ZERO-DENSITY ESTIMATES
FOR DIRICHLET L-FUNCTIONS

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1. INTRODUCTION

$\chi \pmod q$ primitive, $q \geqslant 3$,

$$L(s,\chi) = \sum_1^\infty \chi(n) n^{-s} = \prod_p \left(1-\chi(p)p^{-s}\right)^{-1},$$

$$L(\rho,\chi) = 0, \quad \rho = \beta + i\gamma, \quad 0 < \beta < 1,$$

$$N(T,\chi) = \#\{\rho \,;\, |\gamma| \leq T\}$$

$$= \frac{T}{\pi} \log \frac{qT}{2\pi e} + O(\log qT), \quad T \geqslant 1,$$

$$N(\alpha,T,\chi) = \#\{\rho \,;\, \beta \geqslant \alpha,\ |\gamma| \leq T\}, \quad \tfrac{1}{2} \leq \alpha \leq 1.$$

Density Estimate :

$$\sum_{q \leq Q} \ \sideset{}{^*}\sum_{\chi \pmod q} N(\alpha,T,\chi) \ll (Q^2T)^{c(\alpha)(1-\alpha)} (\log QT)^A$$

Bohr, Landau, Carlson, Ingham, Linnik, Bombieri, Vinogradov.
Turán, Halász, Montgomery, Huxley, Jutila, Gallagher, ...

$$c(\alpha) = \min\left(\frac{3}{2-\alpha}, \frac{3}{3\alpha-1}\right) \quad \text{Ingham-Montgomery-Huxley}$$

Density Conjecture :

$$c(\alpha) = 2.$$

Montgomery uses

$$\sum_{\chi (\mathrm{mod}\, q)}^{*} |L(s,\chi)|^4 \ll q|s| (\log q|s|)^4$$

on the line $\mathrm{Re}\, s = \frac{1}{2}$ getting

$$c(\alpha) = \frac{3}{2-\alpha}, \qquad A = 9 .$$

Using

$$\sum_{q \le Q} \sum_{\chi (\mathrm{mod}\, q)}^{*} |L(s,\chi)|^8 \ll Q^2 |s|^2 (\log Q|s|)^{16}$$

