury)*

## KEY IDEAS

The basic idea is to compute generic coefficients of general Eisenstein series by reduction to Borel Eisenstein series. As is well known the local calculation reaches a point where it depends only on local data. Thus we see that the local factors for the coefficients are mimicked by the local factors of the Borel series.

We call this the **"Template Method."** The calculation for the Borel Eisenstein series (which has a simpler solution) is used as a template to obtain the Fourier coefficients for other Eisenstein series.

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## $SL(2,\mathbb{Z})$ Eisenstein Series:

$$\mathfrak{h}^2 := \left\{ z = x + iy \;\middle|\; x \in \mathbb{R},\ y > 0 \right\}.$$

$$E(z,s) = \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{y^s}{|cz+d|^s}, \qquad (\Re(s) > 1).$$

## Fourier Expansion of the $SL(2,\mathbb{Z})$ Eisenstein Series

$$E(z,s) = \underbrace{y^s + \phi(s)y^{1-s}}_{\text{Constant Term}}$$

$$+ \underbrace{\frac{2\pi^s\sqrt{y}}{\Gamma(s)\zeta(2s)}\sum_{n\neq 0}\sigma_{1-2s}(n)|n|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|y)}_{n^{th}\ \text{Fourier Coefficient}} \cdot e^{2\pi inx}$$

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## First Coefficient and the $n^{th}$ Hecke Eigenvalue

$$\boxed{1^{st}\ \text{Coeff} := \frac{2\pi^s\sqrt{y}}{\Gamma(s)\zeta(2s)} K_{s-\frac{1}{2}}(2\pi|n|y)} \quad \boxed{\lambda(n,s) := \sigma_{1-2s}(n)|n|^{s-\frac{1}{2}}.}$$

# The $p^{th}$ Hecke eigenvalue is easy to compute

**Action of $p^{th}$ Hecke operator $T_p$ on $F(z)$**

$$T_p F(z) = \frac{1}{\sqrt{p}} \left( F(pz) \; + \sum_{0 \le b < p} F\left(\frac{z+b}{p}\right) \right)$$

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## Easy way to compute Action of $T_p$ on $E(z,s)$

$$T_py^s = \frac{1}{\sqrt{p}}\left((F(pz))py)^s + \sum_{0\le b<p}\left(\frac{y}{p}\right)^s\right)$$

$$= \left(p^{s-\frac{1}{2}} + p^{\frac{1}{2}-s}\right)y^s$$

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$$= \left(p^{s-\frac{1}{2}} + p^{\frac{1}{2}-s}\right) y^s$$

$\Longrightarrow$

$$T_pE(z,s) = \lambda(p,s)\cdot E(z,s)$$

$$\boxed{\lambda(p,s) = \left(p^{s-\frac{1}{2}} + p^{\frac{1}{2}-s}\right).}$$

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## Iwasawa decomposition

Upper half plane: $\mathfrak{h}^2 := GL(2,\mathbb{R})/O(2,\mathbb{R}) \cdot \mathbb{R}^\times$

$$g \in \mathfrak{h}^2 \implies g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, \qquad (x \in \mathbb{R},\ y > 0)$$

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Here

$$\Gamma := SL(2,\mathbb{Z}), \qquad B := \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} = \text{Borel subgroup.}$$

# The $SL(3,\mathbb{Z})$ Eisenstein series

**Observation:** $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}.$

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## The $I_s$ function

$$s = (s_1, s_2) \in \mathbb{C}^2.$$

$$\boxed{I_s(g) := y_1^{s_1+2s_2} y_2^{2s_1+s_2}}$$

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## $SL(3,\mathbb{Z})$ Eisenstein Series associated to $P_{2,1}$

$$\boxed{E_{P_{2,1}}(g,s) := \sum_{\gamma \in (P_{2,1} \cap \Gamma)\backslash\Gamma} I_s(\gamma g, P_{2,1})} \qquad (g \in \mathfrak{h}^3)$$

$$\Gamma = SL(3,\mathbb{Z}).$$

## Langlands $SL(3,\mathbb{Z})$ Eisenstein series twisted by an $SL(2,\mathbb{Z})$ cusp form

We have the Langlands decomposition of the parabolic $P_{2,1}$:

$$\mathcal{P} = P_{2,1} = M^{\mathcal{P}} \cdot N^{\mathcal{P}}$$

with

$$N^{\mathcal{P}} = \left\{ \begin{pmatrix} I_2 & * \\ 0 & I_1 \end{pmatrix} \right\}, \qquad M^{\mathcal{P}} = \left\{ \begin{pmatrix} GL_2 & 0 \\ 0 & GL_1 \end{pmatrix} \right\}.$$

Every $g \in \mathfrak{h}^3$ can be put in the form $g = \mathfrak{n}(g) \cdot \mathfrak{m}(g)$ with

$$\mathfrak{n}(g) \in N^{\mathcal{P}}, \quad \mathfrak{m}(g) = \begin{pmatrix} m_2(g) & 0 \\ 0 & m_1(g) \end{pmatrix} \in N^{\mathcal{P}}.$$

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**Langlands $SL(3,\mathbb{Z})$ Eisenstein Series twisted by $SL(2,\mathbb{Z})$ cusp form**

$$E_{P_{2,1}}(g,s,\phi) := \sum_{\gamma \in \left(P_{2,1} \cap \Gamma\right) \backslash \Gamma} I_s\left(\gamma g,\ P_{2,1}\right) \cdot \phi\big(m_2(\gamma g)\big).$$

