On the Duffin-Schaeffer conjecture

Dimitris Koukoulopoulos[1]

joint work with James Maynard[2]

[1]Université de Montréal
[2]University of Oxford

Second Symposium in Analytic Number Theory
Cetraro, Italy
10 July 2019

The problem

Given $\psi : \mathbb{N} \to [0, +\infty)$ and $\alpha \in [0, 1]$, solve the inequality

$$\left|\alpha - \frac{a}{q}\right| \leqslant \frac{\psi(q)}{q} \qquad \text{with} \qquad a \in \mathbb{Z},\ q \in \mathbb{N} \tag{$*$}$$

The problem

Given $\psi : \mathbb{N} \to [0, +\infty)$ and $\alpha \in [0, 1]$, solve the inequality

$$\left|\alpha - \frac{a}{q}\right| \leqslant \frac{\psi(q)}{q} \qquad \text{with} \qquad a \in \mathbb{Z},\ q \in \mathbb{N} \tag{$*$}$$

(possibly imposing the coprimality condition $\gcd(a, q) = 1$)

The problem

Given $\psi : \mathbb{N} \to [0, +\infty)$ and $\alpha \in [0, 1]$, solve the inequality

$$\left|\alpha - \frac{a}{q}\right| \leqslant \frac{\psi(q)}{q} \qquad \text{with} \qquad a \in \mathbb{Z},\ q \in \mathbb{N} \qquad (*)$$

(possibly imposing the coprimality condition $\gcd(a, q) = 1$)

Dirichlet: when $\psi(q) = 1/q$, then $(*)$ has infinitely many solutions **for all** $\alpha \in [0, 1]$.

The problem

Given $\psi : \mathbb{N} \to [0, +\infty)$ and $\alpha \in [0, 1]$, solve the inequality

$$\left| \alpha - \frac{a}{q} \right| \leqslant \frac{\psi(q)}{q} \qquad \text{with} \qquad a \in \mathbb{Z},\ q \in \mathbb{N} \tag{$*$}$$

(possibly imposing the coprimality condition $\gcd(a, q) = 1$)

Dirichlet: when $\psi(q) = 1/q$, then $(*)$ has infinitely many solutions **for all** $\alpha \in [0, 1]$.

Question: can we solve $(*)$ if ψ is more irregular?

The problem

Given $\psi : \mathbb{N} \to [0, +\infty)$ and $\alpha \in [0, 1]$, solve the inequality

$$\left|\alpha - \frac{a}{q}\right| \leqslant \frac{\psi(q)}{q} \qquad \text{with} \qquad a \in \mathbb{Z},\ q \in \mathbb{N} \tag{$*$}$$

(possibly imposing the coprimality condition $\gcd(a, q) = 1$)

Dirichlet: when $\psi(q) = 1/q$, then $(*)$ has infinitely many solutions **for all** $\alpha \in [0, 1]$.

Question: can we solve $(*)$ if ψ is more irregular?

Caveat: There might be exceptional α's.

The problem

Given $\psi : \mathbb{N} \to [0, +\infty)$ and $\alpha \in [0, 1]$, solve the inequality

$$\left|\alpha - \frac{a}{q}\right| \leqslant \frac{\psi(q)}{q} \qquad \text{with} \qquad a \in \mathbb{Z},\ q \in \mathbb{N} \tag{$*$}$$

(possibly imposing the coprimality condition $\gcd(a, q) = 1$)

Dirichlet: when $\psi(q) = 1/q$, then $(*)$ has infinitely many solutions **for all** $\alpha \in [0, 1]$.

Question: can we solve $(*)$ if ψ is more irregular?

Caveat: There might be exceptional α's.

Goal: understand when set of exceptional α's has null measure

Khinchin's theorem

$$\mathcal{K}_q := \bigcup_{0 \leqslant a \leqslant q} \left[\frac{a}{q} - \frac{\psi(q)}{q}, \frac{a}{q} + \frac{\psi(q)}{q} \right]$$

Khinchin's theorem

$$\mathcal{K}_q := \bigcup_{0 \leqslant a \leqslant q} \left[\frac{a}{q} - \frac{\psi(q)}{q}, \frac{a}{q} + \frac{\psi(q)}{q}\right]$$

$$\mathcal{K} := \limsup_{q \to \infty} \mathcal{K}_q$$

Khinchin's theorem

$$\mathcal{K}_q := \bigcup_{0 \leqslant a \leqslant q} \left[\frac{a}{q} - \frac{\psi(q)}{q}, \frac{a}{q} + \frac{\psi(q)}{q} \right]$$

$$\begin{aligned} \mathcal{K} &:= \limsup_{q \to \infty} \mathcal{K}_q \\ &= \{\alpha \in [0, 1] : \alpha \in \mathcal{K}_q \text{ for infinitely many } q\} \end{aligned}$$

Khinchin's theorem

$$\mathcal{K}_q := \bigcup_{0 \leqslant a \leqslant q} \left[\frac{a}{q} - \frac{\psi(q)}{q}, \frac{a}{q} + \frac{\psi(q)}{q} \right]$$

$$\mathcal{K} := \limsup_{q \to \infty} \mathcal{K}_q$$

$$= \{\alpha \in [0,1] : \alpha \in \mathcal{K}_q \text{ for infinitely many } q\}$$

Note that

$$\lambda(\mathcal{K}_q) \asymp \psi(q) \qquad (\lambda = \text{Lebesgue measure})$$

Khinchin's theorem

$$\mathcal{K}_q := \bigcup_{0 \leqslant a \leqslant q} \left[\frac{a}{q} - \frac{\psi(q)}{q}, \frac{a}{q} + \frac{\psi(q)}{q}\right]$$

$$\mathcal{K} := \limsup_{q \to \infty} \mathcal{K}_q$$

$$= \{\alpha \in [0,1] : \alpha \in \mathcal{K}_q \text{ for infinitely many } q\}$$

Note that

$$\lambda(\mathcal{K}_q) \asymp \psi(q) \qquad (\lambda = \text{Lebesgue measure})$$

- 'easy' direction of Borel-Cantelli : $\displaystyle\sum_q \psi(q) < \infty \quad \Rightarrow \quad \lambda(\mathcal{K}) = 0.$

Khinchin's theorem

$$\mathcal{K}_q := \bigcup_{0 \leqslant a \leqslant q} \left[\frac{a}{q} - \frac{\psi(q)}{q}, \frac{a}{q} + \frac{\psi(q)}{q}\right]$$

$$\mathcal{K} := \limsup_{q \to \infty} \mathcal{K}_q$$

$$= \{\alpha \in [0,1] : \alpha \in \mathcal{K}_q \text{ for infinitely many } q\}$$

Note that

$$\lambda(\mathcal{K}_q) \asymp \psi(q) \qquad (\lambda = \text{Lebesgue measure})$$

- 'easy' direction of Borel-Cantelli : $\displaystyle\sum_q \psi(q) < \infty \quad \Rightarrow \quad \lambda(\mathcal{K}) = 0.$

- Khinchin (1924) proved a partial converse:

$$q\psi(q) \searrow \quad \& \quad \sum_q \psi(q) = \infty \quad \Rightarrow \quad \lambda(\mathcal{K}) = 1.$$

The Duffin-Schaeffer conjecture

Study coprime solutions to $|\alpha - a/q| \leqslant \psi(q)/q$ to avoid over-counting:

The Duffin-Schaeffer conjecture

Study coprime solutions to $|\alpha - a/q| \leqslant \psi(q)/q$ to avoid over-counting:

$$\mathcal{A}_q := \bigcup_{\substack{1\leqslant a\leqslant q \\ \gcd(a,q)=1}} \Big[\frac{a}{q} - \frac{\psi(q)}{q}, \frac{a}{q} + \frac{\psi(q)}{q}\Big], \qquad \mathcal{A} = \limsup_{q\to\infty} \mathcal{A}_q$$

The Duffin-Schaeffer conjecture

Study coprime solutions to $|\alpha - a/q| \leqslant \psi(q)/q$ to avoid over-counting:

$$\mathcal{A}_q := \bigcup_{\substack{1 \leqslant a \leqslant q \\ \gcd(a,q)=1}} \Big[\frac{a}{q} - \frac{\psi(q)}{q}, \frac{a}{q} + \frac{\psi(q)}{q}\Big], \qquad \mathcal{A} = \limsup_{q\to\infty} \mathcal{A}_q$$

- Here $\lambda(\mathcal{A}_q) = \psi(q)\varphi(q)/q$, so the 'easy' Borel-Cantelli lemma yields:

$$\sum_q \frac{\psi(q)\varphi(q)}{q} < \infty \quad \Rightarrow \quad \lambda(\mathcal{A}) = 0$$

The Duffin-Schaeffer conjecture

Study coprime solutions to $|\alpha - a/q| \leqslant \psi(q)/q$ to avoid over-counting:

$$\mathcal{A}_q := \bigcup_{\substack{1 \leqslant a \leqslant q \\ \gcd(a,q)=1}} \Big[\frac{a}{q} - \frac{\psi(q)}{q}, \frac{a}{q} + \frac{\psi(q)}{q}\Big], \qquad \mathcal{A} = \limsup_{q\to\infty} \mathcal{A}_q$$

- Here $\lambda(\mathcal{A}_q) = \psi(q)\varphi(q)/q$, so the 'easy' Borel-Cantelli lemma yields:

$$\sum_q \frac{\psi(q)\varphi(q)}{q} < \infty \quad \Rightarrow \quad \lambda(\mathcal{A}) = 0$$

- Duffin and Schaeffer (1941) conjecture a strong converse is also true:

$$\sum_q \frac{\psi(q)\varphi(q)}{q} = \infty \quad \Rightarrow \quad \lambda(\mathcal{A}) = 1.$$

Results on DSC (Duffin-Schaeffer Conjecture)

Results on DSC (Duffin-Schaeffer Conjecture)

- Duffin-Schaeffer (1941): DSC is true when ψ is 'regular', i.e. when

$$\limsup_{Q\to\infty} \frac{\sum_{q\leqslant Q} \psi(q)\varphi(q)/q}{\sum_{q\leqslant Q} \psi(q)} > 0.$$

Results on DSC (Duffin-Schaeffer Conjecture)

- Duffin-Schaeffer (1941): DSC is true when ψ is 'regular', i.e. when

$$\limsup_{Q\to\infty} \frac{\sum_{q\leqslant Q}\psi(q)\varphi(q)/q}{\sum_{q\leqslant Q}\psi(q)} > 0.$$

- Erdős (1970) & Vaaler (1978) : DSC is true when $\psi(q) = O(1/q)$.

Results on DSC (Duffin-Schaeffer Conjecture)

- Duffin-Schaeffer (1941): DSC is true when ψ is 'regular', i.e. when

$$\limsup_{Q\to\infty} \frac{\sum_{q\leqslant Q}\psi(q)\varphi(q)/q}{\sum_{q\leqslant Q}\psi(q)} > 0.$$

- Erdős (1970) & Vaaler (1978) : DSC is true when $\psi(q) = O(1/q)$.
- Pollington-Vaughan (1990) : DSC is true in all dimensions > 1.

Results on DSC (Duffin-Schaeffer Conjecture)

- Duffin-Schaeffer (1941): DSC is true when ψ is 'regular', i.e. when

$$\limsup_{Q\to\infty} \frac{\sum_{q\leqslant Q} \psi(q)\varphi(q)/q}{\sum_{q\leqslant Q} \psi(q)} > 0.$$

- Erdős (1970) & Vaaler (1978) : DSC is true when $\psi(q) = O(1/q)$.
- Pollington-Vaughan (1990) : DSC is true in all dimensions > 1.
- DSC with 'extra divergence', i.e. when $\sum_q \frac{\psi(q)\varphi(q)}{qL(q)} = \infty$:

Results on DSC (Duffin-Schaeffer Conjecture)

- Duffin-Schaeffer (1941): DSC is true when ψ is 'regular', i.e. when

$$\limsup_{Q\to\infty} \frac{\sum_{q\leqslant Q} \psi(q)\varphi(q)/q}{\sum_{q\leqslant Q} \psi(q)} > 0.$$

- Erdős (1970) & Vaaler (1978) : DSC is true when $\psi(q) = O(1/q)$.
- Pollington-Vaughan (1990) : DSC is true in all dimensions > 1.
- DSC with 'extra divergence', i.e. when $\sum_q \frac{\psi(q)\varphi(q)}{qL(q)} = \infty$:

  Haynes-Pollington-Velani (2012) : $L(q) = (q/\psi(q))^{\varepsilon}$.

Results on DSC (Duffin-Schaeffer Conjecture)

- Duffin-Schaeffer (1941): DSC is true when ψ is 'regular', i.e. when

$$\limsup_{Q\to\infty} \frac{\sum_{q\leqslant Q} \psi(q)\varphi(q)/q}{\sum_{q\leqslant Q} \psi(q)} > 0.$$

- Erdős (1970) & Vaaler (1978) : DSC is true when $\psi(q) = O(1/q)$.
- Pollington-Vaughan (1990) : DSC is true in all dimensions > 1.
- DSC with 'extra divergence', i.e. when $\sum_q \frac{\psi(q)\varphi(q)}{qL(q)} = \infty$:

  Haynes-Pollington-Velani (2012) : $L(q) = (q/\psi(q))^{\varepsilon}$.

  Beresnevich-Harman-Haynes-Velani (2013) :
  $L(q) = \exp\{c(\log\log q)(\log\log\log q)\}$.

Results on DSC (Duffin-Schaeffer Conjecture)

- Duffin-Schaeffer (1941): DSC is true when ψ is 'regular', i.e. when

$$\limsup_{Q\to\infty} \frac{\sum_{q\leqslant Q} \psi(q)\varphi(q)/q}{\sum_{q\leqslant Q} \psi(q)} > 0.$$

- Erdős (1970) & Vaaler (1978) : DSC is true when $\psi(q) = O(1/q)$.
- Pollington-Vaughan (1990) : DSC is true in all dimensions > 1.
- DSC with 'extra divergence', i.e. when $\sum_q \frac{\psi(q)\varphi(q)}{qL(q)} = \infty$:

  Haynes-Pollington-Velani (2012) : $L(q) = (q/\psi(q))^{\varepsilon}$.

  Beresnevich-Harman-Haynes-Velani (2013) :
  $L(q) = \exp\{c(\log\log q)(\log\log\log q)\}$.

  Aistleitner-Lachmann-Munsch-Technau-Zafeiropoulos (2018 preprint) : $L(q) = (\log q)^{\varepsilon}$

Results on DSC (Duffin-Schaeffer Conjecture)

- Duffin-Schaeffer (1941): DSC is true when ψ is 'regular', i.e. when

$$\limsup_{Q\to\infty} \frac{\sum_{q\leqslant Q} \psi(q)\varphi(q)/q}{\sum_{q\leqslant Q} \psi(q)} > 0.$$

- Erdős (1970) & Vaaler (1978) : DSC is true when $\psi(q) = O(1/q)$.
- Pollington-Vaughan (1990) : DSC is true in all dimensions > 1.
- DSC with 'extra divergence', i.e. when $\sum_q \frac{\psi(q)\varphi(q)}{qL(q)} = \infty$:

  Haynes-Pollington-Velani (2012) : $L(q) = (q/\psi(q))^{\varepsilon}$.

