Multiplicative chaos in number theory

Adam J Harper

July 2019

Plan of the talk:

- First thoughts about multiplicative chaos and its number theory counterparts
- Four number theory/analysis examples
- How does multiplicative chaos behave?
- Results/open questions in the examples

Multiplicative chaos is a class of probabilistic objects first studied by Kahane in 1985.

Idea: form a random measure (i.e. a random weighting) by integrating test functions against the exponential of some collection of random variables $(X(h))_{h \in \mathcal{H}}$.

For g a test function, we can look at

$$\int g(h) e^{\gamma X(h)} dh,$$

where $\gamma > 0$ is a real parameter.

One needs to make assumptions on $X(h)$ in order for the random measure to be interesting. It turns out one gets something very interesting if the $X(h)$ are:

- Gaussian random variables;
- with mean zero $\mathbb{E}X(h) = 0$, and the same (or similar) finite non-zero variance $\mathbb{E}X(h)^2$ for all h;
  (This condition implies that the *average* mass $\mathbb{E}e^{\gamma X(h)} = e^{(\gamma^2/2)\mathbb{E}X(h)^2}$ assigned to each point h is roughly the same.)
- and the covariance $\mathbb{E}X(h)X(h')$ (i.e. the dependence between $X(h)$ and $X(h')$) decays *logarithmically* as $|h - h'|$ increases.

Connection with number theory

Suppose we have a family of functions $F_j(s)$, for $j \in \mathcal{J}, s \in \mathbb{C}$, that each have:

- an Euler product structure (either exact or approximate);
- some orthogonality/independence between the contribution from different primes, when we vary over $j \in \mathcal{J}$.

Claim: If we look at

$$\int g(h)|F_j(1/2 + ih)|^\gamma dh$$

as $j \in \mathcal{J}$ varies (giving our "randomness"), this (possibly) has lots of the same structure as multiplicative chaos.

Why?

- If $F_j(s)$ has an (approximate) Euler product structure, then $\log|F_j(s)| = \Re \log F_j(s)$ is (approximately) a sum over primes.
- If the contributions from different primes are orthogonal/independent as j varies, we can expect $\log|F_j(s)|$ to behave like a sum of independent contributions.
- (In many situations) this means that $\log|F_j(1/2 + ih)|$ will behave roughly like Gaussians with mean zero and comparable variances.
- The logarithmic covariance structure emerges because there is a *multiscale structure* in an Euler product: $p^{ih} = e^{ih \log p}$ varies on an h-scale roughly $1/\log p$, so contributions from small primes remain correlated over large h intervals, contributions from larger primes decorrelate more quickly.

Example 1: random Euler products

Let $(f(p))_{p \text{ prime}}$ be independent random variables, each distributed uniformly on $\{|z| = 1\}$. Define

$$F(s) := \prod_{p \leq x} (1 - \frac{f(p)}{p^s})^{-1},$$

where x is a large parameter.

Then we can study the behaviour of

$$\int_{-1/2}^{1/2} g(h) |F(1/2 + ih)|^{\gamma} dh,$$

as the random $f(p)$ vary.

Example 2: shifts of the Riemann zeta function

We can study the behaviour of

$$\int_{-1/2}^{1/2} g(h)|\zeta(1/2+it+ih)|^{\gamma} dh,$$

as $T \leq t \leq 2T$ varies.

Notice that $\zeta(1/2+it+ih)$ is not given by an Euler product, but for many purposes we expect it to behave like an Euler product.

Example 3: random multiplicative functions

Let $(f(p))_{p \text{ prime}}$ be independent random variables as before. We define a *Steinhaus random multiplicative function* by setting

$$f(n) := \prod_{p^a || n} f(p)^a \quad \forall n \in \mathbb{N}.$$

Then there are many interesting questions about the behaviour of $\sum_{n \leq x} f(n)$. If one is interested in $\mathbb{E}|\sum_{n \leq x} f(n)|^{2q}$, it turns out (roughly speaking) that for $0 \leq q \ll \frac{\log x}{\log\log x}$ we have

$$\mathbb{E}|\sum_{n \leq x} f(n)|^{2q} \approx e^{O(q^2)} x^q \mathbb{E}\left(\frac{1}{\log x} \int_{-1/2}^{1/2} |F(1/2 + \frac{q}{\log x} + ih)|^2 dh \right)^q .$$