# Cusp forms for $GL(n, \mathbb{R})$

A cusp form is a complex valued function

$$\phi : \mathfrak{h}^n \to \mathbb{C}$$

satisfying the following conditions:

- $\boxed{\phi(\gamma g) \ = \ \phi(g),} \quad \big(\forall\, \gamma \in SL(n, \mathbb{Z}), \ \ g \in \mathfrak{h}^n\big);$
- *eigenfunction of invariant differential operators;*
- *has moderate growth;*
- *vanishes at the cusps.*

# Invariant Differential Operators on $\mathfrak{h}^n$

Recall that $\mathfrak{h}^n$ which can be identified with the set of matrices $xy$:

$$x \in U_n(\mathbb{R}) = \begin{pmatrix} 1 & & & x_{i,j} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix}$$

$$(x_{i,j} \in \mathbb{R}, \qquad y_i > 0).$$

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$$(x_{i,j} \in \mathbb{R}, \qquad y_i > 0).$$

**The space $\mathcal{D}^n$ of invariant differential operators**

The space $\mathcal{D}^n$ consists of all polynomials (with complex coefficients) in the variables $\left\{ \frac{\partial}{\partial x_{i,j}}, \ \frac{\partial}{\partial y_k} \right\}$ which are invariant under $GL(n, \mathbb{R})$ transformations.

## Langlands Parameters

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Langlands parameters can be used to construct a character of the torus which is an eigenfunction of all $\delta \in \mathcal{D}^n$.

## Construction of an eigenfunction of $\mathcal{D}^n$

**Definition:** Let $\alpha \in \mathbb{C}^n$ denote a set of Langlands parameters. We define a character $I_\alpha : U_n(\mathbb{R})\backslash\mathfrak{h}^n \to \mathbb{C}$ by

$$\boxed{I_\alpha(g) := \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{i,j} \frac{1+\alpha_j - \alpha_{j+1}}{n}}}$$

$$g = xy \in \mathfrak{h}^n, \qquad b_{i,j} = \begin{cases} i \cdot j & \text{if } i + j \leq n, \\ (n-i)(n-j) & \text{if } i + j \geq n. \end{cases}$$

# The eigenfunction $I_\alpha$

For Langlands parameters $\alpha \in \mathbb{C}^n$ we have

$$\boxed{\delta I_\alpha = \lambda_\delta \cdot I_\alpha \qquad \big(\forall\, \delta \in \mathcal{D}^n\big)}$$

and for the Laplacian $\Delta$ we have

$$\lambda_\Delta = \frac{n^3 - n}{24} - \frac{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2}{2}.$$

## Character of the unipotent group

Let $u \in U_n$ be given by

$$u = \begin{pmatrix} 1 & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ & 1 & u_{2,3} & \cdots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & u_{n-1,n} \\ & & & & 1 \end{pmatrix}$$

For $L = (\ell_1, \ell_2, \ldots, \ell_{n-1}) \in \mathbb{Z}^{m-1}$, we may define a character:

$$\boxed{\psi_L(u) := e^{2\pi i \left(\ell_1 u_{1,2} + \cdots + \ell_{n-1} u_{n-1,n}\right)}.}$$

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Here

$$\psi_L(u \cdot u') = \psi_L(u)\psi_L(u'), \qquad (u, u' \in U_n).$$

# Whittaker Functions

Given Langlands parameters $\alpha \in \mathbb{C}^n$ and a character $\psi$ of $U_n(\mathbb{R})$ there is a unique Whittaker function

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## Whittaker Function Properties

- $\delta W_\alpha = \lambda_\delta \cdot W_\alpha, \qquad \big(\forall\, \delta \in \mathcal{D}^n\big),$
- $W_\alpha(ug) = \psi(u) \cdot W_\alpha(g), \quad (\forall u \in U_n(\mathbb{R}),\, g \in GL(n, \mathbb{R})),$
- $W_\alpha$ is invariant under all permutations of $\alpha = \{\alpha_1, \ldots, \alpha_n\}$,
- $W_\alpha$ has holomorphic continuation to all $\alpha \in \mathbb{C}^n$,
- $W_\alpha(y)$ has rapid decay in $y_i \to \infty$ where $y = \mathrm{diag}(y_1, y_2, \cdots y_n)$,
- $W_\alpha(y)$ has prescribed polynomial asymptotics as all $y_i \to 0$.
- Simpler but analogous situation over $\mathbb{Q}_p$ by Casselman-Shalika.
    - Here we have an additional normalization $W_\alpha(e) = 1$.

## Example 1 (GL(2) Whittaker function)

$$g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in \mathfrak{h}^2$$

Langlands parameter: $\alpha = (\nu, -\nu) \in \mathbb{C}^2$

$$W_\alpha(g) = \frac{\Gamma(1/2+\nu)}{\pi^{1/2+\nu}} \int_{-\infty}^{\infty} \left( \frac{y}{(x+u)^2 + y^2} \right)^{1/2+\nu} e^{-2\pi i u} \, dx$$

$$= 2\sqrt{y}\, K_\nu(2\pi y) \cdot e^{2\pi i x}$$

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- $W_\alpha(g)$ is an explicit product of Γ's times the "Jacquet integral".
- The small $y$ asymptotics have leading terms $\pi^{\pm\nu}\Gamma(\mp\nu)y^{1/2\pm\nu}$.
- These facts easily generalize to general Chevalley groups.

