Arithmetic Statistics and Mixed Moments of Elliptic Curve L-Functions over Function Fields

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Arithmetic Statistics

Arithmetic statistics: Statistical properties of arithmetic functions, e.g. $\Lambda(n)$, $\mu(n)$, $d_k(n)$.

$L$-function statistics: Statistical properties of $L$-functions, e.g. zero-statistics, moments, etc.

Arithmetic and $L$-function statistics in function fields: Statistical properties of arithmetic functions and $L$-functions related to function fields defined over $\mathbb{F}_q$; e.g. $\Lambda(f)$, $\mu(f)$, $d_k(f)$ for $f$ monic of degree $k$.

Limits: $k \to \infty$, or as the degree of the $L$-functions (considered as polynomials) grows, vs $q \to \infty$. 

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Primes in Short Intervals

\[ \Lambda(n) = \begin{cases} 
\log p & \text{if } n = p^k \text{ for some prime } p \text{ and integer } k \geq 1, \\
0 & \text{otherwise.}
\end{cases} \]

so that

\[ \frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \]

Conjecture (Goldston & Montgomery, 1987; Montgomery & Soundararajan 2004)

For \( X^\delta < H < X^{1-\delta} \), as \( X \to \infty \)

\[ \frac{1}{X} \int_2^X \left| \sum_{n \in [x-H/2, x+H/2]} \Lambda(n) - H \right|^2 \, dx \sim H \left( \log(X/H) - (\gamma_E + \log 2\pi) \right) \]
Remark 1. The main motivation for proving these theorems comes from the fact, shown in Sections 3 and 4, that the Selberg Orthogonality Conjecture and the ratios conjecture for $F$ imply that

\[
e_F(F(T, T) = (T \log X \pi + O(T/d_F + \epsilon)) + O(T^{1/2} + \epsilon)) \quad \text{if} \quad \ll < d_F,
\]

for a smoothed form of the pair correlation $e_F(F(X, T) = \int_{F(0,F)} X \ll F, 0 \ll F, T X i (F(0,F)))^2$. We expect that $F(X, T)$ and $e_F(F(X, T)$ satisfy the same estimates, at least up to some power saving error term, and these are the forms that appear in the theorems quoted above.

Alternatively, if we were to replace $F(X, T)$ by $e_F(F(X, T))$ in the statements of the above theorems, we would obtain correspondingly smoothed forms of the variances $V_F(X, h)$ instead; that is, variances involving averages with weight-functions whose mass is concentrated on $(1, X)$. We establish the form of the ratios conjecture we need in Section 3 and from this obtain the above formulae for $e_F(F(X, T))$ in Section 4.

Figure 1. $\tilde{V}_F(X, h)/(hX)$ plotted against $\log h$ when $F$ is the Riemann zeta-function and $X = 15000000$ ($\bullet$). The line is given by $y = -x + \log 15000000 - \gamma_0 - \log 2\pi$, which corresponds to $d_F = 1$ and $q_F = 1$. 

\[
\tilde{V}_F(X, h)/(hX)
\]
Let $\mathbb{F}_q$ be a finite field of $q$ elements and $\mathbb{F}_q[t]$ the ring of polynomials with coefficients in $\mathbb{F}_q$. 
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Define the norm of a polynomial $0 \neq f \in \mathbb{F}_q[t]$ to be

$$||f|| := q^{\deg f}$$
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$$\| f \| := q^{\deg f}$$

Denote monic irreducible polynomials by $P$.
Define

\[ \Lambda(f) = \begin{cases} 
\deg P & \text{if } f = P^k \text{ for some irreducible polynomial } P, \\
0 & \text{otherwise.} 
\end{cases} \]

and

\[ I(A; h) := \{ f : \|f - A\| \leq q^h \} = A + \mathcal{P}_{\leq h} \]
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**Theorem (variance in intervals – JPK & Z. Rudnick, 2014)**