one gets

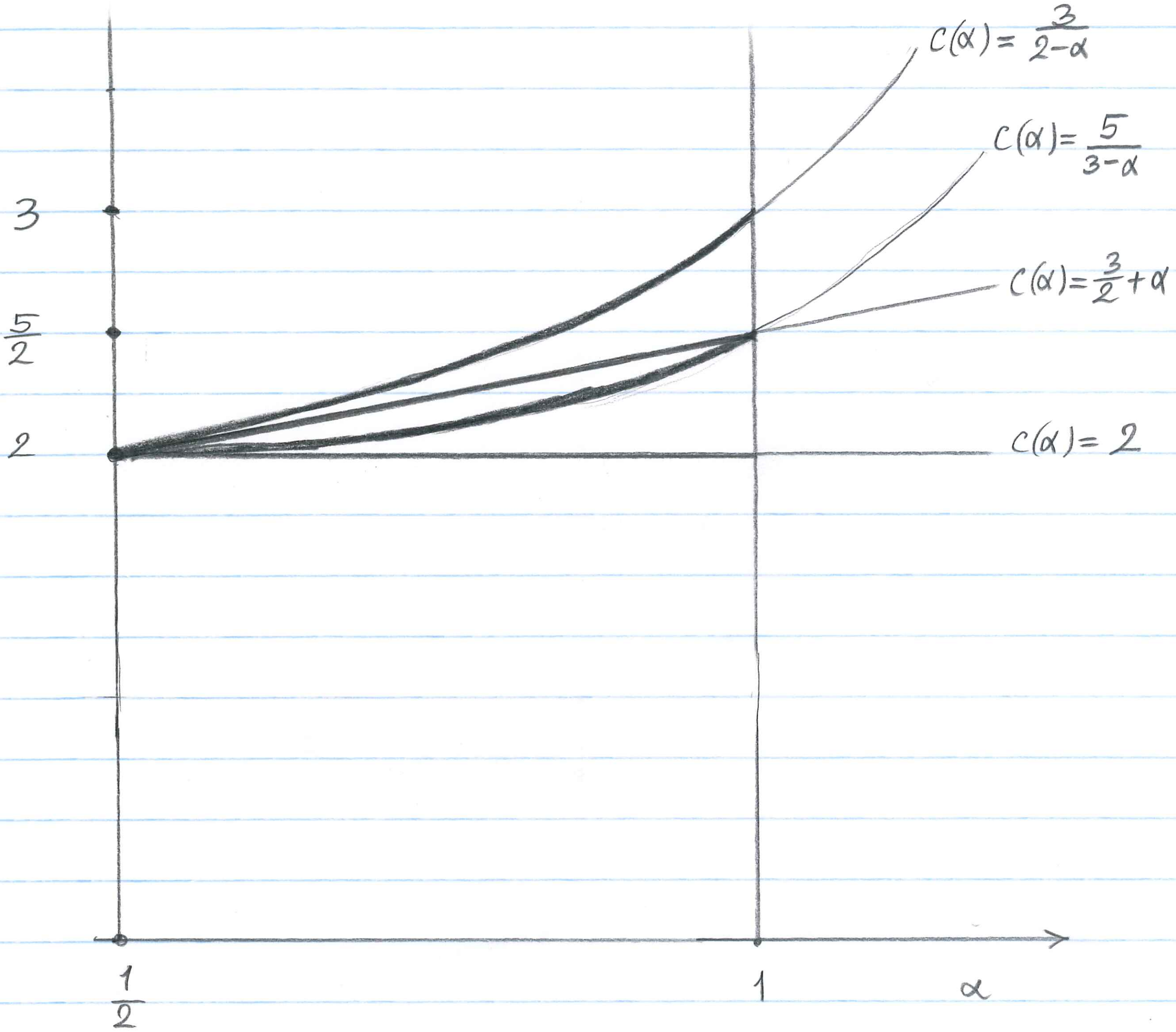
$$c(\alpha) = \frac{5}{3-\alpha}, \qquad A \geq 8 .$$

GOAL. Get a density estimate which is strong for zeros near the central line in the q aspect ($c(\alpha)$ close to 2 and $A \geq 1$ as small as possible).

THEOREM. Let $Q \geq 3$, $T \geq 1$, $\frac{1}{2} \le \alpha \le 1$. We have

$$\sum_{q \le Q} \sum_{\chi (\mathrm{mod}\, q)}^{*} N(\alpha, T, \chi) \ll T^{\frac{9}{8}} Q^{(3+2\alpha)(1-\alpha)} (\log Q)^{3+6(2\alpha-1)} .$$

ZERO-DENSITY EXPONENTS



2. MOLLIFICATION

$$M(s,\chi) = \sum_{m \le X} \chi(m)\mu(m)m^{-s}$$

$$P(s,\chi) = L(s,\chi)M(s,\chi) = 1 + \sum_{n>X} a_n \chi(n) n^{-s}$$

$$a_n = \sum_{m|n,\ m\le X} \mu(m) \ll \tau(n)$$

Formula of Ramachandra

$$P(s,\chi) = e^{-1/Y} + \sum_{n>X} a_n \chi(n) n^{-s} e^{-n/Y}$$

$$- \frac{1}{2\pi i} \int_{-\infty}^{\infty} P(\tfrac{1}{2}+iu,\chi)\Gamma(\tfrac{1}{2}+iu-s)Y^{\frac{1}{2}+iu-s}\, du$$

for $s=\sigma+it$ with $\frac{1}{2}<\sigma<1$, where $Y \ge X$ at our disposal.

We apply this for the set S_χ of the zeros $s=\beta+i\gamma$ with $\beta \ge \alpha$ and $|\gamma - T| \le 1$. No spacing of zeros is required. We have

$$|S_\chi| \le V \ll \log QT \quad \text{for every } \chi.$$

We separate n from s in n^{-s} using the Sobolev-Gallagher inequality (hence loosing one logarithm factor).