## Fourier-Whittaker expansion of cusp forms $\phi$ (Shalika-Piatetski-Shapiro)

Assume $\delta\,\phi = \lambda_\alpha \cdot \phi$ for all $\delta \in \mathcal{D}^n$.

Let $M = (m_1, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1}$ and $\Gamma_{n-1} = \mathrm{SL}(n-1, \mathbb{Z})$.

$$\phi(g) = \sum_{\gamma \in U_{n-1} \backslash \Gamma_{n-1}} \sum_{M \neq 0} \frac{A(M)}{\prod\limits_{k=1}^{n-1} |m_k|^{\frac{k(n-k)}{2}}} \, W_\alpha\left( M^* \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right)$$

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where $g \in \mathfrak{h}^n$ and

$$M^* = \begin{pmatrix} m_1 \cdots m_{n-2}|m_{n-1}| & & & \\ & \ddots & & \\ & & m_1 & \\ & & & 1 \end{pmatrix}.$$

*$A(m_1, \ldots, m_{n-1})$ is called the $M^{th}$ Fourier coefficient of $\phi$.*

# L-function associated to a Hecke cusp form $\phi$

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## L-function

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$$L(s,\phi) = \sum_{m=1}^{\infty} \frac{A(m,1,\ldots,1)}{m^s}$$

## Euler Product

$$L(s,\phi) =$$

$$= \prod_p \left(1 - \frac{A(p,1,\ldots,1)}{p^s} + \frac{A(1,p,1,\ldots,1)}{p^{2s}} - \frac{A(1,1,p,\ldots,1)}{p^{3s}} \right.$$

$$\left. + \quad \cdots \quad + (-1)^{n-1}\frac{A(1,,\ldots,1,p)}{p^{(n-1)s}} + \frac{(-1)^n}{p^{ns}}\right)^{-1}$$

# Functional Equation of $L(s, \phi)$

$L(s, \phi)$ is a degree $n$ L-function. This means the completed L-function has $n$ Gamma factors and satisfies the functional equation

$$\boxed{L^*(s, \phi) := \pi^{-\frac{ns}{2}} \prod_{i=1}^{n} \Gamma\left(\frac{s - \alpha_i}{2}\right) L(s, \phi) = L^*\left(1 - s, \widetilde{\phi}\right)}$$

where $\widetilde{\phi}$ denotes the dual form which has $M^{th}$ Fourier coefficient (for $M = (m_1, m_2, \ldots, m_{n-1})$) given by $A(m_{n-1}, m_{n-2}, \ldots, m_1)$.

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Let $M = (m_1, \ldots, m_{n-1})$ and $\mathbf{1} = (1, \ldots, 1)$. Then

$$A_\phi(M) = \underbrace{A_\phi(\mathbf{1})}_{\text{first coeff.}} \cdot \underbrace{\lambda_\phi(M)}_{M^{th}\text{ Hecke coeff.}}$$

where $\lambda_\phi(\mathbf{1}) = 1$.

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## First Coefficient of a cusp form $\phi$

$$\left|A_\phi(\mathbf{1})\right|^2 = \frac{\langle \phi,\ \phi \rangle}{\mathrm{Vol}\left(\Gamma \backslash \mathfrak{h}^n\right) \cdot L(1, \phi_j, \mathrm{Ad}) \prod\limits_{1 \le j \ne k \le n} \Gamma\left(\frac{1+\alpha_j - \alpha_k}{2}\right)}$$

where $L(s, \phi_j, \mathrm{Ad}) := \frac{L\left(s,\, \phi_j \times \overline{\phi_j}\right)}{\zeta(s)}$.

# Langlands Eisenstein Series for $GL(n, \mathbb{R})$

## Parabolic subgroups

Associated to a partition $n = n_1 + \cdots + n_r$, we have a standard parabolic subgroup

$$\mathcal{P} := \mathcal{P}_{n_1, n_2, \ldots, n_r} := \left\{ \begin{pmatrix} GL_{n_1} & * & \cdots & * \\ 0 & GL_{n_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & GL_{n_r} \end{pmatrix} \right\} = N^{\mathcal{P}} \cdot M^{\mathcal{P}}$$

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with nilpotent radical and Levi subgroup

$$N^{\mathcal{P}} := \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * \\ 0 & I_{n_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_r} \end{pmatrix} \right\}, \quad M^{\mathcal{P}} := \left\{ \begin{pmatrix} GL_{n_1} & 0 & \cdots & 0 \\ 0 & GL_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & GL_{n_r} \end{pmatrix} \right\}.$$

# Langlands Eisenstein Series for $GL(n, \mathbb{R})$

Fix a parabolic subgroup $\mathcal{P}$. Every $g \in \mathfrak{h}^n$ can be put in the form

$$\boxed{g = \mathfrak{n}(g) \cdot \mathfrak{m}(g)} \qquad \Big(\mathfrak{n}(g) \in N^{\mathcal{P}},\ \ m(g) \in M^{\mathcal{P}}\Big).$$

Here

$$\mathfrak{m}(g) = \begin{pmatrix} \mathfrak{m}_1(g) & 0 & \cdots & 0 \\ 0 & \mathfrak{m}_2(g) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathfrak{m}_r(g) \end{pmatrix}$$

where $\mathfrak{m}_i(g) \in GL_{n_i}$.