Let \( h < n - 3 \). Then

\[
\lim_{q \to \infty} \frac{1}{q^{h+1}} \text{Var}_A \left( \sum_{\substack{f \in I(A, h) \\ f(0) \neq 0}} \Lambda(f) \right) = \int_{U(n-h-2)} |\text{tr}U^n|^2 dU = n - h - 2
\]

This is the exact analogue of the Goldston-Montgomery-Soundararajan formula.
In the same vein:

**Theorem (variance in arithmetic progressions – JPK & Z. Rudnick, 2014)**

*Given a sequence of finite fields $\mathbb{F}_q$ and square-free polynomials $Q \in \mathbb{F}_q[t]$ with $2 \leq \deg(Q) \leq n + 1$,*

\[
\sum_{A \mod Q \gcd(A,Q) = 1} \left| \sum_{f \equiv A \mod Q \ f \in M_n} \Lambda(f) - \frac{q^n}{\Phi(Q)} \right|^2 \sim q^n(\deg(Q) - 1)
\]

*as $q \to \infty$.*

This is the exact analogue of a formula in the number field setting conjectured by Hooley.
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recognize/evaluate the integrals
Let $S$ denote the Selberg class $L$-functions. For $F \in S$ primitive,

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s} = \prod_p \exp \left( \sum_{l=1}^{\infty} \frac{b_F(p^l)}{p^{ls}} \right).$$

satisfying the functional equation

$$\Phi(s) := Q^s \left( \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) \right) F(s) = \varepsilon_F \overline{\Phi}(1 - s),$$

with some $Q > 0$, $\lambda_j > 0$, $\text{Re}(\mu_j) \geq 0$ and $|\varepsilon_F| = 1$. This determines the degree $d_F$ and the conductor $q_F$:

$$d_F = 2 \sum_{j=1}^{r} \lambda_j \quad \text{and} \quad q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}.$$
Define

\[ \frac{F'(s)}{F(s)} = - \sum_{n=1}^{\infty} \frac{\Lambda_F(n)}{n^s} \quad (\text{Re}(s) > 1). \]

Then for \( X^{1-1/d_F} < H < X^{1-\delta} \), as \( X \to \infty \)

\[
\frac{1}{X} \int_{\frac{n-H}{2}}^{\frac{n+H}{2}} \left| \sum_{n \in [x-H/2, x+H/2]} \Lambda_F(n) \right|^2 \, dx \sim H \left( \log(X/H) + \log q_F - (\gamma_E - \log 2\pi) d_F \right)
\]
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\]

Then for \( X^{1-1/d_F} < H < X^{1-\delta} \), as \( X \to \infty \)

\[
\frac{1}{X} \int_{2}^{X} \left| \sum_{n \in [x-H/2, x+H/2]} \Lambda_F(n) \right|^2 dx \sim H \left( \log(X/H) + \log q_F - (\gamma_E - \log 2\pi) d_F \right)
\]

and for \( X^\delta < H < X^{1-1/d_F} \), as \( X \to \infty \)

\[
\frac{1}{X} \int_{2}^{X} \left| \sum_{n \in [x-H/2, x+H/2]} \Lambda_F(n) \right|^2 dx \sim \frac{1}{6} H \left( 6 \log X - (3 + 8 \log 2) \right)
\]
In the first case, shown in Figure 1 above, $F$ is the Riemann zeta-function (so $\mathcal{L}$ is just the von Mangoldt function) and $X = 15,000,000$. This is, of course, an example with $d_F = 1$ and so one sees a single regime that is well described by (1.4).

By way of contrast, we plot in Figure 2 data for two $L$-functions with $d_F = 2$. In these examples $X = 1,000,000$. The lines correspond to the formulae for the two regimes described by Theorems C1 and C2.

