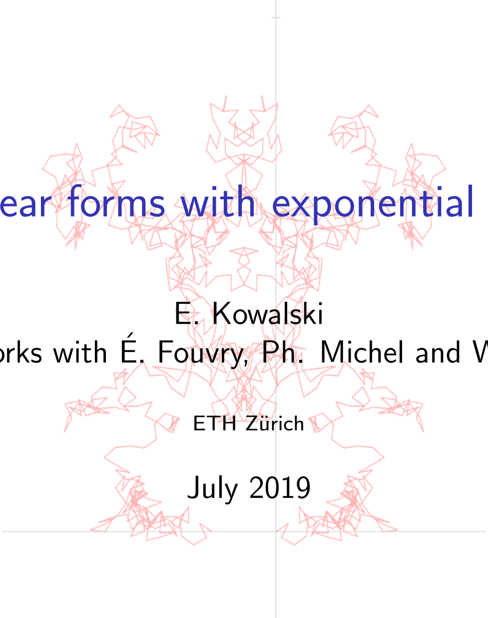

Bilinear forms with exponential sums

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(joint works with É. Fouvry, Ph. Michel and W. Sawin)

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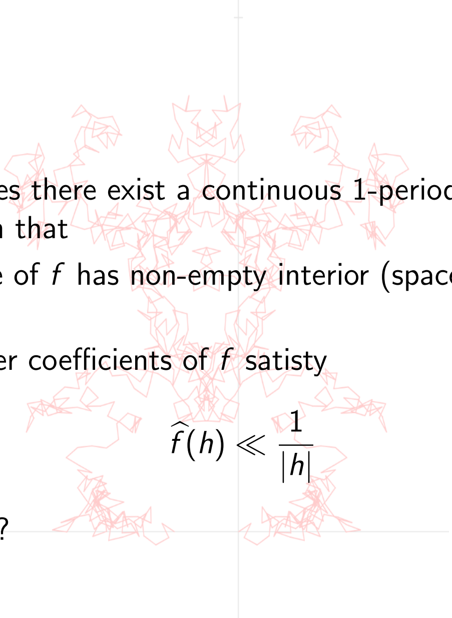
A digression

Question. Does there exist a continuous 1-periodic function $f : \mathbf{R} \to \mathbf{C}$ such that

1. The image of f has non-empty interior (space-filling curve);
2. The Fourier coefficients of f satisty

$$\widehat{f}(h) \ll \frac{1}{|h|}$$

for $h \neq 0$?



Bilinear forms

We will consider the problem of finding good estimates for general bilinear forms of the type

$$\sum_{m\sim M}\sum_{n\sim N}\alpha_m\beta_n K(mn)$$

for some (explicit) function K, where the coefficients (α_m) and (β_n) are arbitrary complex numbers.

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Smooth bilinear form (both variables are smooth):

$$\sum_{m\sim M}\sum_{n\sim N}K(mn).$$

General remarks

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Our main goal is to obtain *non-trivial bounds* that are valid for M and N as small as possible ("short sums"). For the applications we have in mind, the *strength* of the saving is usually not as important as the *range*.

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The critical range is then when M and N are both close to $\sqrt{q}$, even slightly smaller.

Why is it difficult?

If $K(mn) = K_1(m)K_2(n)$ then

$$\sum_m \sum_n \alpha_m \beta_n K(mn) = \left(\sum_m \alpha_m K_1(m)\right) \left(\sum_n \beta_n K_2(n)\right).$$

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So a non-trivial bound implies that K is strongly non-multiplicative.

Moreover, if K is q-periodic and $MN < q$, then there is no repetition of the values of $K(mn)$ that can be used to exclude multiplicativity.

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The coefficients α_m and β_n are not really unknown, but it is almost impossible to exploit their specific features.

A recent application

Let f a fixed modular form (say of level 1). For $q \geq 1$, we want to obtain an asymptotic formula for

$$\frac{1}{\varphi^*(q)} \sum_{\chi \pmod{q}}^{*} |L(f \times \chi, \tfrac{1}{2})|^2,$$

with power-saving error term; this allows us to further implement mollification, amplification, resonance, etc.