Let $n > 2$, $n = n_1 + n_2 + \cdots + n_r$ and $\mathcal{P} := \mathcal{P}_{n_1, n_2, \ldots, n_r}$.

Let $s = (s_1, s_2, \ldots, s_r) \in \mathbb{C}$ where $\sum_{i=1}^{r} n_i s_i = 0$.

# Langlands Eisenstein Series for $GL(n, \mathbb{R})$

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Let $s = (s_1, s_2, \ldots, s_r) \in \mathbb{C}$ where $\sum_{i=1}^{r} n_i s_i = 0$.

## Multiplicative character on a parabolic subgroup

Define the function $| \ |_{\mathcal{P}}^{s} : \mathfrak{h}^n \to \mathbb{C}$ by

$$\boxed{|g|_{\mathcal{P}}^{s} := \prod_{j=1}^{r} \Big| \det(\mathfrak{m}_j(g)) \Big|^{s_j}} \qquad \Big( g = \mathfrak{n}(g)\, \mathfrak{m}(g)\, k \in GL(n, \mathbb{R}) \Big).$$

Here $K = O(n, \mathbb{R})$. Note that $\sum_{i=1}^{r} n_i s_i = 0$ guarantees that $| \ |_{\mathcal{P}}^{s}$ is invariant under scalar multiplication.

# Langlands Eisenstein Series for $GL(n, \mathbb{R})$

Let $\phi_i : \mathfrak{h}^{n_i} \to \mathbb{C}$ be automorphic forms for $\mathrm{SL}(n_i, \mathbb{Z})$ for $i = 1, 2, \ldots, r$.

# Langlands Eisenstein Series for $GL(n, \mathbb{R})$

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**(Automorphic form $\Phi$ associated to a parabolic $\mathcal{P}$)**

Define an automorphic form $\Phi := (\phi_1, \ldots, \phi_r)$ on $\mathfrak{h}^n$ by the recipe

$$\boxed{\Phi(\mathfrak{n}\,\mathfrak{m}\,k) := \prod_{i=1}^{r} \phi_i(\mathfrak{m}_i)}$$

where $\mathfrak{n} \in N^{\mathcal{P}}$, $\mathfrak{m} \in M^{\mathcal{P}}$, $k \in K = O(n, \mathbb{R})$, and

$$\mathfrak{m} = \begin{pmatrix} \mathfrak{m}_1 & 0 & \cdots & 0 \\ 0 & \mathfrak{m}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathfrak{m}_r \end{pmatrix}, \qquad \Big(\mathfrak{m}_i \in GL(n_i, \mathbb{R})\Big),$$

# Langlands Eisenstein Series for $GL(n, \mathbb{R})$

## DEFINITION: Langlands Eisenstein Series

Let $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ with $n > 2$. Consider a partition $n = n_1 + \cdots + n_r$ with associated parabolic subgroup $\mathcal{P}$. Let $s = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$ where $\sum_{i=1}^{r} n_i s_i = 0$.

The Langlands Eisenstein series determined by this data is defined by:

$$\boxed{E_{\mathcal{P},\Phi}(g, s) := \sum_{\gamma \in (\mathcal{P} \cap \Gamma) \backslash \Gamma} \Phi(\gamma g) \cdot |\gamma g|_{\mathcal{P}}^{s}}$$

# Langlands Spectral Decomposition

**Theorem (Langlands)**

*Let $\phi_1, \phi_2, \ldots$ denote an orthogonal basis of Maass forms for $\mathrm{SL}(n,\mathbb{Z})$. Assume that $F \in \mathcal{L}^2(\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n)$ is orthogonal to the residual spectrum. Then for $g \in \mathrm{GL}(n,\mathbb{R})$ we have*

$$F(g) = \sum_{j=1}^{\infty} \langle F, \phi_j \rangle \frac{\phi_j(g)}{\langle \phi_j, \phi_j \rangle} + \sum_{\mathcal{P}} \sum_{\Phi} c_{\mathcal{P}} \cdot \int_{\mathrm{Re}(s_1)=0} \cdots \int_{\mathrm{Re}(s_{r-1})=0} \Big\langle F, E_{\mathcal{P},\Phi}(*\,, s) \Big\rangle E_{\mathcal{P},\Phi}(g\,, s)\, ds_1 \cdots ds_{r-1}$$

*where the sum over $\mathcal{P}$ ranges over parabolics associated to partitions $n_1 + \cdots + n_r = n$, and the sum over $\Phi$ ranges over an orthonormal basis of Maass forms associated to $\mathcal{P}$. Here $s = (s_1, \ldots, s_r)$ where $\sum_{k=1}^{r} n_k s_k = 0$ for the partition $\sum_{k=1}^{r} n_k = n$.*

# Minimal Parabolic Eisenstein Series

$\mathcal{P}_{\mathrm{Min}}$ corresponds to the partition $n = 1 + 1 + \cdots + 1$. Let $s = (s_1, \ldots, s_n)$ with $\sum_{i=1}^{n} s_i = 0$.

$$\boxed{E_{\mathcal{P}_{\mathrm{Min}}}(g, s) := \sum_{\gamma \in (\mathcal{P}_{\mathrm{Min}} \cap \Gamma) \backslash \Gamma} |\gamma g|_{\mathcal{P}_{\mathrm{Min}}}^{s}} \quad \Big( g \in GL(n, \mathbb{R}), \;\; \Re(s) \gg 1 \Big).$$