  Beresnevich-Harman-Haynes-Velani (2013) :
  $L(q) = \exp\{c(\log\log q)(\log\log\log q)\}$.

  Aistleitner-Lachmann-Munsch-Technau-Zafeiropoulos (2018 preprint) : $L(q) = (\log q)^{\varepsilon}$

  Aistleitner (unpublished) : $L(q) = (\log\log q)^{\varepsilon}$.

Results on DSC (Duffin-Schaeffer Conjecture)

- Duffin-Schaeffer (1941): DSC is true when ψ is 'regular', i.e. when

$$\limsup_{Q\to\infty} \frac{\sum_{q\leqslant Q} \psi(q)\varphi(q)/q}{\sum_{q\leqslant Q}\psi(q)} > 0.$$

- Erdős (1970) & Vaaler (1978) : DSC is true when $\psi(q) = O(1/q)$.
- Pollington-Vaughan (1990) : DSC is true in all dimensions > 1.
- DSC with 'extra divergence', i.e. when $\sum_q \frac{\psi(q)\varphi(q)}{qL(q)} = \infty$:

  Haynes-Pollington-Velani (2012) : $L(q) = (q/\psi(q))^{\varepsilon}$.

  Beresnevich-Harman-Haynes-Velani (2013) : $L(q) = \exp\{c(\log\log q)(\log\log\log q)\}$.

  Aistleitner-Lachmann-Munsch-Technau-Zafeiropoulos (2018 preprint) : $L(q) = (\log q)^{\varepsilon}$

  Aistleitner (unpublished) : $L(q) = (\log\log q)^{\varepsilon}$.
- Aistleitner (2014) : DSC when ψ is not 'too concentrated', so that $\sum_{2^{2^j}<q\leqslant 2^{2^{j+1}}} \psi(q)\varphi(q)/q = O(1/j)$.

New results

Theorem (K.-Maynard (2019))

The Duffin-Schaeffer conjecture is true

New results

Theorem (K.-Maynard (2019))

The Duffin-Schaeffer conjecture is true

Corollary (Catlin's conjecture)

$\mathcal{K} := \{\alpha \in [0,1] : |\alpha - a/q| \leqslant \psi(q)/q$ *for infinitely many* $0 \leqslant a \leqslant q\}$

$S := \sum_q \varphi(q) \min_{m \geqslant 1}(\psi(qm)/qm)$

We then have $\lambda(\mathcal{K}) = 1$ *when* $S = \infty$, *whereas* $\lambda(\mathcal{K}) = 0$ *when* $S < \infty$.

New results

Theorem (K.-Maynard (2019))

The Duffin-Schaeffer conjecture is true

Corollary (Catlin's conjecture)

$\mathcal{K} := \{\alpha \in [0,1] : |\alpha - a/q| \leqslant \psi(q)/q$ *for infinitely many* $0 \leqslant a \leqslant q\}$

$S := \sum_q \varphi(q) \min_{m \geqslant 1}(\psi(qm)/qm)$

We then have $\lambda(\mathcal{K}) = 1$ *when* $S = \infty$, *whereas* $\lambda(\mathcal{K}) = 0$ *when* $S < \infty$.

Using a theorem of Beresnevich-Velani we also obtain:

Corollary

$\mathcal{A} := \{\alpha \in [0,1] : |\alpha - a/q| \leqslant \psi(q)/q$ *for inf. many* **coprime** $1 \leqslant a \leqslant q\}$

Assuming $0 \leqslant \psi \leqslant 1/2$, *let* $s = \inf\{\beta \geqslant 0 : \sum_q \varphi(q)(\psi(q)/q)^\beta < \infty\}$.
Then

$$\dim_{Hausdorff}(\mathcal{A}) = \min\{s, 1\}.$$

Inverting Borel-Cantelli

Set-up : $$\mathcal{A}_q = \bigcup_{\substack{1 \leqslant a \leqslant q \\ \gcd(a,q)=1}} \Big[\frac{a-\psi(q)}{q}, \frac{a+\psi(q)}{q}\Big], \qquad \mathcal{A} = \limsup_{q\to\infty} \mathcal{A}_q,$$

$$\lambda(\mathcal{A}_q) = \varphi(q)\psi(q)/q, \qquad \sum_q \lambda(\mathcal{A}_q) = \infty.$$

Inverting Borel-Cantelli

Set-up : $\mathcal{A}_q = \bigcup_{\substack{1\leqslant a\leqslant q \\ \gcd(a,q)=1}} \Big[\frac{a-\psi(q)}{q}, \frac{a+\psi(q)}{q}\Big],\qquad \mathcal{A} = \limsup_{q\to\infty} \mathcal{A}_q,$

$$\lambda(\mathcal{A}_q) = \varphi(q)\psi(q)/q, \qquad \sum_q \lambda(\mathcal{A}_q) = \infty.$$

Working heuristic: the sets $\mathcal{A}_q$ are quasi-independent events of the probability space $[0,1]$ and should thus have limited overlap if the sum of their measures is $\leqslant 1$.

Inverting Borel-Cantelli

Set-up : $\mathcal{A}_q = \bigcup_{\substack{1\leqslant a\leqslant q \\ \gcd(a,q)=1}} \left[\frac{a-\psi(q)}{q}, \frac{a+\psi(q)}{q}\right], \qquad \mathcal{A} = \limsup_{q\to\infty} \mathcal{A}_q,$

$$\lambda(\mathcal{A}_q) = \varphi(q)\psi(q)/q, \qquad \sum_q \lambda(\mathcal{A}_q) = \infty.$$

Working heuristic: the sets $\mathcal{A}_q$ are quasi-independent events of the probability space $[0,1]$ and should thus have limited overlap if the sum of their measures is $\leqslant 1$.

Goal : $\displaystyle\sum_{x\leqslant q\leqslant y} \lambda(\mathcal{A}_q) \asymp 1 \quad \Longrightarrow \quad \lambda(\bigcup_{x\leqslant q\leqslant y} \mathcal{A}_q) \asymp 1.$

Inverting Borel-Cantelli

Set-up : $\mathcal{A}_q = \bigcup_{\substack{1\leqslant a\leqslant q\\ \gcd(a,q)=1}} \Big[\frac{a-\psi(q)}{q}, \frac{a+\psi(q)}{q}\Big]$, $\mathcal{A} = \limsup_{q\to\infty} \mathcal{A}_q$,

$$\lambda(\mathcal{A}_q) = \varphi(q)\psi(q)/q, \qquad \sum_q \lambda(\mathcal{A}_q) = \infty.$$

Working heuristic: the sets $\mathcal{A}_q$ are quasi-independent events of the probability space $[0,1]$ and should thus have limited overlap if the sum of their measures is $\leqslant 1$.

Goal : $\sum_{x\leqslant q\leqslant y} \lambda(\mathcal{A}_q) \asymp 1 \quad \Longrightarrow \quad \lambda(\bigcup_{x\leqslant q\leqslant y} \mathcal{A}_q) \asymp 1.$

This is enough because it implies $\lambda(\mathcal{A}) > 0$ and we know that $\lambda(\mathcal{A}) \in \{0,1\}$ by Gallagher's 0-1 law.