3. LARGE SIEVE OF GALLAGHER

$$\sum_{q\leq Q}\frac{q}{\varphi(q)}\sum^{*}_{\chi(\mathrm{mod}\, q)}\int_0^1\Big|\sum_n a_n\chi(n)n^{iu}\Big|^2 du \ll \sum_n (Q^2+n)|a_n|^2$$

Separating n from p we lost one logarithm, but retained the integration over $0\leq u\leq 1$.

4. MEAN-VALUE OF THE COEFFICIENTS

We have

$$\sum_{n\leq N} a_n^2 n^{-1} \ll \log N$$

by the results of Dress–Iwaniec–Tenenbaum

$$\sum\sum_{m_1,m_2\leq X}\frac{\mu(m_1)\mu(m_2)}{[m_1,m_2]} \sim c = 0.44069\ldots$$

$$\sum\sum_{m_1,m_2\leq X}\frac{\mu(m_1)\mu(m_2)}{[m_1,m_2]}\log[m_1,m_2] \ll 1.$$

5. APPROXIMATE FUNCTIONAL EQUATIONS

$$L(s,\chi)=\sum_n \chi(n)n^{-s}V_s\Big(\frac{n}{X\sqrt{q}}\Big)+\varepsilon(s,\chi)\sum_n\bar{\chi}(n)n^{s-1}V_{1-s}\Big(\frac{nX}{\sqrt{q}}\Big)$$

with any $X>0$ and

$$V_s(y)\ll\Big(1+\frac{y}{\sqrt{|s|}}\Big)^{-2}.$$

6. MEAN-VALUE OF $|P(z,\chi)|^2$

$$\sum_{q\le Q}\frac{q}{\varphi(q)}\sum_{\chi(\mathrm{mod}\, q)}^{*}|P(z,\chi)|^2 \ll \left(Q^2+X\sqrt{Q|z|}\right)\left(\log XQ|z|\right)^6$$

for $z=\frac{1}{2}+iu$, by the large sieve inequality.

Here the estimate of DIT is not applicable, so there is a chance to save one logarithm by other arguments.

7. CONCLUSION

$$\sum_{q\le Q}\sum_{\chi(\mathrm{mod}\, q)}^{*} N(\alpha,T,\chi) \ll T\left(Q^2X^{1-2\alpha}+Y^{2-2\alpha}\right)(\log QT)^3 + TY^{1-2\alpha}\left(Q^2+X\sqrt{QT}\right)(\log QT)^7(\log\log QT)^2.$$

Choose

$$X=Q^{1+\alpha}Z^{\alpha-1},\qquad Y=Q^{\frac{3}{2}+\alpha}Z^{\alpha-\frac{1}{2}}$$

where

$$Z=T(\log Q)^8(\log\log Q)^4.$$

COROLLARY 1. For $Q \geqslant 3$ and $T \geqslant 1$ we have

$$\sum_{Q<q\leq 2Q} \frac{1}{\varphi(q)} \sum_{\chi(\bmod q)}^{*} \sum_{\rho_\chi} q^{(1+\beta)(2\beta-1)} \ll T^{\frac{9}{8}} Q (\log Q)^5$$

where $\rho_\chi = \beta + i\gamma$ runs over the zeros of $L(s,\chi)$ with

$$\frac{1}{2} \leq \beta \leq 1, \qquad |\gamma| \leq T.$$

COROLLARY 2. Let $Q \geqslant 3$ and $T=1$. There are primes p with $Q < p \leq 2Q$ such that

$$\frac{1}{p-2} \sum_{\chi(\bmod p)}^{*} \sum_{\rho_\chi} p^{(1+\beta)(2\beta-1)} \ll (\log p)^6 .$$

PROBLEM. Establish a large sieve inequality for prime moduli.

D. WOLKE (1971). If $N \leqslant Q^{2-\theta}$, then

$$\sum_{p\leq Q} \sum_{a(\bmod p)}^{*} \Big|\sum_{M}^{M+N} a_n e\Big(\frac{an}{p}\Big)\Big|^2 \ll \frac{Q^2}{\theta \log Q} (\log\log Q) \sum_{M}^{M+N} |a_n|^2 .$$