Figure 2. $\tilde{V}_F(X, h)/(hX)$ plotted against $\log h$ when $F$ is associated with the Ramanujan $\tau$-function (●) and with an elliptic curve of conductor 37 (▲). Here $X = 1000000$. The horizontal line is given by $y = \log 1000000 - \frac{(3 + 8 \log 2)}{6}$. The slanted lines are given by $y = -2x + 2(\log 1000000 - \gamma_0 - \log 2\pi)$, which corresponds to the case $d_F = 2$ and $q_F = 1$ for the Ramanujan $\tau$-function, and $y = -2x + 2(\log 1000000 - \gamma_0 - \log 2\pi) + \log 37$, which corresponds to the case $d_F = 2$ and $q_F = 37$ for the elliptic curve.

Remark 3. The analogues of Theorems A1, B1, C1 and D1 in the case when $F$ is the Riemann zeta-function were obtained by Chan [3]. Note that, unlike the case of the Riemann zeta-function [6, 3], the A Theorems above are not exactly the converse of the B Theorems, and the C Theorems are not exactly the converse of the D Theorems. They are close to being the converse of each other, but with the power saving errors we have here, the intervals of uniformity do not match precisely. The proofs of the theorems within each pair are essentially identical, so we only give the proofs of Theorems A1, B1 and C1. Likewise, the proofs of Theorems D1 and D2 are similar to the proofs of C1 and C2, so we omit them too.
Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N$ defined over $\mathbb{Q}$ with $L$-function

$$L(s, E) = \prod_{p | N} (1 - a_p p^{-s-1/2})^{-1} \prod_{p \nmid N} (1 - a_p p^{-s-1/2} + p^{-2s})^{-1}$$

where $a_p = p + 1 - \#\tilde{E}(\mathbb{F}_p)$. When $p \mid N$, then $a_p$ is either 1, $-1$, or 0. In general, $|a_p| < 2\sqrt{p}$. Hence can write

$$\frac{a_p}{p^{1/2}} = 2 \cos(\theta_p) = \alpha_p + \beta_p$$

where, for $p \nmid N$, one has $\alpha_p = e^{i\theta_p}$ and $\beta_p = e^{-i\theta_p}$ with $\theta_p \in [0, \pi]$ and for $p \mid N$, one has $\alpha_p = a_p$, and $\beta_p = 0$. 
Define $\Lambda_E$ by

$$\frac{L(s, E)'}{L(s, E)} = - \sum_{n=1}^{\infty} \Lambda_E(n) n^{-s},$$

so that for $e \geq 1$

$$\Lambda_E(n) = \begin{cases} 
\log p \cdot (\alpha_p^e + \beta_p^e) & \text{if } n = p^e \text{ with } p \text{ prime} \\
0 & \text{otherwise.}
\end{cases}$$

Denote the variance of the sum of $\Lambda_E$ in arithmetic progressions, for example, by

$$S_{x,c,E}(A) := \sum_{n \leq x \atop n = A \mod c} \Lambda_E(n).$$
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$$S_{x, c, E}(A) := \sum_{n \leq x \atop n = A \mod c} \Lambda_E(n).$$

Then for $x^\epsilon < c < x^{1-\epsilon}$, $\epsilon > 0$ the prediction is

$$\text{Var}_A(S_{x, c, E}) \sim \frac{x}{\phi(c)} \min\{\log x, 2 \log c\}.$$
The corresponding formulae for the variance of the sum of $\Lambda_E(n)$ over arithmetic progressions (Hall, K & Roditty-Gershon) and short intervals (Sawin) hold (unconditionally) when $q \to \infty$ for the function field analogues of elliptic curve $L$-functions.
Suppose $q$ is an odd prime power, and let $E_{\text{Leg}}/\mathbb{F}_q(t)$ be the Legendre curve, that is, the elliptic curve with affine model

$$y^2 = x(x - 1)(x - t).$$

Over the ring $\mathbb{F}_q[t]$, this curve has conductor $s = t(t - 1)$. The corresponding $L$-function is given by the Euler product

$$L(T, E_{\text{Leg}}/\mathbb{F}_q(t)) = \prod_{P} L(T^{\deg(P)}, E_{\text{Leg}}/\mathbb{F}_P)^{-1}$$

where $\mathbb{F}_P$ is the residue field $\mathbb{F}_q[t]/P\mathbb{F}_q[t]$.