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## Non-constant Fourier coefficients of $E_{\mathcal{P}_{\mathrm{Min}}}$

Let $M = (m_1, \ldots, m_{n-1})$. Then

$$A_{\mathcal{P}_{\mathrm{Min}}}(M,s) = A_{\mathcal{P}_{\mathrm{Min}}}\big((1,\ldots,1),s\big) \cdot \lambda_{\mathcal{P}_{\mathrm{Min}}}(M,s),$$

$$A_{\mathcal{P}_{\mathrm{Min}}}(M,s) \;=\; \underbrace{A_{\mathcal{P}_{\mathrm{Min}}}(\mathbf{1},s)}_{\text{first coeff.}} \cdot \underbrace{\lambda_{\mathcal{P}_{\mathrm{Min}}}(M,s)}_{M^{th}\text{ Hecke coeff.}}$$

# Fourier coefficients of Minimal Parabolic Eisenstein Series

**Theorem (Selberg, Maass, Terras, Langlands, Shahidi)**

*Let* $E_{\mathcal{P}_{\mathrm{Min}}}(g,s)$ *have Langlands parameters* $\alpha = (\alpha_1(s), \ldots, \alpha_n(s))$. *Then*

$$\boxed{\lambda_{\mathcal{P}_{\mathrm{Min}}}\big((m,1,\ldots,1),s\big) = \sum_{c_1 c_2 \cdots c_n = m} c_1^{\alpha_1(s)} c_2^{\alpha_2(s)} \cdots c_n^{\alpha_n(s)}}$$

for $m = 0, 1, 2, 3, \ldots,$ and

$$\boxed{A_{\mathcal{P}_{\mathrm{Min}}}\big((1,\ldots,1),s\big) = \prod_{1 \le j < k \le n} \zeta^*\big(1 + \alpha_j(s) - \alpha_k(s)\big)^{-1}}$$

where

$$\zeta^*(w) = \pi^{-\frac{w}{2}} \Gamma\left(\frac{w}{2}\right) \zeta(w) = \zeta^*(1-w)$$

for $w \in \mathbb{C}$.

# The $m^{th}$ Hecke eigenvalue of $E_{\mathcal{P},\Phi}(g,s)$

## Theorem (G)

- Partition: $n = n_1 + \cdots n_r$.
- $s = (s_1, \ldots, s_r) \in \mathbb{C}^r$ with $n_1 s_1 + \cdots + n_r s_r = 0$.
- Hecke operator: $T_m$ for $m = 1, 2, 3, \ldots$

Then

$$\boxed{T_m \, E_{\mathcal{P},\Phi}(g,s) \; = \; \lambda_{E_{P,\Phi}}\big((m,1,\ldots,1),s\big) \cdot E_{\mathcal{P},\Phi}(g,s)}$$

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where

$$\lambda_{E_{P,\Phi}}\big((m, 1, \ldots, 1), s\big) = \sum_{\substack{1 \le c_1, c_2, \ldots, c_r \in \mathbb{Z} \\ c_1 c_2 \cdots c_r = m}} \lambda_{\phi_1}(c_1) \cdots \lambda_{\phi_r}(c_r)$$

$$\cdot\, c_1^{s_1 + N_1 + \frac{n_1 - n}{2}}\, c_2^{s_2 + N_2 + \frac{n_2 - n}{2}} \cdots c_r^{s_r + N_r + \frac{n_r - n}{2}},$$

and $N_1 = 0$, $N_i = n_1 + n_2 + \cdots n_{i-1}$ for $i \ge 1$. In the above $\lambda_{\phi_i}(c_i)$ denotes the eigenvalue of the $\mathrm{SL}(n_i, \mathbb{Z})$ Hecke operator $T_{c_i}$ acting on $\phi_i$ which may also be viewed as the $(c_i, 1, 1, \ldots, 1)$ Fourier coefficient of $\phi_i$.

# Langlands Eisenstein series for $\mathrm{SL}(4, \mathbb{Z})$

There are 4 standard non-associate parabolic subgroups associated to the partitions:

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

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which depend on $s$ and the Langlands parameters of $\Phi$. Set $M = (m_1, m_2, m_3)$. Then

$$\int\limits_{U_4(\mathbb{Z})\backslash U_4(\mathbb{R})} E_{\mathcal{P}}(ug, s)\,\overline{\psi_M(u)}\,du \;=\; \frac{A_{E_{\mathcal{P},\Phi}}(M, s)}{|m_1|^{\frac{3}{2}}|m_2|^2|m_3|^{\frac{3}{2}}}\; W_\alpha(Mg).$$

$$\boxed{A_{E_{\mathcal{P},\Phi}}(M, s) = A_{E_{\mathcal{P},\Phi}}\big((1,1,1), s\big) \cdot \lambda_{E_{\mathcal{P},\Phi}}(M, s)}$$

# Notation for L-functions on GL(1), GL(2), GL(2) × GL(2)

## GL(1)

$$\zeta^*(w) = \pi^{-\frac{w}{2}} \Gamma\left(\frac{w}{2}\right) \zeta(w) = \zeta^*(1-w), \qquad (w \in \mathbb{C}).$$

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## GL(2)

$\phi =$ Masss cusp form on GL(2) with Laplace eigenvalue $1 - v^2$ $(v \in \mathbb{C})$ and Langlands parameter $\alpha = \{\alpha_1, \alpha_2\} = \{v, -v\}$.

