Results on the standard twist of L-functions

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General notation.

S — the Selberg class.

$S^\sharp$ — the extended Selberg class.

For $F \in S^\sharp$, $\sigma > 1$

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$

Functional equation

$$\Phi(s) = \omega \overline{\Phi(1-\bar{s})},$$

where

$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s)$, and $r \geqslant 0$, $Q > 0$, $\lambda_j > 0$, $\Re \mu_j \geqslant 0$, $|\omega| = 1$.

Invariants

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ξ- and θ-invariants:

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The root number:

$$\omega_F := \omega \prod_{j=1}^{r} \lambda_j^{-2i\Im(\mu_j)}$$

The standard twist

$$S_d^\sharp := \{F \in S^\sharp : d_F = d\}$$

Definition

Let $F \in S_d^{\#}$, $d > 0$. For a real $\alpha > 0$ and $\sigma > 1$ we define the *standard twist* by the formula

$$F(s,\alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-n^{1/d}\alpha).$$

$$(e(\theta) := \exp(2\pi i\theta))$$

The standard twist

1. We write

$$n_\alpha = q_F d^{-d} \alpha^d$$

and

$$a(n_\alpha) = \begin{cases} a(n) & \text{if} \quad n_\alpha = n \in \mathbb{N} \\ 0 & \quad \text{otherwise.} \end{cases}$$

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2. $$\mathrm{Spec}(F) := \{\alpha > 0 : a(n_\alpha) \neq 0\}$$

Theorem (J.K.-A.P. – 2005)

$F(s,\alpha)$ has meromorphic continuation to $\mathbb{C}$. Moreover, $F(s,\alpha)$ is entire if $\alpha \notin Spec(F)$. Otherwise, $F(s,\alpha)$ has at most simple poles at the points

$$s_k = \frac{d+1}{2d} - \frac{k}{d} - i\frac{\theta_F}{d}, \quad k \geqslant 0$$

with

$$\mathrm{res}_{s=s_0} F(s,\alpha) = c_F \frac{\overline{a(n_\alpha)}}{n_\alpha^{1-s_0}} \quad (c_F \neq 0).$$

Some applications of the standard twist

The standard twist proved to be the central object in the Selberg class theory.

Some applications of the standard twist

I. $S_d^\sharp = \emptyset$ if $0 < d < 1$. [E. Richert, S. Bochner, B. Conrey-A. Ghosh, G. Molteni]
Proof: Suppose there exists $F \in S_d^\sharp$ with $0 < d < 1$. Take $\alpha \in Spec(F)$. Then $F(s,\alpha)$ has a pole at $s = s_0$. But $\Re(s_0) = \frac{1}{2} + \frac{1}{2d} > 1$ – a contradiction.

Some applications of the standard twist

II. Description of the structure of $S_1^\sharp$ (J.K.-A.P. – 1999).
If $d = 1$, the standard twist is linear

$$F(s, \alpha) = \sum_{n=1}^{\infty} \frac{a(n)e(-\alpha n)}{n^s}$$

Thus

$$F(s, \alpha + 1) = F(s, \alpha)$$

Comparing residues at $s = s_0$ we see that coefficients $a(n)$ are q-periodic (q - the conductor of F). $\implies$ $F(s)$ is a linear combination of Dirichlet L-functions $(\bmod\, q)$.

Some applications of the standard twist

III. $S^\sharp_d = \emptyset$ for $1 < d < 2$ (J.K.-A.P. – 2011).
Main idea of the proof. Suppose that $F \in S^\sharp$ has degree $1 < d < 2$, and consider

$$F(s,f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-f(n,\alpha))$$

where

$$f(\xi,\alpha) = \sum_{j=0}^{N} \alpha_j \xi^{\kappa_j} \quad , \quad \alpha = (\alpha_0, \dots, \alpha_N) \quad , \quad \alpha_0 > 0$$

$$\kappa_0 > \kappa_1 > \ldots > \kappa_N > 0 \qquad \kappa_0 > 1/d$$

Some applications of the standard twist

Two basic operations

$$T : F(s, f) \mapsto F(s^*, f^*)$$

where f^* denotes (suitably defined) 'conjugated' exponent of a similar form as f but possibly with different exponents and coefficients.

$$S : F(s, f) \mapsto F(s, \xi + f)$$

Consider the group $\mathfrak{S} =< S, T >$ which acts on twists $F(s, f)$.