Cauchy-Schwarz

Goal : $$\sum_{x\leqslant q\leqslant y} \lambda(\mathcal{A}_q) \asymp 1 \quad \Longrightarrow \quad \lambda\Big(\bigcup_{x\leqslant q\leqslant y} \mathcal{A}_q\Big) \asymp 1.$$

Cauchy-Schwarz

$$\textbf{Goal}: \quad \sum_{x \leqslant q \leqslant y} \lambda(\mathcal{A}_q) \asymp 1 \quad \Longrightarrow \quad \lambda\Big(\bigcup_{x \leqslant q \leqslant y} \mathcal{A}_q\Big) \asymp 1.$$

Cauchy-Scwarz $\rightsquigarrow$ enough to show $\displaystyle\sum_{x \leqslant q, r \leqslant y} \lambda(\mathcal{A}_q \cap \mathcal{A}_r) \ll 1.$

Cauchy-Schwarz

Goal : $$\sum_{x\leqslant q\leqslant y} \lambda(\mathcal{A}_q) \asymp 1 \quad \Longrightarrow \quad \lambda(\bigcup_{x\leqslant q\leqslant y} \mathcal{A}_q) \asymp 1.$$

Cauchy-Scwarz $\rightsquigarrow$ enough to show $$\sum_{x\leqslant q,r\leqslant y} \lambda(\mathcal{A}_q \cap \mathcal{A}_r) \ll 1.$$

Simplifying assumptions:

Cauchy-Schwarz

$$\textbf{Goal}: \quad \sum_{x \leqslant q \leqslant y} \lambda(\mathcal{A}_q) \asymp 1 \quad \Longrightarrow \quad \lambda\Big(\bigcup_{x \leqslant q \leqslant y} \mathcal{A}_q\Big) \asymp 1.$$

Cauchy-Scwarz $\rightsquigarrow$ enough to show $\displaystyle\sum_{x \leqslant q, r \leqslant y} \lambda(\mathcal{A}_q \cap \mathcal{A}_r) \ll 1.$

Simplifying assumptions:

- $\mathrm{supp}(\psi) \subset \{\text{square-free integers}\}$;

Cauchy-Schwarz

Goal : $$\sum_{x\leqslant q\leqslant y} \lambda(\mathcal{A}_q) \asymp 1 \quad \Longrightarrow \quad \lambda\Big(\bigcup_{x\leqslant q\leqslant y} \mathcal{A}_q\Big) \asymp 1.$$

Cauchy-Scwarz $\rightsquigarrow$ enough to show $$\sum_{x\leqslant q,r\leqslant y} \lambda(\mathcal{A}_q \cap \mathcal{A}_r) \ll 1.$$

Simplifying assumptions:

- supp$(\psi) \subset \{$square-free integers$\}$;
- $\psi(q) \in \{0, q^{-c}\}$ for some $c \in (0, 1]$ ($c = 1$ is Erdős-Vaaler);

Cauchy-Schwarz

Goal : $$\sum_{x\leqslant q\leqslant y} \lambda(\mathcal{A}_q) \asymp 1 \quad \Longrightarrow \quad \lambda\Big(\bigcup_{x\leqslant q\leqslant y} \mathcal{A}_q\Big) \asymp 1.$$

Cauchy-Scwarz $\rightsquigarrow$ enough to show $$\sum_{x\leqslant q,r\leqslant y} \lambda(\mathcal{A}_q \cap \mathcal{A}_r) \ll 1.$$

Simplifying assumptions:

- supp$(\psi) \subset \{$square-free integers$\}$;
- $\psi(q) \in \{0, q^{-c}\}$ for some $c \in (0, 1]$ ($c = 1$ is Erdős-Vaaler);
- there is a *sparse* infinite set of x s.t. $\sum_{x\leqslant q\leqslant 2x} \psi(q)\varphi(q)/q \asymp 1$

Cauchy-Schwarz

$$\textbf{Goal}: \quad \sum_{x\leqslant q\leqslant y} \lambda(\mathcal{A}_q) \asymp 1 \quad \Longrightarrow \quad \lambda\Big(\bigcup_{x\leqslant q\leqslant y} \mathcal{A}_q\Big) \asymp 1.$$

Cauchy-Scwarz $\rightsquigarrow$ enough to show $\displaystyle\sum_{x\leqslant q,r\leqslant y} \lambda(\mathcal{A}_q \cap \mathcal{A}_r) \ll 1.$

Simplifying assumptions:

- $\mathrm{supp}(\psi) \subset \{$square-free integers$\}$;
- $\psi(q) \in \{0, q^{-c}\}$ for some $c \in (0,1]$ $\quad$ ($c = 1$ is Erdős-Vaaler);
- there is a *sparse* infinite set of x s.t. $\sum_{x\leqslant q\leqslant 2x} \psi(q)\varphi(q)/q \asymp 1$

  $\rightsquigarrow \quad \sum_{q\in\mathcal{S}} \varphi(q)/q \asymp x^c, \quad$ where $\quad \mathcal{S} := [x, 2x] \cap \mathrm{supp}(\psi)$

Cauchy-Schwarz

Goal : $$\sum_{x\leqslant q\leqslant y} \lambda(\mathcal{A}_q) \asymp 1 \quad \Longrightarrow \quad \lambda\Big(\bigcup_{x\leqslant q\leqslant y} \mathcal{A}_q\Big) \asymp 1.$$

Cauchy-Scwarz $\rightsquigarrow$ enough to show $$\sum_{x\leqslant q,r\leqslant y} \lambda(\mathcal{A}_q \cap \mathcal{A}_r) \ll 1.$$

Simplifying assumptions:

- $\mathrm{supp}(\psi) \subset \{\text{square-free integers}\}$;
- $\psi(q) \in \{0, q^{-c}\}$ for some $c \in (0, 1]$ $\quad$ ($c = 1$ is Erdős-Vaaler);
- there is a *sparse* infinite set of x s.t. $\sum_{x\leqslant q\leqslant 2x} \psi(q)\varphi(q)/q \asymp 1$

$\rightsquigarrow \quad \sum_{q\in\mathcal{S}} \varphi(q)/q \asymp x^c,$ $\quad$ where $\quad \mathcal{S} := [x, 2x] \cap \mathrm{supp}(\psi)$

Pollington-Vaughan: for $q, r \in \mathcal{S}$, we have

$$\frac{\lambda(\mathcal{A}_q \cap \mathcal{A}_r)}{\lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r)} \ll 1 + \mathbf{1}_{\gcd(q,r)\leqslant x^{1-c}} \prod_{\substack{p \mid \frac{\mathrm{lcm}[q,r]}{\gcd(q,r)} \\ p > x^{1-c}/\gcd(q,r)}} \Big(1 + \frac{1}{p}\Big).$$

Revised goal: if $\displaystyle\sum_{q\in\mathcal{S}} \frac{\varphi(q)}{q} \asymp x^c, \quad \mathcal{S} \subset \{x \leqslant q \leqslant 2x : q \text{ square-free}\},$

show that
$$\sum_{\substack{q,r\in\mathcal{S} \\ \gcd(q,r)\leqslant x^{1-c}}} \frac{\varphi(q)}{q} \cdot \frac{\varphi(r)}{r} \prod_{\substack{p \mid \frac{\operatorname{lcm}[q,r]}{\gcd(q,r)} \\ p > x^{1-c}/\gcd(q,r)}} \left(1 + \frac{1}{p}\right) \ll x^{2c}.$$