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If f is a suitable Eisenstein series then this expression is

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \pmod{q}} |L(\chi, \tfrac{1}{2})|^4$$

(M. Young, 2006, for q prime).

Reduction to bilinear forms

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Strategy: use the approximate functional equation and the orthogonality of Dirichlet characters to reduce to sums

$$\sum_{\substack{m \sim M, \\ m \equiv \pm n \pmod q}} \sum_{n \sim N} \frac{\lambda_f(m)\lambda_f(n)}{\sqrt{mn}}$$

with $1 \leq M \leq N$ and $MN \ll q^2$. We need to show that such sums are $\ll q^{-\delta}$ for some $\delta > 0$.

(Blomer, Fouvry, K., Michel, Milićević, *"On moments of twisted L-functions"*)

Reduction to bilinear forms

Recall

$$\mathop{\sum\sum}_{\substack{m\sim M,\ n\sim N\\ m\equiv\pm n \pmod q}} \frac{\lambda_f(m)\lambda_f(n)}{\sqrt{mn}} \approx \frac{1}{\sqrt{MN}} \mathop{\sum\sum}_{\substack{m\sim M,\ n\sim N\\ m\equiv\pm n \pmod q}} \lambda_f(m)\lambda_f(n)$$

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We use different methods depending on M and N.

For instance, write $m = n + qr$ and view

$$\sum_n \lambda_f(n+qr)\lambda_f(n)$$

as a shifted convolution sum. This succeeds in wide ranges using automorphic techniques; if q has suitable factorization, it can succeed in general (Blomer–Milićević).

The irreducible case

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Applying the Voronoi summation formula to the n-variable, the sums become

$$\frac{1}{\sqrt{q^3 M/N}} \sum_{m\sim M} \sum_{n\sim q^2/N} \lambda_f(m)\lambda_f(n)\,\mathrm{Kl}_2(\pm mn, q).$$

(Hyper-)Kloosterman sums

Let $k \geq 2$, q a prime number, $\boldsymbol{\chi} = (\chi_1, \ldots, \chi_k)$ Dirichlet characters modulo q. For $a \in \mathbf{F}_q^\times$, define

$$\mathrm{Kl}_k(a, \boldsymbol{\chi}; q) = \frac{1}{q^{(k-1)/2}} \sum_{y_1 \cdots y_k = a} \chi_1(y_1) \cdots \chi_k(y_k) e\Big(\frac{y_1 + \cdots + y_k}{q}\Big).$$

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For all χ trivial, write $\mathrm{Kl}_k(a; q) = \mathrm{Kl}_k(a, (1, \ldots, 1); q)$. So

$$\mathrm{Kl}_2(a; q) = \mathrm{Kl}_2(a, (1, 1); q) = \frac{1}{\sqrt{q}} \sum_{x \in \mathbf{F}_q} e\Big(\frac{ax + \bar{x}}{q}\Big).$$

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Weil $(k = 2)$/**Deligne** $(k \geq 3)$ bounds: for all $a \in \mathbf{F}_q^\times$, we have

$$|\,\mathrm{Kl}_k(a, \boldsymbol{\chi}; q)| \leq k.$$

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We do not know how to exploit the oscillations of the Hecke eigenvalues! So we view this as a value of a general bilinear form

$$\sum_{m \sim M} \sum_{n \sim N} \alpha_m \beta_n \, \mathrm{Kl}_2(\pm mn, q),$$

and try to exploit the oscillations of the Kloosterman sums.

A general “abstract” bound

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Applying the Cauchy-Schwarz inequality we get

$$|B(\boldsymbol{\alpha}, \boldsymbol{\beta})|^2 \leq \Delta \, \|\boldsymbol{\alpha}\|^2 \, \|\boldsymbol{\beta}\|^2$$

where

$$\Delta = \max_{m_1 \sim M} \sum_{m_2 \sim M} \left| \sum_{n \sim N} K(m_1 n) \overline{K(m_2 n)} \right|.$$

General bound for trace functions

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If K is a geometrically irreducible trace function modulo q and M, $N \leq q$, then the Riemann Hypothesis (and the underlying formalism) give

$$\Delta \ll N + Mq^{1/2} \log q$$

where the implied constant depends on the *conductor* $c(K)$, *except* if $K(n) = c\chi(n)e(an/q)$.