Some applications of the standard twist

We take $\alpha_0 \in Spec(F)$ and the exponent $f_0(\xi) = \alpha_0 \xi^{1/d}$, so that $F(s, f_0)$ is the standard twist of F. Then it is proved that there exists

$$g = STS^{m_N} S \dots S^{m_1} TS^{m_1} TS \in \mathfrak{S}$$

such that

$$g(F(s, f_0)) = F(s^*, f^*)$$

with

$$\Re(s_0^*) > 1$$

Now, $F(s, f_0)$ has a pole at $s = s_0 \implies F(s, f^*)$ has a pole at $s = s_0^*$- a contradiction.

Some applications of the standard twist

IV. Let $F(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ has meromorphic continuation to $\mathbb{C}$ with at most one singularity, a pole at $s = 1$. Moreover, let

$$\Phi(s) = \omega\overline{\Phi(1-\bar{s})} \qquad (|\omega| = 1)$$

$$\Phi(s) = \left(\frac{\sqrt{5}}{2\pi}\right)^{s} \Gamma(s+\mu)F(s) \qquad (\Re(\mu) \geqslant 0)$$

$$a(n) \ll n^{\varepsilon}$$

$$\log F(s) = \sum_{n=2}^{\infty} \frac{b(n)\Lambda(n)}{n^{s}} \qquad (b(n) \ll n^{\theta}, \theta < 1/2).$$

Some applications of the standard twist

Theorem (J.K.-A.P. – 2018)

There exists $k \in \mathbb{N}$, $\chi(\mathrm{mod} 5)$ *such that*

$$\Re(\mu) = \frac{k-1}{2} \qquad \chi(-1) = (-1)^k$$

and either

$$F(s) = \zeta(s)L(s,\chi) \qquad \text{(if } F(s) \text{ is polar)}$$

or

$$F(s) = L(s+\mu, f)$$

for certain newform $f \in S_k(\Gamma_0(5),\chi)$.

The general problem:

Problem: Describe finer properties of the standard twist. In particular:
(1) does it satisfy functional equation relating s to $1-s$?
(2) What is the polar structure of $F(s,\alpha)$ when $\alpha \in Spec(F)$?
(3) Give precise convexity bounds for the Lindelöff μ-function

$$\mu(s,\alpha) = \inf\{\lambda | F(\sigma+it) = O(|t|^{\lambda}) \text{ as } t \to \infty\}.$$

(4) Determine location of the zeros (trivial, nontrivial).
(5) Other.

The case of half-integral weight cusp forms L-functions.

Let f be a cusp form of half-integral weight $\kappa = k/2$ and level N, where $k > 0$ is an odd integer and $4|N$, and $L_f(s)$ be the associated Hecke L-function. Then $L_f(s)$ is entire and satisfies the functional equation

$$\Lambda_f(s) = \omega \Lambda_{f^*}(\kappa - s)$$

where

$$\Lambda_f(s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L_f(s)$$

$|\omega| = 1$ and f^* is related to f by the slash operator. Note that $L_{f^*}(s)$ is also entire and has properties similar to $L_f(s)$.

The case of half-integral weight cusp forms L-functions.

Extra notation

$$c_l^*(\nu^2) = \begin{cases} -e^{i\pi\mu} a^*(\nu^2) & \text{if } \nu \geqslant 1 \\ e^{i\pi(\frac{1}{2}+l-\mu)} a^*(\nu^2) & \text{if } -\nu_\alpha < \nu < -1 \\ e^{-i\pi\mu} a^*(\nu^2) & \text{if } \nu < -\nu_\alpha \end{cases}$$

$$\nu_\alpha = \sqrt{n_\alpha} = \frac{1}{2}\sqrt{N}\alpha \quad , \quad \nu = \sqrt{n} \qquad (n \geqslant 1)$$

$$F_l^+(s,\nu) = \sum_{\nu > -\nu_\alpha} \frac{c^*(\nu^2)}{|\nu|^{\frac{1}{2}+l} |\nu + \nu_\alpha|^{2s-\frac{1}{2}-l}}$$

$$F_l^-(s,\nu) = \sum_{\nu < -\nu_\alpha} \frac{c^*(\nu^2)}{|\nu|^{\frac{1}{2}+l} |\nu + \nu_\alpha|^{2s-\frac{1}{2}-l}}$$

$$F_l^*(s,\alpha) = e^{-i\pi s} F_l^+(s) + e^{i\pi s} F_l^-(s)$$

The case of half-integral weight cusp forms L-functions.

