Two sieve identities revisited

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Part I: The combinatorial sieve, revisited

The (classical) Inclusion-Exclusion inequalities

$$\sum_{j=0}^{J} \binom{k}{j}(-1)^j = \binom{k-1}{J}(-1)^J$$

for all integers $k \geq 1$ and $0 \leq J \leq k$.
(When $k = 0$ we interpret $\binom{-1}{J} = (-1)^J$.)

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for all integers $k \geq 1$ and $0 \leq J \leq k$.
(When $k = 0$ we interpret $\binom{-1}{J} = (-1)^J$.)
Therefore

$$\sum_{j=0}^{2J+1} \binom{k}{j} (-1)^j \leq (1-1)^k \leq \sum_{j=0}^{2J} \binom{k}{j} (-1)^j$$

Here $(1-1)^k = 0$ if $k \geq 1$, and $(1-1)^0 = 1$.

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become

$$\sum_{\substack{d|(m,n) \\ \omega(d)\leq 2J+1}} \mu(d) \leq 1_{(\cdot,m)=1}(n) \leq \sum_{\substack{d|(m,n) \\ \omega(d)\leq 2J}} \mu(d).$$

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Sum the above equation over all $n \in A$ to obtain

$$\sum_{\substack{d|m\\ \omega(d)\le 2J+1}} \mu(d)\#A_d \le \#\{n \in A : (n,m) = 1\} \le \sum_{\substack{d|m\\ \omega(d)\le 2J}} \mu(d)\#A_d.$$

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We now assume

$$\#A_d = \frac{g(d)}{d}\#A + r_d(A)$$

(e.g. with each $g(d) = 1$), and $m = \prod_{p\le y} p$.

We deduce that

$$\sum_{\substack{d|m \\ \omega(d)\leq 2J+1}} \frac{\mu(d)g(d)}{d}\#A - E$$

$$\leq \#\{n \in A : (n,m) = 1\}$$

$$\leq \sum_{\substack{d|m \\ \omega(d)\leq 2J}} \frac{\mu(d)g(d)}{d}\#A + E$$

where

$$E = \sum_{d\leq D} |r_d(A)|$$

and $D = y^{2J+1}$, since each d has no more than $2J+1$ prime factors, all $\leq y$.

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This means that E is typically well bounded, but we have to think more about the main term.

Let y be fixed and $A = \{n \le x\}$, and, by the CRT, remove $g(p)$ residues classes mod p. Then as $x \to \infty$, in the inequalities

$$x \sum_{\substack{d|m \\ \omega(d) \le 2J+1}} \frac{\mu(d)g(d)}{d} - E$$

$$\le \#\{n \le x : (n, m) = 1\}$$

$$\le x \sum_{\substack{d|m \\ \omega(d) \le 2J}} \frac{\mu(d)g(d)}{d} + E$$

the error term E is bounded, and so the three terms are, asymptotically, x times

$$\sum_{\substack{d|m \\ \omega(d) \le 2J+1}} \frac{\mu(d)g(d)}{d} \le \prod_{p \le y} \left(1 - \frac{g(p)}{p}\right) \le \sum_{\substack{d|m \\ \omega(d) \le 2J}} \frac{\mu(d)g(d)}{d}.$$

Now if we write

$$\sum_{\substack{d|m \\ \omega(d)\leq 2J+1}} \frac{\mu(d)g(d)}{d} \leq e^H := \prod_{p\leq y}\left(1-\frac{g(p)}{p}\right) \leq \sum_{\substack{d|m \\ \omega(d)\leq 2J}} \frac{\mu(d)g(d)}{d}.$$

as $L \leq P \leq U$ and show that $U - L \leq \Delta P$ then $L, U = (1 + O(\Delta))P$.

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This is only small when $J \gg H \sim \kappa \log\log y$. But $E = \sum_{d\leq D} |r_d(A)|$ with $x^{1-\epsilon} \geq D = y^{2J+1} \approx y^{C \log\log y}$.

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$$y \le x^{c/\log\log x}.$$

First Combinatorial Sieve result

We have proved that if $x = y^u$ then $\#\{n \in A : (n, m) = 1\} =$

$$= \prod_{p \leq y} \left(1 - \frac{g(p)}{p}\right) \#A\{1 + O(1/u^u)\} + O\left(\sum_{d \leq D} |r_d(A)|\right)$$

provided $y \leq x^{c/\log\log x}$.