Each Euler factor is the reciprocal of a polynomial and satisfies

$$T \frac{d}{dT} \log L(T, E_{\text{Leg}}/\mathbb{F}_P)^{-1} = \sum_{m=1}^{\infty} a_{P,m} T^m \in \mathbb{Z}[[T]].$$
Define $\Lambda_{\text{Leg}}(f)$ by

$$
\Lambda_{\text{Leg}}(f) = \begin{cases} 
\deg(P) \cdot a_{P,m} & \text{if } f = P^m \\
0 & \text{otherwise,}
\end{cases}
$$

then the $L$-function satisfies

$$
T \frac{d}{dT} \log(L(T, E_{\text{Leg}}/\mathbb{F}_q(t))) = \sum_{n=1}^{\infty} \left( \sum_{f \in M_n} \Lambda_{\text{Leg}}(f) \right) T^n.
$$

Let $Q \in \mathbb{F}_q[t]$ be monic and square-free. For each $n \geq 1$ and each $A$ in $\Gamma(Q) = (\mathbb{F}_q[t]/Q \mathbb{F}_q[t])^\times$, consider the sum

$$
S_{n,Q}(A) := \sum_{f \in M_n} \Lambda_{\text{Leg}}(f) \text{ for } f \equiv A \pmod{Q}.
$$
Let $A$ vary uniformly over $\Gamma(Q)$.

**Theorem (C. Hall, JPK, E. Roditty-Gershon)**

If $\gcd(Q, s) = t$ and if $\deg(Q)$ is sufficiently large, then

$$\lim_{q \to \infty} \frac{|\Gamma(Q)|}{q^{2n}} \cdot \text{Var}_A[S_{n, Q}(A)] = \min\{n, 2 \deg(Q) - 1\}.$$
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- The fact that the expression for the variance depends on $2 \deg(Q)$ is a direct consequence of the fact that the associated $L$-functions have degree two. (For an $L$-function of degree $r$, one will get a leading term of $r \deg(Q)$ instead.) This then leads to there being two ranges.
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- (NB the same multiple-range behaviour arises in sufficiently high moments of the zeta function (c.f. Brian Conrey’s talk))
Derivatives of Elliptic Curve $L$-functions and Applications to Rank Correlations

The first moment of the family of derivatives of $L$–functions of quadratic twists of a fixed modular form was computed by Bump-Friedberg & Hoffstein, Iwaniec, and Murty & Murty in 1990, with applications to non-vanishing; e.g. for a fixed elliptic curve with root number equal to 1 there are infinitely many fundamental discriminants $d < 0$ such that its twist by $d$ has analytic rank equal to 1.

The second moment of this family was considered by Soundararajan & Young in 2010. Petrow recently obtained several asymptotic formulas for moments of derivatives of these $GL(2)$–functions when the sign of the functional equation is $-1$ (on GRH).
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Petrow recently obtained several asymptotic formulas for moments of derivatives of these \( GL(2) \) \( L \)-functions when the sign of the functional equation is \(-1\) (on GRH).
Fix a prime power $q$ with $(q, 6) = 1$ and $q \equiv 1 \pmod{4}$. Let $K = \mathbb{F}_q(t)$ be the rational function field and $\mathcal{O}_K = \mathbb{F}_q[t]$. Let $E/K$ be an elliptic curve defined by $y^2 = x^3 + ax + b$, with $a, b \in \mathcal{O}_K$ and discriminant $\Delta = 4a^3 + 27b^2$ such that $\deg_t(\Delta)$ is minimal among models of $E/K$ of this form.
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The associated $L$–function may be written (for $\Re(s) > 1$),

$$L(E, s) := \mathcal{L}(E, u) = \sum_{f \in \mathcal{M}} \lambda(f)u^{\deg(f)}$$