Theorem ((J.K.-A.P. – 2018))

(1) The functions $F_l^*(s,\alpha)$ *are entire.*

(2) We have

$$F(s,\alpha) = \frac{\omega}{i\sqrt{2\pi}} \left(\frac{\sqrt{N}}{4\pi}\right)^{1-2s} \sum_{l=0}^{h^*} a_l \Gamma(2(1-s) - \frac{1}{2} - l) F_l^*(1-s,\alpha)$$

$$((h^* = \max(0,[\kappa]-1))$$

The case of half-integral weight cusp forms L-functions.

Corollary

For $\alpha \in Spec(F)$, the standard twist $F(s, \alpha)$ has a finite number of poles. They could be at the points

$$s = s_l = \frac{3}{4} - \frac{l}{2} \qquad (l = 0, \ldots, h^*)$$

The case of half-integral weight cusp forms L-functions.

Remark. A closer analysis of the proof reveals that these statements are consequences of the very special form of the functional equation of the L-functions associated to the half-integer cusp forms. This is due to the fact that the argument is based on the explicit expression of the Mellin-Barnes integral

$$\frac{1}{2\pi i}\int_{(c)}\Gamma(\xi-w)\Gamma(w)\eta^{-w}\,\mathrm{d}w=\Gamma(\xi)(1+\eta)^{-\xi},$$

where $0<c<\Re(\xi)$ and $|\arg(\eta)|<\pi$. The method works for some other γ-factors but fails in general. In particular, the above statements are FALSE even for the L-functions of the Hecke cusp forms, notwithstanding the similarity of functional equations.

The general case.

Let $F \in S_d^\sharp$, $d > 0$.

$$Spec(F) = \left\{ \left(\frac{m}{q}\right)^{1/d} : m \in \mathbb{N} \text{ with } a(m) \neq 0 \right\}$$

$$s_l = \frac{d+1}{2d} - \frac{l}{d} \qquad (l = 0, 1, 2, \ldots)$$

$$\gamma(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)$$

For simplicity we assume that $F(s)$ is entire and normalized:

$$\theta_F := \Im\left(\sum_{j=1}^{r} \mu_j\right) = 0$$

The general case.

Definition

$$S_F(s) = 2^r \prod_{j=1}^{r} \sin(\pi(\lambda_j s + \mu_j)) = \sum_{j=-N}^{N} a_j e^{i\pi d\omega_j s}$$

$$-\frac{1}{2} = \omega_{-N} < \ldots < \omega_N = \frac{1}{2}$$

$$h_F(s) = \frac{\omega}{(2\pi)^r} Q^{1-2s} \prod_{j=1}^{r} (\Gamma(\lambda_j(1-s) + \overline{\mu_j})\Gamma(1 - \lambda_j s - \mu_j))$$

Remark. Both functions $S_F(s)$ and $h_F(s)$ are invariants. Moreover,

$$F(s) = h_F(s) S_F(s) \overline{F}(1-s)$$

The general case.

Definition

For $l \geqslant 0$ we define

$$\overline{F}_l(s,\alpha) = \sum_{j=-N}^{N} a_j e^{i\pi d\omega_j(1-s)} \sum_{n\geqslant 1}^{\flat} \frac{a(n)}{n^s} \left(1 + e^{i\pi(\frac{1}{2}-\omega_j)} \left(\frac{n_\alpha}{n}\right)^{1/d}\right)^{d(1-s-s_l)},$$

where $^\flat$ indicates that if $j=-N$ then the term $n=n_\alpha$ is omitted.

The general case.