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What was the cause of this restriction?

We needed $y^{2J+1} \le x$; that is, $u \ge 2J + 1 \gg \sum_{p \le y} \frac{g(p)}{p}$. In general if we sieve we a set of primes $\mathcal{P}$, all $\le y$ then we need

$$\sum_{p \in \mathcal{P}} \frac{g(p)}{p} \ll u.$$

To sieve the integers up to $x = y^u$ with a set $\mathcal{P}$ of primes, all $\leq y$ then we need

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How to use inclusion-exclusion?

Write $1_{(\cdot,m)=1}(n) = \prod_{i=1}^{\ell} 1_{(\cdot,m_i)=1}(n)$ with squarefree $m = m_1 \cdots m_\ell$.

$$\prod_{i=1}^{\ell} (1-1)^{k_i} \leq \sum_{\substack{0 \leq j_i \leq 2J_i \\ \text{for } i=1,\ldots,\ell}} \prod_{i=1}^{\ell} (-1)^{j_i} \binom{k_i}{j_i} = \prod_{i=1}^{\ell} \binom{k_i - 1}{2J_i},$$

which is obviously ≥ 0.

Lower bound inequality (Ford & Halberstam)

$$\prod_{i=1}^{\ell}(1-1)^{k_i} \geq \sum_{\substack{0\leq j_i\leq 2J_i+1 \\ \text{for } i=1,\ldots,\ell \\ j_i=2J_i+1 \text{ for at most one } i}} \prod_{i=1}^{\ell}(-1)^{j_i}\binom{k_i}{j_i}.$$

If all $k_i = 0$ then only non-zero term is all $j_i = 0$, so equals 1.
The above sum can be rewritten as

$$\sum_{0\leq j_i\leq 2J_i \text{ for all } i} + \sum_{h=1}^{\ell} \sum_{\substack{0\leq j_i\leq 2J_i \text{ for all } i\neq h \\ j_h=2J_h+1}}$$

to obtain,

$$\prod_{i=1}^{\ell}\binom{k_i-1}{2J_i} + \sum_{h=1}^{\ell}\prod_{i=1,\ i\neq h}^{\ell}\binom{k_i-1}{2J_i}\cdot(-1)\binom{k_h}{2J_h+1}$$

Our lower bound on $\prod_{i=1}^{\ell}(1-1)^{k_i}$ is

$$\prod_{i=1}^{\ell}\binom{k_i-1}{2J_i}+\sum_{h=1}^{\ell}\prod_{i=1,\ i\neq h}^{\ell}\binom{k_i-1}{2J_i}\cdot(-1)\binom{k_h}{2J_h+1}$$

We may assume that each $k_i=0$ or $\geq 2J_i+1$ (else each term equals 0) to obtain

$$=\prod_{i=1}^{\ell}\binom{k_i-1}{2J_i}\left(1-\sum_{h=1}^{\ell}\frac{k_i}{2J_i+1}\right).$$

This equals 1 if each $k_i=0$, and is ≤ 0 otherwise.

When we substitute in our new inclusion-exclusion identity we obtain $\#\{n \in A : (n,m) = 1\} =$

$$= \prod_{p \le y} \left(1 - \frac{g(p)}{p}\right) \#A\{1 + O(\Delta)\} + O\left(\sum_{d \le D} |r_d(A)|\right)$$

where $\Delta := \Delta_1 + \cdots + \Delta_\ell$ and

$$\Delta_i := \sum_{\substack{d | m_i \\ \omega(d) = 2J_i + 1}} \frac{g(d)}{d} \Big/ \prod_{p | m_i} \left(1 - \frac{g(p)}{p}\right) \le e^{L_i} \frac{{L_i}^{2J_i+1}}{(2J_i + 1)!}$$

for $L_i := \log \prod_{p|m_i} (1 - \frac{g(p)}{p})^{-1}$, with $D = \prod_i y_i^{2J_i+1}$.

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A painful optimization problem

The Fundamental Lemma of Sieve Theory

Choices: Write $y_i = y^{u_i}$ for each i.
$2J_1 + 1$ is the largest odd integer $\leq u - 2u/(\log u)^2$, with $u_1 = 1$.
$2J_k$ be the largest even integer $\leq u + (k-2)/u$ and
$u_k = \frac{1}{(\log u)^3}(1 - \frac{1}{\log u})^{k-2}$ for all $k \geq 2$.