Revised goal: if $\displaystyle\sum_{q\in\mathcal{S}} \frac{\varphi(q)}{q} \asymp x^c, \quad \mathcal{S} \subset \{x \leqslant q \leqslant 2x : q \text{ square-free}\},$

show that
$$\sum_{\substack{q,r\in\mathcal{S} \\ \gcd(q,r)\leqslant x^{1-c}}} \frac{\varphi(q)}{q} \cdot \frac{\varphi(r)}{r} \prod_{\substack{p \mid \frac{\operatorname{lcm}[q,r]}{\gcd(q,r)} \\ p > x^{1-c}/\gcd(q,r)}} \Big(1 + \frac{1}{p}\Big) \ll x^{2c}.$$

Divide range according to largest t such that
$$L_t(q,r) := \sum_{\substack{p\mid qr,\ p\nmid \gcd(q,r) \\ p>t}} \frac{1}{p} \geqslant 100.$$

Revised goal: if $\displaystyle\sum_{q\in\mathcal{S}} \frac{\varphi(q)}{q} \asymp x^c, \quad \mathcal{S} \subset \{x \leqslant q \leqslant 2x : q \text{ square-free}\},$

show that $$\sum_{\substack{q,r\in\mathcal{S} \\ \gcd(q,r)\leqslant x^{1-c}}} \frac{\varphi(q)}{q}\cdot\frac{\varphi(r)}{r} \prod_{\substack{p \mid \frac{\operatorname{lcm}[q,r]}{\gcd(q,r)} \\ p > x^{1-c}/\gcd(q,r)}} \Big(1+\frac{1}{p}\Big) \ll x^{2c}.$$

Divide range according to largest t such that

$$L_t(q,r) := \sum_{\substack{p\mid qr,\ p\nmid\gcd(q,r) \\ p>t}} \frac{1}{p} \geqslant 100.$$

Re-revised goal: assuming that $\sum_{q\in\mathcal{S}} \varphi(q)/q \asymp x^c$, show that

$$\sum_{\substack{q,r\in\mathcal{S},\ L_t(q,r)\geqslant 100 \\ x^{1-c}/t\leqslant\gcd(q,r)\leqslant x^{1-c}}} \frac{\varphi(q)}{q}\cdot\frac{\varphi(r)}{r} \ll \frac{x^{2c}}{t}.$$

Two conditions

$$\sum_{q\in\mathcal{S}} \frac{\varphi(q)}{q} \asymp x^c \quad \overset{?}{\Longrightarrow} \quad \sum_{\substack{q,r\in\mathcal{S} \\ \gcd(q,r)\geqslant x^{1-c}/t \\ L_t(q,r)\geqslant 100}} \frac{\varphi(q)}{q}\cdot\frac{\varphi(r)}{r} \ll \frac{x^{2c}}{t}$$

where $L_t(q,r) = \sum_{p|qr,\, p\nmid\gcd(q,r)} \frac{1_{p>t}}{p}$.

Two conditions

$$\sum_{q \in \mathcal{S}} \frac{\varphi(q)}{q} \asymp x^c \quad \stackrel{?}{\Longrightarrow} \quad \sum_{\substack{q,r \in \mathcal{S} \\ \gcd(q,r) \geqslant x^{1-c}/t \\ L_t(q,r) \geqslant 100}} \frac{\varphi(q)}{q} \cdot \frac{\varphi(r)}{r} \ll \frac{x^{2c}}{t}$$

where $L_t(q,r) = \sum_{p \mid qr,\, p \nmid \gcd(q,r)} \frac{1_{p>t}}{p}$.

(1) $\gcd(q,r) \geqslant x^{1-c}/t$ is a **structural condition**. The heart of the proof is understanding how often it occurs.

Two conditions

$$\sum_{q\in\mathcal{S}} \frac{\varphi(q)}{q} \asymp x^c \quad \overset{?}{\Longrightarrow} \quad \sum_{\substack{q,r\in\mathcal{S} \\ \gcd(q,r)\geqslant x^{1-c}/t \\ L_t(q,r)\geqslant 100}} \frac{\varphi(q)}{q}\cdot\frac{\varphi(r)}{r} \ll \frac{x^{2c}}{t}$$

where $L_t(q,r) = \sum_{p|qr,\, p\nmid \gcd(q,r)} \frac{1_{p>t}}{p}$.

(1) $\gcd(q,r) \geqslant x^{1-c}/t$ is a **structural condition.** The heart of the proof is understanding how often it occurs.

(2) $L_t(q,r) \geqslant 100$ is an **anatomical condition** and is easily analyzed:

$$\#\{x \leqslant q, r \leqslant 2x : L_t(q,r) \geqslant 100\} \ll x^2 e^{-t}$$

Two conditions

$$\sum_{q\in\mathcal{S}} \frac{\varphi(q)}{q} \asymp x^c \quad \overset{?}{\Longrightarrow} \quad \sum_{\substack{q,r\in\mathcal{S} \\ \gcd(q,r)\geqslant x^{1-c}/t \\ L_t(q,r)\geqslant 100}} \frac{\varphi(q)}{q}\cdot\frac{\varphi(r)}{r} \ll \frac{x^{2c}}{t}$$

where $L_t(q,r) = \sum_{p|qr,\, p\nmid\gcd(q,r)} \frac{1_{p>t}}{p}$.

(1) $\gcd(q,r) \geqslant x^{1-c}/t$ is a **structural condition.** The heart of the proof is understanding how often it occurs.

(2) $L_t(q,r) \geqslant 100$ is an **anatomical condition** and is easily analyzed:

$$\#\{x \leqslant q, r \leqslant 2x : L_t(q,r) \geqslant 100\} \ll x^2 e^{-t}$$

When $c = 1$, condition (1) is vacuous and we can complete the proof:

$$\sum_{\substack{q,r\in\mathcal{S} \\ L_t(q,r)\geqslant 100}} \frac{\varphi(q)}{q}\cdot\frac{\varphi(r)}{r} \ll \frac{x^2}{e^t} \leqslant \frac{x^2}{t}$$

Two conditions

$$\sum_{q\in\mathcal{S}} \frac{\varphi(q)}{q} \asymp x^c \quad \overset{?}{\Longrightarrow} \quad \sum_{\substack{q,r\in\mathcal{S} \\ \gcd(q,r)\geqslant x^{1-c}/t \\ L_t(q,r)\geqslant 100}} \frac{\varphi(q)}{q}\cdot\frac{\varphi(r)}{r} \ll \frac{x^{2c}}{t}$$

where $L_t(q,r) = \sum_{p|qr,\, p\nmid\gcd(q,r)} \frac{1_{p>t}}{p}$.

(1) $\gcd(q,r) \geqslant x^{1-c}/t$ is a **structural condition**. The heart of the proof is understanding how often it occurs.

(2) $L_t(q,r) \geqslant 100$ is an **anatomical condition** and is easily analyzed:

$$\#\{x \leqslant q, r \leqslant 2x : L_t(q,r) \geqslant 100\} \ll x^2 e^{-t}$$

When $c = 1$, condition (1) is vacuous and we can complete the proof:

$$\sum_{\substack{q,r\in\mathcal{S} \\ L_t(q,r)\geqslant 100}} \frac{\varphi(q)}{q}\cdot\frac{\varphi(r)}{r} \ll \frac{x^2}{e^t} \leqslant \frac{x^2}{t} \qquad \text{(Erdős-Vaaler)}$$

Analysis of the structural condition $\gcd(q, r) \geqslant x^{1-c}/t$

Analysis of the structural condition $\gcd(q,r) \geqslant x^{1-c}/t$

$$\sum_{\substack{x \leqslant q \leqslant 2x \\ \gcd(q,r) \geqslant x^{1-c}/t}} 1 \leqslant \sum_{\substack{d \mid r \\ d \geqslant x^{1-c}/t}} \sum_{\substack{x \leqslant q \leqslant 2x \\ d \mid q}} 1$$