Theorem

For every $l \geqslant 0$ and $\alpha > 0$, the function $\overline{F}_l(s,\alpha)$ is entire and not identically vanishing. Moreover, uniformly for σ in any bounded interval, as $|t| \to \infty$ we have

$$\overline{F}_l(s,\alpha) \ll e^{\frac{\pi}{2}d|t|}|t|^{c(\sigma)}$$

with a certain $c(\sigma) \geqslant 0$ independent of l and α, satisfying $c(\sigma) = 0$ for $\sigma > 1$.

The general case.

Definition (Structural coefficients of $F(s)$)

For $|\arg(-s)| < \pi - \delta$ we have

$$h_F(s) \sim \frac{\omega_F}{\sqrt{2\pi}} \left(\frac{q^{1/d}}{2\pi d} \right)^{d(\frac{1}{2}-s)} \sum_{l=0}^{\infty} d_l \Gamma(d(s_l - s))$$

The invariants d_l are called the *structural coefficients* of $F(s)$.

The general case.

Theorem (J.K.-A.P. – 2019)

For any integer $k \geqslant 0$ and s in the strip $s_{k+1} < \sigma < s_k$ we have

$$F(s,\alpha) = \frac{\omega_F}{\sqrt{2\pi}} \left(\frac{q^{1/d}}{2\pi d} \right)^{d(\frac{1}{2}-s)} \sum_{l=0}^{k} d_l \Gamma(d(1-s)-s_l) \overline{F}_l(1-s,\alpha) + H_k(s,\alpha),$$

where the function $H_k(s,\alpha)$ is holomorphic in the above strip and meromorphic over $\mathbb{C}$. Moreover, there exists $\theta = \theta(d) > 0$ such that for any $\sigma \in [s_{k+1}, s_k] \cap (-\infty, 0)$ we have

$$H_k(s,\alpha) \ll |t|^{-\theta} \qquad \text{as} \quad |t| \to \infty.$$

The general case.

Theorem (J.K.-A.P. – 2019)

For $\alpha \in Spec(F)$ we have

$$\mathrm{Res}_{s=s_l} F(s,\alpha) = \frac{d_l}{d} \frac{\omega_F}{\sqrt{2\pi}} e^{-i\frac{\pi}{2}(\xi_F + ds_l)} \left(\frac{q^{1/d}}{2\pi d} \right)^{\frac{d}{2} - ds_l} \frac{\overline{a(n_\alpha)}}{n_\alpha^{1-s_l}}.$$

In particular, the set of poles of $F(s,\alpha)$ is independent of α and equals $\{s_l : d_l \neq 0\}$.

The general case.

Definition

We say that $F(s,\alpha)$ satisfies a strict functional equation if there exists an integer h such that $H_k(s,\alpha) \equiv 0$ for every $k \geqslant h$ and $\alpha > 0$.

Remark. Obviously the strict functional equation of $F(s,\alpha)$ has the following form

$$F(s,\alpha) = \frac{\omega_F}{\sqrt{2\pi}} \left(\frac{q^{1/d}}{2\pi d} \right)^{d(\frac{1}{2}-s)} \sum_{l=0}^{h} d_l \Gamma(d(1-s)-s_l) \overline{F}_l(1-s,\alpha).$$

The general case.

Definition

Let $N \geqslant 1$ and $n_j \geqslant 0$, $j = 1, \ldots, N$, be integers. We say that $n_1, \ldots, n_N$ form a *compatible system* if

(1) $n_i \not\equiv n_j (\mathrm{mod} 2N)$ for every $i \neq j$

(2) $n_i \not\equiv 1 - n_j (\mathrm{mod} 2N)$ for every i, j.

The general case.

Theorem (J.K.-A.P. – 2019)

The following statements are equivalent.
(i) $F(s,\alpha)$ *satisfies a strict functional equation.*
(ii) For every $\alpha \in Spec(F)$ *all the poles of* $F(s,\alpha)$ *are at there points* s_l *where* $0 \leqslant l \leqslant h$, $d_l \neq 0$.
(iii) $F(s)$ *has a* γ*-factor of the form*

$$\gamma(s) = Q^s \prod_{j=1}^{N} \Gamma\left(\frac{d}{2N}s + \frac{2n_j - d - 1}{4N}\right),$$

where $Q > 0$, $N \geqslant 1$ *and the integers* n_j *satisfy* $n_j \geqslant (d+1)/2$ *and form a compatible system.*

Grazie per l'attenzione!