$$= \prod_{P|\Delta} \left( 1 - \lambda(P)u^{\deg(P)} \right)^{-1} \prod_{P|\Delta} \left( 1 - \lambda(P)u^{\deg(P)} + u^{2\deg(P)} \right)^{-1},$$

where we set $u := q^{-s}$, and $\mathcal{M}$ denotes the set of monic polynomials over $\mathbb{F}_q[t]$. 
The $L$–function is a polynomial in $u$ with integer coefficients of degree

$$n := \deg (L(E, u)) = \deg(M) + 2 \deg(A) - 4,$$

where we denote by $M$ the product of the finite primes where $E$ has multiplicative reduction and by $A$ the product of the finite primes where $E$ has additive reduction.
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has additive reduction.
The $L$–function satisfies the functional equation
\[ \mathcal{L}(E, u) = \epsilon(E)(\sqrt{qu})^n \mathcal{L}(E, \frac{1}{qu}) \]
where $\epsilon(E) \in \{ \pm 1 \}$. 
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\[ L(E, u) = \epsilon(E)(\sqrt{qu})^n L(E, \frac{1}{qu}) \]
where $\epsilon(E) \in \{\pm 1\}$.

For $D \in \mathcal{O}_K$ with $D$ square-free, monic of odd degree and $(D, \Delta) = 1$, consider the twisted elliptic curve $E \otimes \chi_D/K$ with the affine model
\[ y^2 = x^3 + D^2ax + D^3b. \]
The $L$–function is a polynomial in $u$ with integer coefficients of degree

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$$L(E, u) = \epsilon(E)(\sqrt{qu})^n L \left( E, \frac{1}{qu} \right)$$

where $\epsilon(E) \in \{ \pm 1 \}$.

For $D \in \mathcal{O}_K$ with $D$ square-free, monic of odd degree and $(D, \Delta) = 1$, consider the twisted elliptic curve $E \otimes \chi_D/K$ with the affine model $y^2 = x^3 + D^2 ax + D^3 b$. The $L$–function corresponding to the twisted elliptic curve $L(E \otimes \chi_D, u)$ has the Dirichlet series and Euler product

$$\sum_{f \in \mathcal{M}} \lambda(f) \chi_D(f) u^{\deg(f)} =$$

$$\prod_{P|\Delta} \left( 1 - \lambda(P)\chi_D(P) u^{\deg(P)} \right)^{-1} \prod_{P|\Delta D} \left( 1 - \lambda(P)\chi_D(P) u^{\deg(P)} + u^{2\deg(P)} \right)^{-1}$$
The new $L$–function is a polynomial of degree $(n + 2 \deg(D))$ and satisfies the functional equation

$$L(E \otimes \chi_D, u) = \epsilon (\sqrt{qu})^{n+2\deg(D)} L\left(E \otimes \chi_D, \frac{1}{qu}\right),$$

where

$$\epsilon = \epsilon(E \otimes \chi_D) = \epsilon_{\deg(D)}\epsilon(E)\chi_D(M),$$

with $\epsilon_{\deg(D)} \in \{\pm 1\}$ is an integer which only depends on the degree of $D$. 
The new $L$–function is a polynomial of degree $(n + 2 \deg(D))$ and satisfies the functional equation

$$\mathcal{L}(E \otimes \chi_D, u) = \epsilon \left( \sqrt{qu} \right)^{n + 2 \deg(D)} \mathcal{L} \left( E \otimes \chi_D, \frac{1}{qu} \right),$$

where

$$\epsilon = \epsilon(E \otimes \chi_D) = \epsilon_{\deg(D)} \epsilon(E) \chi_D(M),$$

with $\epsilon_{\deg(D)} \in \{ \pm 1 \}$ is an integer which only depends on the degree of $D$.