$$L^*(w,\phi) := \pi^{-w}\Gamma\left(\frac{w+\alpha_1}{2}\right)\Gamma\left(\frac{w+\alpha_2}{2}\right)L(w,\phi) = L^*(1-w,\phi).$$

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### GL(2) × GL(2) (Rankin-Selberg convolution)

$\phi_1, \phi_2$ with Laplace eigenvalues $\frac{1}{4} - v^2$, $\frac{1}{4} - v'^2$, respectively.

$$L^*(w,\phi_1\times\phi_2) = \pi^{-2w}\left(\prod_{i=1}^{2}\prod_{j=1}^{2}\Gamma\left(\frac{w+\alpha_i+\alpha_j}{2}\right)\right)L(w,\phi_1\times\phi_2)$$
$$= L^*(1-w,\phi_1\times\phi_2).$$

# Notation for L-functions on GL(3)

## GL(3) Maass Form

Finally, for a Maass form $\phi$ on GL(3) with Langlands parameter

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) = (v + 2v',\, v - v', -2v - v'), \quad (v, v' \in \mathbb{C})$$

define the completed L-function $L^*(w, \phi)$ associated to $\phi$ by

$$L^*(w,\phi) := \pi^{-\frac{3w}{2}} \Gamma\left(\frac{w+\alpha_1}{2}\right) \Gamma\left(\frac{w+\alpha_2}{2}\right) \Gamma\left(\frac{w+\alpha_3}{2}\right) L(w,\phi)$$
$$= L^*\big(1-w, \overline{\phi}\,\big).$$

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$$= L^*\left(1-w, \overline{\phi}\right).$$

## Adjoint L-function

The adjoint L-function of a Maass form $\phi$ on GL($n$) is defined by

$$L(w, \mathrm{Ad}\ \phi) := L(w, \phi \times \overline{\phi})/\zeta(w).$$

# Langlands Eisenstein series for the 4=2+2 partition

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$$\boxed{\Phi = (\phi_1, \phi_2).}$$

$$E_{\mathcal{P},\Phi}(g,s) := \sum_{\gamma \in (\mathcal{P}\cap\Gamma)\backslash\Gamma} \Phi(\gamma g)\cdot |\gamma g|_{\mathcal{P}}^{s}$$

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## The minimal parabolic Eisenstein series $E_{\mathcal{P}_{2,2},\Phi^*}$

KEY IDEA: Replace $\Phi = (\phi_1, \phi_2)$ by $\Phi^* = (E_1^*, E_2^*)$ where $E_1^*, E_2^*$ are minimal parabolic Eisenstein series for $\mathrm{SL}(2,\mathbb{Z})$ with the same Langlands parameters as $\phi_1, \phi_2$. Then compute the first Fourier coefficient of $E_{\mathcal{P}_{2,2},\Phi^*}$

# The minimal parabolic Eisenstein series $E_{\mathcal{P}_{2,2},\Phi^*}$

$$y = \begin{pmatrix} y_1y_2y_3 & & & \\ & y_1y_2 & & \\ & & y_1 & \\ & & & 1 \end{pmatrix}$$

## The replacements for $\phi_1, \phi_2$

$$E_1^*(y) = E^*_{P_{\mathrm{Min}}}\left( \begin{pmatrix} y_1y_2y_3 & \\ & y_1y_2 \end{pmatrix}, \nu \right) = \zeta^*(1+2v) E_{P_{\mathrm{Min}}}\left( \begin{pmatrix} y_3 & \\ & 1 \end{pmatrix}, \nu \right)$$

$$= \zeta^*(1+2v) \sum_{\gamma \in \backslash U_2(\mathbb{Z})\,\mathrm{SL}(2,\mathbb{Z})} y_3^{\frac{1}{2}+v} \Big|_\gamma,$$

$$E_2^*(y) = E^*_{P_{\mathrm{Min}}}\left( \begin{pmatrix} y_1 & \\ & 1 \end{pmatrix}, \nu' \right) = \zeta^*(1+2v') E_{P_{\mathrm{Min}}}\left( \begin{pmatrix} y_1 & \\ & 1 \end{pmatrix}, \nu' \right)$$

$$= \zeta^*(1+2v') \sum_{\gamma \in \backslash U_2(\mathbb{Z})\,\mathrm{SL}(2,\mathbb{Z})} y_1^{\frac{1}{2}+v'} \Big|_\gamma.$$

**Important:** *$E_1^*, E_2^*$ are normalized to have first Fourier coeff. $= 1$.*

# The minimal parabolic Eisenstein series $E_{\mathcal{P}_{2,2},\Phi^*}$

Then $E_{\mathcal{P}_{2,2},\Phi^*}(g,(s_1,s_2))$ has Langlands parameters

$$\alpha = (s_1+v, s_1-v, -s_1+v', -s_1-v').$$

## The first Fourier coefficient of $E_{\mathcal{P}_{2,2},\Phi^*}$

$$\boxed{A_{E_{P_{2,2},\Phi^*}}((1,1,1),s)} = \zeta^*(1+2v)\,\zeta^*(1+2v')\cdot\Big(\zeta^*(1+2v) \cdot\zeta^*(1+2s_1-v-v')\zeta^*(1+2s_1+v-v')\zeta^*(1+2s_1-v+v') \cdot\zeta^*(1+2s_1+v+v')\zeta^*(1+2v')\Big)^{-1}$$