Analysis of the structural condition $\gcd(q, r) \geqslant x^{1-c}/t$

$$\sum_{\substack{x \leqslant q \leqslant 2x \\ \gcd(q,r) \geqslant x^{1-c}/t}} 1 \leqslant \sum_{\substack{d \mid r \\ d \geqslant x^{1-c}/t}} \sum_{\substack{x \leqslant q \leqslant 2x \\ d \mid q}} 1$$

$$\ll \sum_{\substack{d \mid r \\ d \geqslant x^{1-c}/t}} \frac{x}{d}$$

Analysis of the structural condition $\gcd(q, r) \geqslant x^{1-c}/t$

$$\sum_{\substack{x \leqslant q \leqslant 2x \\ \gcd(q,r) \geqslant x^{1-c}/t}} 1 \leqslant \sum_{\substack{d \mid r \\ d \geqslant x^{1-c}/t}} \sum_{\substack{x \leqslant q \leqslant 2x \\ d \mid q}} 1$$

$$\ll \sum_{\substack{d \mid r \\ d \geqslant x^{1-c}/t}} \frac{x}{d}$$

$$\ll t x^{c} \cdot \#\{d \mid r\}$$

Analysis of the structural condition $\gcd(q, r) \geqslant x^{1-c}/t$

$$\sum_{\substack{x \leqslant q \leqslant 2x \\ \gcd(q,r) \geqslant x^{1-c}/t}} 1 \leqslant \sum_{\substack{d \mid r \\ d \geqslant x^{1-c}/t}} \sum_{\substack{x \leqslant q \leqslant 2x \\ d \mid q}} 1$$

$$\ll \sum_{\substack{d \mid r \\ d \geqslant x^{1-c}/t}} \frac{x}{d}$$

$$\ll t x^{c} \cdot \#\{d \mid r\}$$

$$\rightsquigarrow \qquad \sum_{\substack{q,r \in \mathcal{S} \\ \gcd(q,r) \geqslant x^{1-c}/t \\ L_t(q,r) \geqslant 100}} \frac{\varphi(q)}{q} \cdot \frac{\varphi(r)}{r} \ll t x^{2c+o(1)} = t^2 \cdot x^{o(1)} \cdot \frac{x^{2c}}{t}$$

Analysis of the structural condition $\gcd(q,r) \geqslant x^{1-c}/t$

$$\sum_{\substack{x \leqslant q \leqslant 2x \\ \gcd(q,r) \geqslant x^{1-c}/t}} 1 \leqslant \sum_{\substack{d \mid r \\ d \geqslant x^{1-c}/t}} \sum_{\substack{x \leqslant q \leqslant 2x \\ d \mid q}} 1$$
$$\ll \sum_{\substack{d \mid r \\ d \geqslant x^{1-c}/t}} \frac{x}{d}$$
$$\ll t x^{c} \cdot \#\{d \mid r\}$$

$$\rightsquigarrow \qquad \sum_{\substack{q,r \in \mathcal{S} \\ \gcd(q,r) \geqslant x^{1-c}/t \\ L_t(q,r) \geqslant 100}} \frac{\varphi(q)}{q} \cdot \frac{\varphi(r)}{r} \ll t x^{2c+o(1)} = t^2 \cdot x^{o(1)} \cdot \frac{x^{2c}}{t}$$

- Hope to remove factor t^2 by exploiting the anatomical condition $L_t(q,r) \geqslant 100$.

Analysis of the structural condition $\gcd(q,r) \geqslant x^{1-c}/t$

$$\sum_{\substack{x \leqslant q \leqslant 2x \\ \gcd(q,r) \geqslant x^{1-c}/t}} 1 \leqslant \sum_{\substack{d \mid r \\ d \geqslant x^{1-c}/t}} \sum_{\substack{x \leqslant q \leqslant 2x \\ d \mid q}} 1$$

$$\ll \sum_{\substack{d \mid r \\ d \geqslant x^{1-c}/t}} \frac{x}{d}$$

$$\ll t x^{c} \cdot \#\{d \mid r\}$$

$$\rightsquigarrow \qquad \sum_{\substack{q,r \in \mathcal{S} \\ \gcd(q,r) \geqslant x^{1-c}/t \\ L_t(q,r) \geqslant 100}} \frac{\varphi(q)}{q} \cdot \frac{\varphi(r)}{r} \ll t x^{2c+o(1)} = t^2 \cdot x^{o(1)} \cdot \frac{x^{2c}}{t}$$

- Hope to remove factor t^2 by exploiting the anatomical condition $L_t(q,r) \geqslant 100$.
- But: how to remove the factor $x^{o(1)}$?

A guiding model problem

Recall: $\mathcal{S} \subset [x, 2x]$ and $\sum_{q \in \mathcal{S}} \varphi(q)/q \asymp x^c$.

A guiding model problem

Recall: $\mathcal{S} \subset [x, 2x]$ and $\sum_{q \in \mathcal{S}} \varphi(q)/q \asymp x^c$.

For simplicity, ignore the arithmetic weights $\varphi(q)/q$.

A guiding model problem

Recall: $\mathcal{S} \subset [x, 2x]$ and $\sum_{q\in\mathcal{S}} \varphi(q)/q \asymp x^c$.

For simplicity, ignore the arithmetic weights $\varphi(q)/q$. This leads to:

Question

Let $\mathcal{S} \subset [x, 2x]$ satisfy $|\mathcal{S}| \asymp x^c$ and be such that there are $\geqslant |\mathcal{S}|^2/t$ pairs $(q, r) \in \mathcal{S}^2$ with $\gcd(q, r) \geqslant x^{1-c}/t$. Must it be the case that there is an integer $d \geqslant x^{1-c}/t$ that divides $\gg |\mathcal{S}|t^{-O(1)}$ elements of $\mathcal{S}$?

A guiding model problem

Recall: $\mathcal{S} \subset [x, 2x]$ and $\sum_{q \in \mathcal{S}} \varphi(q)/q \asymp x^c$.

For simplicity, ignore the arithmetic weights $\varphi(q)/q$. This leads to:

Question

Let $\mathcal{S} \subset [x, 2x]$ satisfy $|\mathcal{S}| \asymp x^c$ and be such that there are $\geqslant |\mathcal{S}|^2/t$ pairs $(q, r) \in \mathcal{S}^2$ with $\gcd(q, r) \geqslant x^{1-c}/t$. Must it be the case that there is an integer $d \geqslant x^{1-c}/t$ that divides $\gg |\mathcal{S}| t^{-O(1)}$ elements of $\mathcal{S}$?

If yes, we are done: we may replace the factor $x^{o(1)}$ with $t^{O(1)}$. We may then kill this new factor using the anatomical condition $L_t(q, r) \geqslant 100$.

Compressing GCD graphs

- $G = (\mathcal{V}, \mathcal{W}, \mathcal{E})$ bipartite graph;
- $\mathcal{V}, \mathcal{W} \subset \mathcal{S}$;
- $\mathcal{E} \subset \{(v, w) \in \mathcal{V} \times \mathcal{W} : \gcd(v, w) \geqslant x^{1-c}/t,\ L_t(v, w) \geqslant 100\}$;
- vertex v weighted with $\mu(v) = \varphi(v)/v$;
- edge (v, w) weighted with $\mu(v)\mu(w)$.

Compressing GCD graphs

- $G = (\mathcal{V}, \mathcal{W}, \mathcal{E})$ bipartite graph;
- $\mathcal{V}, \mathcal{W} \subset \mathcal{S}$;
- $\mathcal{E} \subset \{(v, w) \in \mathcal{V} \times \mathcal{W} : \gcd(v, w) \geqslant x^{1-c}/t,\ L_t(v, w) \geqslant 100\}$;
- vertex v weighted with $\mu(v) = \varphi(v)/v$;
- edge (v, w) weighted with $\mu(v)\mu(w)$.