Remarks about Example 3

- One needs a non-trivial, but not too difficult, conditioning argument to establish this connection between $\mathbb{E}|\sum_{n \leq x} f(n)|^{2q}$ and the Euler product integral.
- *We see here that the exponent* $\gamma = 2$ *has some special significance.*

Example 4: moments of character sums

Let r be a large prime and $x \leq r$. We can study the behaviour of

$$\frac{1}{r-2} \sum_{\chi \neq \chi_0 \text{ mod } r} |\sum_{n \leq x} \chi(n)|^{2q},$$

where the sum is over all the non-principal Dirichlet characters mod r.

Key properties of multiplicative chaos

- As γ increases, $\mathbb{E}e^{\gamma X(h)} = e^{(\gamma^2/2)\mathbb{E}X(h)^2}$ increases, and $\int g(h)e^{\gamma X(h)}dh$ is dominated more and more by very large values of $X(h)$.
- There is a *critical value* γ_c of γ at which, with very high probability, one no longer finds any values of h for which $X(h)$ is large enough to overcome $\mathbb{E}e^{\gamma X(h)}$.
- When $\gamma < \gamma_c$, one see non-trivial behaviour after rescaling $\int g(h)e^{\gamma X(h)}dh$ by $e^{(\gamma^2/2)\mathbb{E}X(h)^2}$.
- When $\gamma = \gamma_c$, one sees non-trivial behaviour after rescaling by

$$\frac{e^{(\gamma^2/2)\mathbb{E}X(h)^2}}{\sqrt{\mathbb{E}X(h)^2}}.$$

Key properties of multiplicative chaos (continued)

- In the examples considered above, it turns out that the critical exponent $\gamma_c = 2$, and $\sqrt{\mathbb{E}X(h)^2} \asymp \sqrt{\log\log x}$. So this quantity will come up a lot!
- One word about the proofs: restrict everything to the case where $X(h)$ and its "subsums" are all below a certain barrier, for all h. Such an event can be found that occurs with very high probability, but decreases the size of various averages in the proofs (by factors like $\sqrt{\mathbb{E}X(h)^2}$).

Theorem 1 (H., 2017, 2018)

If $f(n)$ is a Steinhaus random multiplicative function, then uniformly for all large x and real $0 \leq q \leq 1$ we have

$$\mathbb{E}|\sum_{n \leq x} f(n)|^{2q} \asymp \left(\frac{x}{1 + (1-q)\sqrt{\log \log x}} \right)^q .$$

For $1 \leq q \leq \frac{c \log x}{\log \log x}$, we have

$$\mathbb{E}|\sum_{n \leq x} f(n)|^{2q} = e^{-q^2 \log q - q^2 \log \log(2q) + O(q^2)} x^q \log^{(q-1)^2} x.$$

In particular, $\mathbb{E}|\sum_{n \leq x} f(n)| \asymp \frac{\sqrt{x}}{(\log \log x)^{1/4}}$. "Better than squareroot cancellation"

Related work/open problems:
One can look instead at

$$\mathbb{E}|\sum_{n\leq x}\frac{f(n)}{\sqrt{n}}|^{2q}=\lim_{T\to\infty}\frac{1}{T}\int_0^T|\sum_{n\leq x}\frac{1}{n^{1/2+it}}|^{2q}dt,$$

which are sometimes called the *pseudomoments* of the zeta function. They have been studied by Conrey and Gamburd (2006); Bondarenko–Heap–Seip (2015); Bondarenko–Brevig–Saksman–Seip–Zhao (2018); Heap (2018); Brevig–Heap (2019). *Correspond to* $\gamma = 2$

More generally, one can look at $\mathbb{E}|\sum_{n\leq x} f(n)d_\alpha(n)|^{2q}$ or $\mathbb{E}|\sum_{n\leq x}\frac{f(n)d_\alpha(n)}{\sqrt{n}}|^{2q}$, where $d_\alpha(n)$ is the α divisor function.
Correspond to $\gamma = \alpha$

Open problem (so far...): what is the order of magnitude of $\mathbb{E}|\sum_{n\leq x} \frac{f(n)}{\sqrt{n}}|^{2q}$ for $0 < q \leq 1/2$?
We might suspect it should be $\log^{q^2} x$ (as for a unitary L-function).