$$= \Big(\zeta^*(1+2s_1-v-v')\zeta^*(1+2s_1+v-v') \cdot\zeta^*(1+2s_1-v+v')\zeta^*(1+2s_1+v+v')\Big)^{-1}$$

$$= \boxed{L^*\left(1+2s_1, E_1^*\times E_2^*\right)^{-1}.}$$

# The first Fourier coefficient of $E_{\mathcal{P}_{2,2},\Phi}$

## Proposition (Shahidi-Woodbury-G)

The first coefficient of $E_{\mathcal{P}_{2,2},\Phi}$, where $\phi_1, \phi_2$ are Maass forms of norm 1 on GL(2) with spectral parameters $\frac{1}{2} + v$, $\frac{1}{2} + v'$, is given by

$$\left( L(1, \mathrm{Ad}\ \phi_1)^{\frac{1}{2}} L(1, \mathrm{Ad}\ \phi_2)^{\frac{1}{2}} \Gamma\left(\frac{1}{2} + v\right) \Gamma\left(\frac{1}{2} + v'\right) \cdot L^*\left(1 + 2s_1, \phi_1 \times \phi_2\right) \right)^{-1}$$

up to a constant factor.

# The first Fourier coefficient of $E_{\mathcal{P}_{2,1,1},\Phi}$

$$s = \left(1 + s_1, -\frac{1}{2} + s_2, s_3\right), \quad \alpha = (s_1 + v, s_1 - v, s_2, -2s_1 - s_2)$$

## Proposition

The first coefficient of $E_{\mathcal{P}_{2,1,1},\Phi}(g, s)$, where $\phi$ is a Maass form of norm 1 on GL(2) with spectral parameter $\frac{1}{2} + v$, is given by

$$\Bigg( L(1, \mathrm{Ad}\ \phi)^{\frac{1}{2}}\ \Gamma\,(1/2 + v)\,\zeta^*(1 + 2s_1 + 2s_2) L^*(1 + s_1 - s_2, \phi) \cdot L^*(1 + 3s_1 + s_2, \phi) \Bigg)^{-1}$$

up to a constant factor.

# The first Fourier coefficient of $E_{\mathcal{P}_{3,1},\Phi}$

$$s = \left(1/2 + s_1,\ -3/2 - 3s_1\right)$$
$$\alpha = (s_1 + 2v + v', s_1 - v + v', s_1 - v - 2v', -3s_1)$$

## Proposition

The first coefficient of $E_{\mathcal{P}_{3,1},\Phi}(g, s)$, where $\phi$ is a Maass form of norm 1 on GL(3) with Langlands parameter $(2v + v', -v + v', -v - 2v')$, is given by

$$\left( L(1, \mathrm{Ad}\ \phi)^{\frac{1}{2}}\, \Gamma\left(\frac{1+3v}{2}\right) \Gamma\left(\frac{1+3v'}{2}\right) \Gamma\left(\frac{1+3v+3v'}{2}\right) \cdot L^*(1 + 4s_1, \phi) \right)^{-1}$$

up to a constant factor.

# THE END !!

# From $GL(n)$ to Chevalley groups

Chevalley groups $G$ are specific realizations of Lie groups that have a friendly integral structure.

**They can be viewed as algebraic subgroups of $GL(M)$ such that**

- $T$=maximal torus consists of all diagonal matrices
- $N$ consists of all unipotent upper triangular matrices
- $B$ consists of all upper triangular matrices
- $K$ consists of all orthogonal matrices.

Examples: $SL(n)$, $PGL(n)$.

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Examples: $SL(n)$, $PGL(n)$.

**Chevalley groups are associated to root systems $\Delta \subset \mathbb{R}^r$, $r$=rank**

- $T$ acts by adjoint action on $\mathfrak{g}$.
- Diagonalized by *root vectors* $X_\alpha$: $Ad(t)X_\alpha = t^\alpha \cdot X_\alpha$,
- where $t \mapsto t^\alpha$ is notation for a character of $T$ and $\alpha \in \Delta$.

# Root systems

## Example of $SL(n)$, rank $r = n - 1$:

The adjoint action of $T$ is:

- trivial on diagonal matrices
- diagonal on the span of the elementary matrices $E_{i,j}$, $i \neq j$.

If $t = \mathrm{diag}(t_1, \ldots, t_n)$, the corresponding root satisfies $t^\alpha = t_i/t_j$.

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If $t = \text{diag}(t_1, \ldots, t_n)$, the corresponding root satisfies $t^\alpha = t_i / t_j$.

*Positive* roots $\alpha$ (i.e., $X_\alpha \in \mathfrak{n}$) are nonnegative integral combinations of *simple* roots $\Sigma = \{\alpha_1, \ldots, \alpha_r\}$.

The root system comes endowed with an inner product $(\cdot, \cdot)$ from its ambient vector space $\mathbb{R}^r$. The coroot $\alpha^\vee = \frac{2}{(\alpha,\alpha)}\alpha$.

For simply-laced root systems (i.e., $SL(n)$, $SO(2n)$, $E_6$, $E_7$, $E_8$) we may scale so that $\alpha^\vee = \alpha$.

Via $(\cdot, \cdot)$, there exists a basis of *fundamental weights* $\varpi_1, \ldots, \varpi_r$ and a pairing $\langle \cdot, \cdot \rangle$ such that

$$\langle \varpi_i, \alpha_j^\vee \rangle = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

# General Eisenstein series

Start with a parabolic $P = LU \subset G$.