(Fouvry, K., Michel, *"Algebraic trace functions over the primes"*)

The Riemann Hypothesis

Theorem (Deligne). Let q be prime, let K_1 and K_2 be geometrically irreducible trace functions, of weight 0, modulo q. *Either* K_1 is proportional to K_2 (with a proportionality constant of modulus 1), *or*

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Moreover, if $K_1 = \alpha K_2$, then

$$\Big| \sum_{x \pmod q} K_1(x)\overline{K_2(x)} - \alpha q \Big| \leq c(K_1)c(K_2)\sqrt{q}.$$

Examples

Recall

$$|B(\boldsymbol{\alpha}, \boldsymbol{\beta})|^2 \ll (N + Mq^{1/2} \log q) \|\boldsymbol{\alpha}\|^2 \ \|\boldsymbol{\beta}\|^2,$$

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   - or g mod q is of degree ≥ 2.
3. $K(n) = \mathrm{Kl}_k(f(n), \boldsymbol{\chi}; q)$ if $k \geq 2$ and f mod q non-constant, with $c(K) \ll_k \deg(f)$.

Quality of the bound

Recall

$$B(\boldsymbol{\alpha}, \boldsymbol{\beta}) \ll (N^{1/2} + M^{1/2} q^{1/4} \log q) \, \|\boldsymbol{\alpha}\| \, \|\boldsymbol{\beta}\|,$$

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Assuming that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are essentially bounded, the bound becomes

$$B(\boldsymbol{\alpha}, \boldsymbol{\beta}) \ll M^{1/2} N + M N^{1/2} q^{1/4} \log q$$

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This bound can only be non-trivial if $N > q^{1/2}$. *This is a fundamental Fourier-theoretic constraint.*

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For smooth bilinear forms ($\alpha_m = 1 = \beta_n$) and $MN < q$, we have

$$\sum_{m \sim M} \sum_{n \sim N} K(mn) \ll (MN)^{1/2} q^{1/2-1/8+\varepsilon}$$

for any $\varepsilon > 0$ if K is not proportional to an additive character. This bound is non-trivial as long as $MN > q^{3/4}$.

(Fouvry, K., Michel, *"Algebraic trace functions over the primes"*)

Bilinear forms with (generalized) hyper-Kloosterman sums

Main Theorem. Let $k \geq 2$, let a be coprime with q. Suppose that for some $\delta > 0$, we have

$$M,\, N \geq q^{\delta}, \qquad MN \geq q^{3/4+\delta}.$$

Then there exists $\eta > 0$ such that

$$\sum_{m \sim M} \sum_{n \sim N} \alpha_m \beta_n \, \mathrm{Kl}_k(amn; q) \ll (MN)^{1/2-\eta} \, \|\alpha\| \, \|\beta\|$$

(K., Michel, Sawin: *"Bilinear forms with Kloosterman sums and applications"* and *"Stratification and averaging for exponential sums: bilinear forms with generalized Kloosterman sums"*)

Some highlights of the proof

The strategy goes back to Friedlander–Iwaniec and Fouvry–Michel, but the implementation is much more complicated on the algebraic-geometric side.

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The strategy goes back to Friedlander–Iwaniec and Fouvry–Michel, but the implementation is much more complicated on the algebraic-geometric side.

1. Reduction to square-root cancellation in two-variable complete exponential sums of "sums of products" type (analytic number theory).
2. *Sheaf-theoretic interpretation* of the summands, investigation of the local structure of the resulting objects (algebraic geometry).
3. *Deligne's Riemann Hypothesis* (Weil 2) implies a representation-theoretic interpretation of square-root cancellation (algebra).
4. *Diophantine interpretation of certain properties of étale cohomology* are used to extract basic information on the "sum-product" sheaves (analytic number theory).