Bailey and Keating (2018): look at the analogue of $\mathbb{E}\left(\int_{-1/2}^{1/2} |F(1/2+ih)|^{\gamma} dh\right)^q$ for characteristic polynomials of random unitary matrices, obtain asymptotics when $q \in \mathbb{N}, \gamma \in 2\mathbb{N}$.

Saksman and Webb (2016): prove convergence of the random measure coming from Euler products to a "genuine" multiplicative chaos measure (for $\gamma \leq 2$).

It is possible to "derandomise" some of these arguments. Derandomising the passage from $\mathbb{E}|\sum_{n \leq x} f(n)|^{2q}$ to an integral average, one can show:

Theorem 2 (H.)

Let r be a large prime. Then uniformly for any $1 \leq x \leq r$ and $0 \leq q \leq 1$, if we set $L := \min\{x, r/x\}$ we have

$$\frac{1}{r-2} \sum_{\chi \neq \chi_0 \text{ mod } r} |\sum_{n \leq x} \chi(n)|^{2q} \ll \left(\frac{x}{1 + (1-q)\sqrt{\log\log 10L}} \right)^q .$$

Because of the "duality" between $\sum_{n \leq x} \chi(n)$ and $\sum_{n \leq r/x} \chi(n)$ (coming from Poisson summation), this bound involving L is the natural analogue of Theorem 1.

Open problem (probably hard): obtain a corresponding lower bound.

By a different combinatorial method, La Bretèche, Munsch and Tenenbaum recently proved that for $1 \leq x < r/2$,

$$\frac{1}{r-2} \sum_{\chi \neq \chi_0 \bmod r} \Big| \sum_{n \leq x} \chi(n) \Big| \gg \frac{\sqrt{x}}{\log^{c+o(1)} x}, \qquad c \approx 0.04304.$$

If one could obtain a lower bound that matched Theorem 2 (for $x \leq r^{1/2+o(1)}$), this would (essentially) imply a positive proportion non-vanishing result for Dirichlet theta functions $\theta(1;\chi)$.

Derandomising the analysis of the integral average, one can show:

Theorem 3 (H., 2019)

Uniformly for all large T and all $0 \leq q \leq 1$, we have

$$\frac{1}{T}\int_T^{2T}\left(\int_{-1/2}^{1/2}|\zeta(1/2+it+ih)|^2 dh\right)^q \ll \left(\frac{\log T}{1+(1-q)\sqrt{\log\log T}}\right)^q .$$

Open problem (probably doable): obtain a matching lower bound.

Arguin, Ouimet and Radziwiłł (2019): estimates for $\int_{-1/2}^{1/2}|\zeta(1/2+it+ih)|^\gamma dh$ up to factors $\log^\epsilon T$, for almost all $T \leq t \leq 2T$.

$\int_{-1/2}^{1/2} |\zeta(1/2+it+ih)|^2 dh$ is usually dominated by h for which $\log|\zeta(1/2+it+ih)| \approx \log\log T - \Theta(\sqrt{\log\log T})$. These values are atypical, but they don't correspond to the very largest values of $\log|\zeta(1/2+it+ih)|$ that one expects on an interval of length 1.

By biasing the integral to only include (roughly speaking) very large values, one can prove (roughly): $\max_{|h|\leq 1/2} \log|\zeta(1/2+it+ih)|$ is

$$\leq \log\log T - (3/4)\log\log\log T + (3/2)\log\log\log\log T$$

for "almost all" $T \leq t \leq 2T$. This matches the first two terms in a conjecture of Fyodorov–Hiary–Keating (2012, 2014).

Arguin, Bourgade, Radziwiłł and Soundararajan: in forthcoming work, give an independent (different) proof of this upper bound.