Then $\#\{n \in A : (n, m) = 1\} =$

$$= \prod_{p \leq y} \left(1 - \frac{g(p)}{p}\right) \#A\,(1 + O(E(u)) + O\left(\sum_{d \leq y^u} |r_d(A)|\right)$$

where $E(u) = \exp(-u(\log u + O(\log\log\log u)))$.

And now: Something completely different

Part II: Vaughan's identity, revisited

Vaughan's identity

For appropriate choices of U and V we have

$$\Lambda = \Lambda_{<V} + \mu_{<U} * L - \mu_{<U} * \Lambda_{<V} * 1 + \mu_{\geq U} * \Lambda_{\geq V} * 1.$$

For example, if we wish to bound a sum $\sum_{n \leq x} \Lambda(n) F(n)$ (eg BV Theorem, Integers as sums of three primes), with $F(\cdot)$ "arithmetic", then we decompose using the above. The reason this is useful is that we get bilinear sums, with longish sums in two variables, and then we can apply results that require little arithmetic information to get decent bounds:

Bounding bilinear sums

$$\text{Define} \qquad f(n) := \sum_{\ell m = n} \alpha_\ell \beta_m$$

where $\{\alpha_\ell\}$ and $\{\beta_m\} \in \mathbb{C}$, for which

- The $\{\alpha_\ell\}$ satisfy the Siegel-Walfisz criterion;
- The $\{\alpha_\ell\}$ are only supported in the range $L_0 \leq \ell \leq x/y$;
- $\sum_{\ell \leq L} |\alpha_\ell|^2 \leq aL$ and $\sum_{m \leq M} |\beta_m|^2 \leq bM$ for all $L, M \leq x$.

Then, for any $B > 0$ we have

$$\sum_{q \leq Q} \max_{a:\ (a,q)=1} \left| \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} f(n) - \frac{1}{\phi(q)} \sum_{\substack{n \leq x \\ (n,q)=1}} f(n) \right| \ll (ab)^{1/2} Q x^{1/2} \log x,$$

with $Q = \frac{x^{1/2}}{(\log x)^B}$, $x/y \leq \frac{Q^2}{(\log x)^2}$, $L_0 \geq y + \exp((\log x)^\epsilon)$.

Bounding bilinear sums: Key features

To use this result to determine $\sum_{n\le x} f(n)$ we need to be able to write $f(n) = \sum_i f_i(n)$ where each

$$f_i(n) := \sum_{\ell m = n} \alpha_\ell \beta_m$$

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with the $\{\alpha_\ell\}$ satisfying Siegel-Walfisz criterion.
Therefore Vaughan's identity has been the standard tool:

$$\Lambda = \Lambda_{<V} + \mu_{<U} * L - \mu_{<U} * \Lambda_{<V} * 1 + \mu_{\ge U} * \Lambda_{\ge V} * 1,$$

The last term contains the key difficulty in proving B-V Theorem.

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$$1_{\mathbb{P}}(n) = 1 - \sum_{\substack{\ell m = n \\ \ell \text{ prime} \leq z}} \frac{1}{1 + \omega_z(m)};$$

where $\mathbb{P}$ is the set of primes $> z$, and $\omega_z(m) = \sum_{p|m,\ p \leq z} 1$.

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$$\log n = \sum_{\ell m = n} \Lambda(\ell),$$

multiply through by $1_y(n) = 1_y(\ell) 1_y(m)$, and re-organize, to get

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$$\Lambda(n) 1_y(n) = \log n \cdot 1_y(n) - \sum_{\substack{\ell m = n \\ \ell, m > 1}} \Lambda(\ell) 1_y(\ell) \cdot 1_y(m).$$

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Let $y \geq \exp((\log x)^\epsilon)$, so if $1_y(\ell)1_y(m) \neq 0$ then $\ell, m \geq \exp((\log x)^\epsilon)$:

Second term: Can apply theorem for bounding bilinear sums!

First term: For y smallish can use the combinatorial sieve in each arithmetic progression.

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New Bombieri-Vinogradov Theorem

If $x^{1/2}/(\log x)^B \leq Q \leq x^{1/2}$ then

$$\sum_{q \leq Q} \max_{(a,q)=1} \left| \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \right| \ll Q\,(x \log\log x)^{1/2}.$$

(This improves previous best that had a few extra powers of log.)