Sums of products

The sums to handle are of the form

$$\sum_{r \in \mathbf{F}_q^\times} \sum_{\substack{s_1, s_2 \in \mathbf{F}_q^\times \\ s_1 \neq s_2}} \prod_{i=1}^{l} \mathrm{Kl}_k(s_1(r+b_i))\overline{\mathrm{Kl}_k(s_1(r+b_{i+l}))} \times \prod_{i=1}^{l} \mathrm{Kl}_k(s_2(r+b_i))\overline{\mathrm{Kl}_k(s_2(r+b_{i+l}))}$$

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We need (at least) generic square-root cancellation. Opening the Kloosterman sums is out of the question!

Sum-product sheaves

Fix $\boldsymbol{b} = (b_1, \ldots, b_{2l})$. Define

$$L(r) = \frac{1}{\sqrt{q}} \sum_{s \in \mathbf{F}_q^\times} \prod_{i=1}^{l} \mathrm{Kl}_k(s(r + b_i))\overline{\mathrm{Kl}_k(s(r + b_{i+l}))}.$$

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Theorem (Deligne, Katz, FKM, "Goursat–Kolchin–Ribet criterion"**)**
(1) Unless the b_i for $1 \leq i \leq l$ "pair" with the b_i with $l+1 \leq i \leq 2l$, we have $|L(r)| \leq C_{k,l}$.

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Theorem (Deligne, Katz, FKM, "Goursat–Kolchin–Ribet criterion"**)**
(1) Unless the b_i for $1 \leq i \leq l$ "pair" with the b_i with $l+1 \leq i \leq 2l$, we have $|L(r)| \leq C_{k,l}$.
(2) The "part of weight 0" of L is a trace function modulo q of a a *sum-product sheaf* $\mathscr{F}_{\boldsymbol{b}}$ with conductor bounded in terms of that of K.

Diophantine cohomology

The goal is then to prove that, generically, the sum-product sheaf $\mathscr{F}_{\boldsymbol{b}}$ is geometrically irreducible; the Riemann Hypothesis then leads to generic square-root cancellation.

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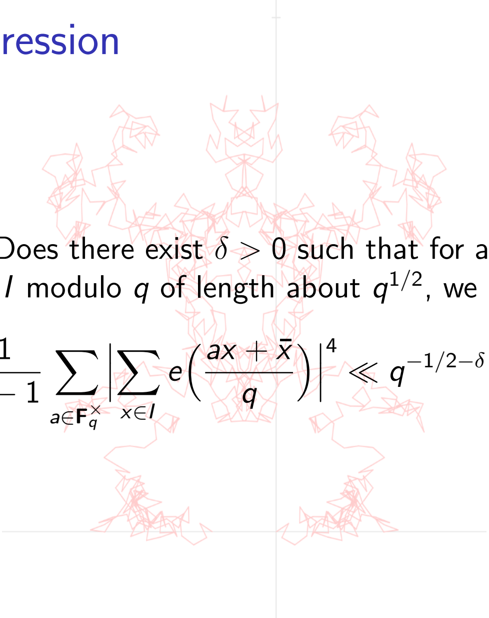
Here is one tool where analytic number theory comes back:

Theorem (Deligne, Katz, "Diophantine criterion for irreducibility**).** If a sheaf $\mathscr{F}$ modulo q, of weight 0, satisfies

$$\limsup_{\nu \to +\infty} \frac{1}{q^\nu} \sum_{x \in \mathbf{F}_{q^\nu}} |K(x;\nu)|^2 = 1,$$

then it is geometrically irreducible.

Another digression



Question. Does there exist $\delta > 0$ such that for any q prime, any interval I modulo q of length about $q^{1/2}$, we have

$$\frac{1}{q-1}\sum_{a\in\mathbf{F}_q^\times}\Big|\sum_{x\in I} e\Big(\frac{ax+\bar{x}}{q}\Big)\Big|^4 \ll q^{-1/2-\delta} \quad ?$$