Let $\mathcal{H}_{2g+1}^*$ denote the set of monic, square free polynomials of degree $(2g + 1)$ coprime to $\Delta$. Our first two theorems concern the first moments of $L(E \otimes \chi_D, 1/2)$ and $L'(E \otimes \chi_D, 1/2)$. 


Theorem (Bui, Florea, JPK, Roditty-Gershon)

Unless \( \epsilon_{2g+1}\epsilon(E) = -1 \) and \( M = 1 \),

\[
\frac{1}{|\mathcal{H}_{2g+1}^*|} \sum_{D \in \mathcal{H}_{2g+1}^*} L(E \otimes \chi_D, \frac{1}{2}) = c_1(M) + O_\epsilon(q^{-g+\epsilon})
\]

with \( c_1(M) \neq 0 \).
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with $c_1(M) \neq 0$.

Theorem (Bui, Florea, JPK, Roditty-Gershon)

Unless $\epsilon_{2g+1}\epsilon(E) = 1$ and $M = 1$,

$$\frac{1}{|\mathcal{H}_{2g+1}^*|} \sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon L'(E \otimes \chi_D, \frac{1}{2}) = c_2(M)L(\text{Sym}^2E, 1)g + c_3(M) + O_\epsilon(q^{-g+\epsilon g})$$

where $c_2(M) \neq 0$. 
Similarly

**Theorem (Bui, Florea, JPK, Roditty-Gershon)**

Unless $\epsilon_{2g+1} \epsilon(E) = -1$ and $M = 1$,

$$\frac{1}{|H^*_{2g+1}|} \sum_{D \in H^*_{2g+1}} L(E \otimes \chi_D, \frac{1}{2})^2 = c_4(M)L(\text{Sym}^2 E, 1)^3 g + O_\epsilon(g^{1/2+\epsilon}),$$

where $c_4(M) \neq 0$. 
Similarly

**Theorem (Bui, Florea, JPK, Roditty-Gershon)**

Unless $\epsilon_{2g+1}\epsilon(E) = -1$ and $M = 1$,

$$
\frac{1}{|H_{2g+1}^*|} \sum_{D \in H_{2g+1}^*} L(E \otimes \chi_D, \frac{1}{2})^2 = c_4(M) L(Sym^2E, 1)^3 g + O_\epsilon(g^{1/2+\epsilon}),
$$

where $c_4(M) \neq 0$.

**Theorem (Bui, Florea, JPK, Roditty-Gershon)**

Unless $\epsilon_{2g+1}\epsilon(E) = -1$ and $M = 1$,

$$
\frac{1}{|H_{2g+1}^*|} \sum_{D \in H_{2g+1}^*} \epsilon^{-L'(E \otimes \chi_D, \frac{1}{2})^2} = c_5(M) L(Sym^2E, 1)^3 g^3 + O_\epsilon(g^{2+\epsilon}),
$$

where $c_5(M) \neq 0$. 

Jon Keating (Bristol) Arithmetic Statistics and Mixed Moments of Elliptic Curve L-Functions over Function Fields July 11, 2019 23 / 28
Let $E_1$ and $E_2$ be two elliptic curves over $K$. Let $\Delta = \Delta_1 \Delta_2$, where, for $i = 1, 2$, $\Delta_i$ denotes the discriminant of $E_i$. Let $M_i$ denote the product of the finite primes where $E_i$ has multiplicative reduction and $\epsilon_i = \epsilon_{\text{deg}(D)} \epsilon(E_i) \chi_D(M_i)$. Define

$$\epsilon_i^+ := \frac{1 + \epsilon_i}{2} \quad \text{and} \quad \epsilon_i^- := \frac{1 - \epsilon_i}{2}.$$
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$$\epsilon_i^+ := \frac{1 + \epsilon_i}{2} \quad \text{and} \quad \epsilon_i^- := \frac{1 - \epsilon_i}{2}.$$ 