Applications

$$\Lambda(n)1_y(n) = \log n \cdot 1_y(n) - \sum_{\substack{\ell m = n \\ \ell, m > 1}} \Lambda(\ell)1_y(\ell) \cdot 1_y(m).$$

- Bombieri-Vinogradov (and Yitang Zhang's) Theorem

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- Interesting suggestions

In more generality

let $f(\cdot)$ be multiplicative with $F(s) = \sum_{n\geq 1} f(n)/n^s$ and

$$-\frac{F'}{F}(s) = \sum_{n\geq 1} \Lambda_f(n) n^s$$

so that $-F' = F \cdot (-F'/F)$, and comparing coefficients, we obtain

$$f(n)\log n = \sum_{\ell m = n} \Lambda_f(\ell) f(m).$$

We can rewrite this

$$\Lambda_f(n) = f(n)\log n - \sum_{\substack{\ell m = n \\ m > 1}} \Lambda_f(\ell) f(m).$$

Now suppose that $f(\cdot)$ is only supported on prime powers that are $> y \geq \exp((\log x)^\epsilon)$.

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Both show that the key to BV is the large primes' behaviour

Removing the support condition

Let $f(\cdot)$ be a multiplicative function and $f_y(\cdot) = f(\cdot)1_y(\cdot)$. Our Theorem above states that

f_y satisfies the BV Theorem if and only if Λ_{f_y} satisfies the BV Theorem.

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We would like to know that

f_y satisfies the BV Theorem if and only if f satisfies the BV Theorem

This essentially means sieving the values of $f(\cdot)$ by its values on the primes $\leq y$

General BV

Suppose that f and g are 1-bounded functions that are supported only on prime powers $> y$. For $n > 1$ we have

$$(f * g)(n) - f(1)g(n) - g(1)f(n) = \sum_{\substack{\ell m = n \\ \ell, m > y}} f(\ell)g(m).$$

If f satisfies the Siegel-Walfisz criterion then, Using our bilinear forms condition, we see that

$$h(n) := (f * g)(n) - f(1)g(n) - g(1)f(n)$$

satisfies the BV Theorem. If, also, $f(1) = g(1) = 0$ then $f * g$ satisfies the BV Theorem. If
If, also, $g(1) = 0$ but $f(1) \neq 0$ then $f * g$ satisfies the BV Theorem if and only if f does.
If f satisfies the BV Theorem then $(f * g)(n) - f(1)g(n)$ satisfies the BV Theorem.

The Fundamental Lemma for complex-valued sequences?

If $f(n) \geq 0$ for all $n \geq 1$ then

$$\sum_{\substack{d|(m,n)\\ \omega(d)\leq 2J+1}} \mu(d)f(n) \leq 1_{(\cdot,m)=1}(n)f(n) \leq \sum_{\substack{d|(m,n)\\ \omega(d)\leq 2J}} \mu(d)f(n).$$

Summing over all $n \leq x$ gives

$$\sum_{\substack{d|m\\ \omega(d)\leq 2J+1}} \mu(d)\Sigma_d \leq \sum_{\substack{n\leq x\\ (n,m)=1}} f(n) \leq \sum_{\substack{d|m\\ \omega(d)\leq 2J}} \mu(d)\Sigma_d.$$

where $\Sigma_d = \sum_{n\leq x:\ d|n} f(n)$, and we can often proceed as in the combinatorial sieve.

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How to proceed in more generality? Even for real f which can be negative

Koukoulopoulos: If each $|f(n)| \leq 1$ then

$$\left| \sum_{\substack{d|m \\ \omega(d)\leq 2J}} \mu(d)\Sigma_d - \sum_{\substack{n\in A \\ (n,m)=1}} f(n) \right|$$

$$= \left| \sum_{n\in A} f(n) \left(\sum_{\substack{d|(m,n) \\ \omega(d)\leq 2J}} \mu(d) - 1_{(\cdot,m)=1}(n) \right) \right|$$

$$\leq \left| \sum_{n\in A} \left(\sum_{\substack{d|(m,n) \\ \omega(d)\leq 2J}} \mu(d) - 1_{(\cdot,m)=1}(n) \right) \right|$$

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This is the error term that we began this talk in bounding

red blue orange purple violet magenta cyan brown black darkgray gray