Goal: start with $G^{\text{start}} = (\mathcal{V}^{\text{start}}, \mathcal{W}^{\text{start}}, \mathcal{E}^{\text{start}})$ where $\mathcal{V}^{\text{start}} = \mathcal{W}^{\text{start}} = \mathcal{S}$ and $\mathcal{E}^{\text{start}} = \{(v, w) \in \mathcal{S}^2 : \gcd(v, w) \geqslant x^{1-c}/t,\ L_t(v, w) \geqslant 100\}$.

Compressing GCD graphs

- $G = (\mathcal{V}, \mathcal{W}, \mathcal{E})$ bipartite graph;
- $\mathcal{V}, \mathcal{W} \subset \mathcal{S}$;
- $\mathcal{E} \subset \{(v, w) \in \mathcal{V} \times \mathcal{W} : \gcd(v, w) \geqslant x^{1-c}/t,\ L_t(v, w) \geqslant 100\}$;
- vertex v weighted with $\mu(v) = \varphi(v)/v$;
- edge (v, w) weighted with $\mu(v)\mu(w)$.

Goal: start with $G^{\text{start}} = (\mathcal{V}^{\text{start}}, \mathcal{W}^{\text{start}}, \mathcal{E}^{\text{start}})$ where $\mathcal{V}^{\text{start}} = \mathcal{W}^{\text{start}} = \mathcal{S}$ and $\mathcal{E}^{\text{start}} = \{(v, w) \in \mathcal{S}^2 : \gcd(v, w) \geqslant x^{1-c}/t,\ L_t(v, w) \geqslant 100\}$.

Arrive at $G^{\text{end}} = (\mathcal{V}^{\text{end}}, \mathcal{W}^{\text{end}}, \mathcal{E}^{\text{end}})$ where there are $a, b \in \mathbb{N}$ s.t.

- all vertices in $\mathcal{V}^{\text{end}}$ are multiples of a;
- all vertices in $\mathcal{W}^{\text{end}}$ are multiples of b;
- all edges in $\mathcal{E}^{\text{end}}$ have $\gcd(v, w) = \gcd(a, b)$.

Compressing GCD graphs

- $G = (\mathcal{V}, \mathcal{W}, \mathcal{E})$ bipartite graph;
- $\mathcal{V}, \mathcal{W} \subset \mathcal{S}$;
- $\mathcal{E} \subset \{(v, w) \in \mathcal{V} \times \mathcal{W} : \gcd(v, w) \geqslant x^{1-c}/t,\ L_t(v, w) \geqslant 100\}$;
- vertex v weighted with $\mu(v) = \varphi(v)/v$;
- edge (v, w) weighted with $\mu(v)\mu(w)$.

Goal: start with $G^{\text{start}} = (\mathcal{V}^{\text{start}}, \mathcal{W}^{\text{start}}, \mathcal{E}^{\text{start}})$ where $\mathcal{V}^{\text{start}} = \mathcal{W}^{\text{start}} = \mathcal{S}$ and $\mathcal{E}^{\text{start}} = \{(v, w) \in \mathcal{S}^2 : \gcd(v, w) \geqslant x^{1-c}/t,\ L_t(v, w) \geqslant 100\}$.

Arrive at $G^{\text{end}} = (\mathcal{V}^{\text{end}}, \mathcal{W}^{\text{end}}, \mathcal{E}^{\text{end}})$ where there are $a, b \in \mathbb{N}$ s.t.

- all vertices in $\mathcal{V}^{\text{end}}$ are multiples of a;
- all vertices in $\mathcal{W}^{\text{end}}$ are multiples of b;
- all edges in $\mathcal{E}^{\text{end}}$ have $\gcd(v, w) = \gcd(a, b)$.

Important requirement: the size of $\mathcal{E}^{\text{start}}$ must be somehow controlled by the size of $\mathcal{E}^{\text{end}}$.

Variations of density-increment arguments

First attempt: consider weighted edge density

$$\delta(G) = \frac{\mu(\mathcal{E})}{\mu(\mathcal{V})\mu(\mathcal{W})}.$$

Variations of density-increment arguments

First attempt: consider weighted edge density

$$\delta(G) = \frac{\mu(\mathcal{E})}{\mu(\mathcal{V})\mu(\mathcal{W})}.$$

Classical density-increment arguments due to Roth, Szemerédi, etc.

Variations of density-increment arguments

First attempt: consider weighted edge density

$$\delta(G) = \frac{\mu(\mathcal{E})}{\mu(\mathcal{V})\mu(\mathcal{W})}.$$

Classical density-increment arguments due to Roth, Szemerédi, etc.

Hard to use here: δ loses control of the size of the vertex sets and thus it is very hard to exploit the anatomical condition $L_t(v, w) \geqslant 100$.

Second attempt: reverse engineer, starting from ‘end graph’**.**

Second attempt: reverse engineer, starting from 'end graph'.

We have $\gcd(a, b) = \gcd(v, w) \geqslant x^{1-c}/t$ and

$$\mu(\mathcal{V}^{\text{end}})\mu(\mathcal{W}^{\text{end}}) \ll \frac{x}{a} \cdot \frac{x}{b} \leqslant t^2 x^{2c} \cdot \frac{\gcd(a, b)^2}{ab}$$

Second attempt: reverse engineer, starting from 'end graph'.

We have $\gcd(a,b) = \gcd(v,w) \geqslant x^{1-c}/t$ and

$$\mu(\mathcal{V}^{\text{end}})\mu(\mathcal{W}^{\text{end}}) \ll \frac{x}{a} \cdot \frac{x}{b} \leqslant t^2 x^{2c} \cdot \frac{\gcd(a,b)^2}{ab}$$

So we could try to increase

$$\tilde{q}(G) := \frac{a_G b_G}{\gcd(a_G, b_G)^2} \cdot \mu(\mathcal{V}) \cdot \mu(\mathcal{W}),$$

where a_G divides everything in $\mathcal{V}$ and b_G everything in $\mathcal{W}$.

Second attempt: reverse engineer, starting from 'end graph'.

We have $\gcd(a,b) = \gcd(v,w) \geqslant x^{1-c}/t$ and

$$\mu(\mathcal{V}^{\text{end}})\mu(\mathcal{W}^{\text{end}}) \ll \frac{x}{a} \cdot \frac{x}{b} \leqslant t^2 x^{2c} \cdot \frac{\gcd(a,b)^2}{ab}$$

So we could try to increase

$$\tilde{q}(G) := \frac{a_G b_G}{\gcd(a_G, b_G)^2} \cdot \mu(\mathcal{V}) \cdot \mu(\mathcal{W}),$$

where a_G divides everything in $\mathcal{V}$ and b_G everything in $\mathcal{W}$.

$$\rightsquigarrow \qquad \mu(\mathcal{E}^{\text{start}}) = \frac{\tilde{q}(G^{\text{start}})}{\delta(G^{\text{start}})} \leqslant \frac{\tilde{q}(G^{\text{end}})}{\delta(G^{\text{start}})} \ll \frac{t^2}{\delta(G^{\text{start}})}$$

Second attempt: reverse engineer, starting from 'end graph'.