**Theorem (Bui, Florea, JPK, Roditty-Gershon)**

Unless $\epsilon_{2g+1} \epsilon(E_1) = -1$ and $M_1 = 1$, or $\epsilon_{2g+1} \epsilon(E_2) = 1$ and $M_2 = 1$, or $\epsilon(E_1) = \epsilon(E_2)$ and $M_1 = M_2$, we have

$$\frac{1}{|\mathcal{H}_{2g+1}^*|} \sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon_2^- L(E_1 \otimes \chi_D, \frac{1}{2}) L'(E_2 \otimes \chi_D, \frac{1}{2})$$

$$= c_6(M_1, M_2) L(\text{Sym}^2 E_1, 1) L(\text{Sym}^2 E_2, 1) L(E_1 \otimes E_2, 1) g$$

$$+ O_\epsilon(g^{1/2+\epsilon}),$$

where $c_6(M_1, M_2) \neq 0$. 
Theorem (Bui, Florea, JPK, Roditty-Gershon)

Unless $\epsilon_{2g+1}\epsilon(E_1) = 1$ and $M_1 = 1$, or $\epsilon_{2g+1}\epsilon(E_2) = 1$ and $M_2 = 1$, or $\epsilon(E_1) = -\epsilon(E_2)$ and $M_1 = M_2$, we have

$$
\frac{1}{|\mathcal{H}_{2g+1}^*|} \sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon_1^{-} \epsilon_2^{-} L'(E_1 \otimes \chi_D, \frac{1}{2}) L'(E_2 \otimes \chi_D, \frac{1}{2})
$$

$$
= c_7(M_1, M_2) L(\text{Sym}^2 E_1, 1) L(\text{Sym}^2 E_2, 1) L(E_1 \otimes E_2, 1) g^2 
$$

$$
+ O_{\epsilon}(g^{1+\epsilon}),
$$

where $c_7(M_1, M_2) \neq 0$. 
Define the analytic rank of a quadratic twist of an elliptic curve $L$-function $L(E \otimes \chi_D, s)$ by

$$r_{E \otimes \chi_D} := \text{ord}_{s=1/2} L(E \otimes \chi_D, s).$$
Define the analytic rank of a quadratic twist of an elliptic curve $L$-function $L(E \otimes \chi_D, s)$ by

$$r_{E \otimes \chi_D} := \text{ord}_{s=1/2} L(E \otimes \chi_D, s).$$

**Corollary**

Unless $\epsilon_{2g+1}\epsilon(E_1) = -1$ and $M_1 = 1$, or $\epsilon_{2g+1}\epsilon(E_2) = 1$ and $M_2 = 1$, or $\epsilon(E_1) = \epsilon(E_2)$ and $M_1 = M_2$, we have

$$\# \left\{ D \in \mathcal{H}^*_{2g+1} : r_{E_1 \otimes \chi_D} = 0, r_{E_2 \otimes \chi_D} = 1 \right\} \gg \varepsilon \frac{q^{2g}}{g^{6+\varepsilon}}$$

as $g \to \infty$. Also, unless $\epsilon_{2g+1}\epsilon(E_1) = 1$ and $M_1 = 1$, or $\epsilon_{2g+1}\epsilon(E_2) = 1$ and $M_2 = 1$, or $\epsilon(E_1) = -\epsilon(E_2)$ and $M_1 = M_2$, we have

$$\# \left\{ D \in \mathcal{H}^*_{2g+1} : r_{E_1 \otimes \chi_D} = r_{E_2 \otimes \chi_D} = 1 \right\} \gg \varepsilon \frac{q^{2g}}{g^{6+\varepsilon}}$$

as $g \to \infty$. 
As far as we are aware, this is the first result in literature where explicit lower bounds concerning the correlations between the ranks of two twisted elliptic curves are obtained.
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Following Harper’s argument for the upper bounds for moments of $L$-functions, one may remove the exponents $\varepsilon$.

We fail to obtain positive proportions in the above results because we are not able to use a mollifier.
Computing mixed moments is a key theme of recent developments in Random Matrix Theory.
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