- $L$ is the product of its center and smaller Chevalley groups.
- Cuspidal automorphic representations of these smaller groups lift to $L$.
- They can be further *twisted* by characters of the center, which can be described by certain $\varpi_i$, to get $\tau_{\vec{s}}$ (more details in examples).
- If $\phi$ is a vector for $\tau$, multiply by character of the center to get vector $\phi_{\vec{s}} \in \tau_{\vec{s}}$.

## Form global Eisenstein series

$$E(g, \phi, \vec{s}) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} \phi_{\vec{s}}(\gamma g).$$

This converges absolutely for large $\Re \vec{s}$ and has a meromorphic continuation.

# General formula for generic coefficient

- At each place $v \leq \infty$, $\phi$ has Langlands/Satake parameters $\mu_v$.
- $E(g, \phi, \vec{s})$ has Langlands/Satake parameters $\lambda_v = \sum s_i \varpi_i + \mu_v$.
- Recall the normalized Whittaker function. It has $p$-adic analogs, hence a global normalized Whittaker function $W_{\mathbb{A}}$.

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## Formula for generic coefficient

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(ng, \phi, \vec{s})\,\overline{\chi(n)}\,dn = \widehat{\phi} \cdot \prod_{v \leq \infty} \prod_{\alpha \in \Delta_U} \zeta_v\Big(\langle \lambda_v, \alpha^\vee \rangle + 1\Big)^{-1} \cdot W_{\mathbb{A}}$$

where $\widehat{\phi}$ is the Fourier coefficient of the inducing cusp form on the Levi and $\Delta_U$ are the roots in the unipotent radical.

**Main point:** Precise control of the constant normalization allows for new applications.

# Example: (2,1,1) parabolic $P \subset SL(4)$

Label the simple roots $\alpha_i$, $i = 1, 2, 3$: $t^{\alpha_i} = t_i / t_{i+1}$.
It is natural to identify

$$
\begin{aligned}
\alpha_1 &= (1, -1, 0, 0), & \varpi_1 &= (1, 0, 0, 0), \\
\alpha_2 &= (0, 1, -1, 0), & \varpi_2 &= (1, 1, 0, 0), \\
\alpha_3 &= (0, 0, 1, -1), & \varpi_3 &= (1, 1, 1, 0).
\end{aligned}
$$

# Example: (2,1,1) parabolic $P \subset SL(4)$

Label the simple roots $\alpha_i$, $i = 1, 2, 3$: $t^{\alpha_i} = t_i / t_{i+1}$.
It is natural to identify

$$\alpha_1 = (1, -1, 0, 0), \qquad \varpi_1 = (1, 0, 0, 0),$$
$$\alpha_2 = (0, 1, -1, 0), \qquad \varpi_2 = (1, 1, 0, 0),$$
$$\alpha_3 = (0, 0, 1, -1), \qquad \varpi_3 = (1, 1, 1, 0).$$

- The Lie algebra of the Levi $L$ contains $X_{\pm\alpha_1}$.
- The Lie algebra of $U$ contains $X_\alpha$ for all $\alpha \in \Delta_U = \{\alpha_2, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_3\}$.
- The fundamental weights $\varpi_2$, $\varpi_3$ are orthogonal to $\alpha_1$, and extend to characters of $L$ which are trivial on its $SL(2)$ block.

# Fourier coefficient for (2,1,1) parabolic

- Let $(\nu_v, -\nu_v)$ as before denote the local Langlands/Satake parameters for $\tau$, assumed to be spherical at all places $v \leq \infty$.
- The induced Eisenstein series has local parameters $\lambda_v = s_2\varpi_2 + s_3\varpi_3 + \nu_v\alpha_1$, for $v \leq \infty$.
- The 5 inner products $\langle \lambda_v, \alpha^\vee \rangle$, for $\alpha \in \Delta_U$, are $s_2 \pm \nu_v$, $s_3$, and $s_2 + s_3 \pm \nu_v$.

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**Conclusion: the $L$-factors give:**

$$\Big( L^*(s_2+1,\tau)L^*(s_2+s_3+1,\tau)\zeta^*(s_3+1) \Big)^{-1}.$$

# Example: $E_8$

## The $E_7$ parabolic

- $U$ is a 57-dimensional Heisenberg group.
- $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2$, where $\dim(\mathfrak{u}_1) = 56$ and $\dim(\mathfrak{u}_2) = 1$.
- The adjoint actions of the Levi $L$ on $\mathfrak{u}_i$ are the standard representation of $E_7 \subset Sp(56)$, and the trivial representation.
- Hence the $L$-factors are $L^*(s+1, \tau, \mathbf{56})^{-1} \cdot \zeta^*(2s+1)^{-1}$.

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### The $Spin(7,7)$ parabolic parabolic

- $U$ is a 78-dimensional 2-step nilpotent group.
- $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2$, where $\dim(\mathfrak{u}_1) = 64$ and $\dim(\mathfrak{u}_2) = 14$.
- The adjoint actions of the Levi $L$ on $\mathfrak{u}_i$ are the spin representation and standard representation.
- Hence the $L$-factors are $L^*(s+1, \tau, Spin)^{-1} L^*(2s+1, \tau, Stan)^{-1}$.

HAPPY BIRTHDAY BILL !!