We have $\gcd(a, b) = \gcd(v, w) \geqslant x^{1-c}/t$ and

$$\mu(\mathcal{V}^{\text{end}})\mu(\mathcal{W}^{\text{end}}) \ll \frac{x}{a} \cdot \frac{x}{b} \leqslant t^2 x^{2c} \cdot \frac{\gcd(a, b)^2}{ab}$$

So we could try to increase

$$\tilde{q}(G) := \frac{a_G b_G}{\gcd(a_G, b_G)^2} \cdot \mu(\mathcal{V}) \cdot \mu(\mathcal{W}),$$

where a_G divides everything in $\mathcal{V}$ and b_G everything in $\mathcal{W}$.

$$\rightsquigarrow \qquad \mu(\mathcal{E}^{\text{start}}) = \frac{\tilde{q}(G^{\text{start}})}{\delta(G^{\text{start}})} \leqslant \frac{\tilde{q}(G^{\text{end}})}{\delta(G^{\text{start}})} \ll \frac{t^2}{\delta(G^{\text{start}})}$$

Can assume $\delta(G^{\text{start}}) \gg 1/t$; factor t^3 can be killed using anatomy.

Second attempt: reverse engineer, starting from ‘end graph’.

We have $\gcd(a, b) = \gcd(v, w) \geqslant x^{1-c}/t$ and

$$\mu(\mathcal{V}^{\text{end}})\mu(\mathcal{W}^{\text{end}}) \ll \frac{x}{a} \cdot \frac{x}{b} \leqslant t^2 x^{2c} \cdot \frac{\gcd(a, b)^2}{ab}$$

So we could try to increase

$$\tilde{q}(G) := \frac{a_G b_G}{\gcd(a_G, b_G)^2} \cdot \mu(\mathcal{V}) \cdot \mu(\mathcal{W}),$$

where a_G divides everything in $\mathcal{V}$ and b_G everything in $\mathcal{W}$.

$$\rightsquigarrow \qquad \mu(\mathcal{E}^{\text{start}}) = \frac{\tilde{q}(G^{\text{start}})}{\delta(G^{\text{start}})} \leqslant \frac{\tilde{q}(G^{\text{end}})}{\delta(G^{\text{start}})} \ll \frac{t^2}{\delta(G^{\text{start}})}$$

Can assume $\delta(G^{\text{start}}) \gg 1/t$;   factor t^3 can be killed using anatomy.

Problem: hard to increase $\tilde{q}$.

Third attempt: consider a hybrid.

Third attempt: consider a hybrid.

The quality of the GCD graph G is defined by

$$q(G) := \delta(G)^{10} \cdot \frac{a_G b_G}{\gcd(a_G, b_G)^2} \cdot \mu(\mathcal{V}) \cdot \mu(\mathcal{W}).$$

Third attempt: consider a hybrid.

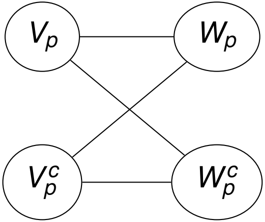
The quality of the GCD graph G is defined by

$$q(G) := \delta(G)^{10} \cdot \frac{a_G b_G}{\gcd(a_G, b_G)^2} \cdot \mu(\mathcal{V}) \cdot \mu(\mathcal{W}).$$

Quality increment can be made to work AND we have control on vertex sets

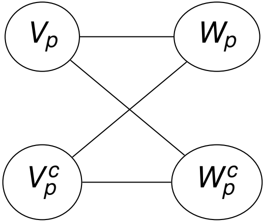
A very rough sketch of the quality-increment argument

$V_p = \{v \in \mathcal{V} : p|v\}, \quad \mathcal{V}_p^c = \{v \in \mathcal{V} : p \nmid v\}$ (square-free integers)



A very rough sketch of the quality-increment argument

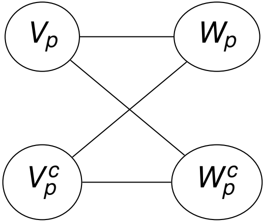
$V_p = \{v \in \mathcal{V} : p | v\}, \quad \mathcal{V}_p^c = \{v \in \mathcal{V} : p \nmid v\}$ (square-free integers)



Goal: focus on one of the four graphs induced by the pairs of vertex sets $(\mathcal{V}_p, \mathcal{W}_p)$, $(\mathcal{V}_p, \mathcal{W}_p^c)$, $(\mathcal{V}_p^c, \mathcal{W}_p)$, $(\mathcal{V}_p^c, \mathcal{W}_p^c)$.

A very rough sketch of the quality-increment argument

$V_p = \{v \in \mathcal{V} : p | v\}, \quad \mathcal{V}_p^c = \{v \in \mathcal{V} : p \nmid v\}$ (square-free integers)

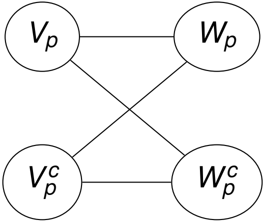


Goal: focus on one of the four graphs induced by the pairs of vertex sets $(\mathcal{V}_p, \mathcal{W}_p)$, $(\mathcal{V}_p, \mathcal{W}_p^c)$, $(\mathcal{V}_p^c, \mathcal{W}_p)$, $(\mathcal{V}_p^c, \mathcal{W}_p^c)$.

In $(\mathcal{V}_p^c, \mathcal{W}_p)$ and in $(\mathcal{V}_p, \mathcal{W}_p^c)$ we gain factor p in quality.

A very rough sketch of the quality-increment argument

$V_p = \{v \in \mathcal{V} : p | v\}, \quad \mathcal{V}_p^c = \{v \in \mathcal{V} : p \nmid v\}$ (square-free integers)



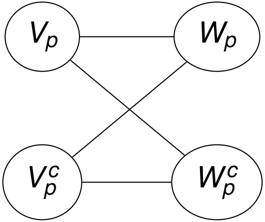
Goal: focus on one of the four graphs induced by the pairs of vertex sets $(\mathcal{V}_p, \mathcal{W}_p)$, $(\mathcal{V}_p, \mathcal{W}_p^c)$, $(\mathcal{V}_p^c, \mathcal{W}_p)$, $(\mathcal{V}_p^c, \mathcal{W}_p^c)$.

In $(\mathcal{V}_p^c, \mathcal{W}_p)$ and in $(\mathcal{V}_p, \mathcal{W}_p^c)$ we gain factor p in quality.

Hard case when $|\mathcal{V}_p|, |\mathcal{W}_p| \sim 1 - O(1/p)$, or when $|\mathcal{V}_p^c|, |\mathcal{W}_p^c| = 1 - O(1/p)$.

A very rough sketch of the quality-increment argument

$V_p = \{v \in \mathcal{V} : p | v\}$, $\mathcal{V}_p^c = \{v \in \mathcal{V} : p \nmid v\}$ (square-free integers)



Goal: focus on one of the four graphs induced by the pairs of vertex sets $(\mathcal{V}_p, \mathcal{W}_p)$, $(\mathcal{V}_p, \mathcal{W}_p^c)$, $(\mathcal{V}_p^c, \mathcal{W}_p)$, $(\mathcal{V}_p^c, \mathcal{W}_p^c)$.

In $(\mathcal{V}_p^c, \mathcal{W}_p)$ and in $(\mathcal{V}_p, \mathcal{W}_p^c)$ we gain factor p in quality.

Hard case when $|\mathcal{V}_p|, |\mathcal{W}_p| \sim 1 - O(1/p)$, or when $|\mathcal{V}_p^c|, |\mathcal{W}_p^c| = 1 - O(1/p)$.

Weight $\mu(v) = \varphi(v)/v$ is of crucial importance to deal with this hard case. Gain factor $1 + 1/p$ in quality.

Thank you!

*Preprint available at dms.umontreal.ca/~koukoulo/documents/publications/DS.pdf after the talk