Möbius disjointness for skew products on $\mathbb{T} \times \Gamma\backslash G$

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Plan

1. Möbius disjointness and skew products
2. Skew products on $\mathbb{T} \times \Gamma\backslash G$
3. Proof of Theorem 2

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A short abstract

Let $\mathbb{T}$ be the unit circle and $\Gamma\backslash G$ the 3-dimensional Heisenberg nilmanifold. We prove that

- a class of skew products on $\mathbb{T} \times \Gamma\backslash G$ are distal ;
- the Möbius function is linearly disjoint from these skew products.

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1. The Möbius disjointness and skew products

The Möbius Disjointness Conjecture

- Let μ be the Möbius function. The behavior of μ is central in the theory of prime numbers.
- Let (X, T) be a flow, namely X is a compact metric space and $T : X \to X$ a continuous map. We say that μ is *linearly disjoint* from (X, T) if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \mu(n) f(T^n x) = 0$$

for any $f \in C(X)$ and any $x \in X$.

The Möbius Disjointness Conjecture, Sarnak 2009

The function μ is linearly disjoint from every (X, T) whose entropy is 0.

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Known examples before 2009

Examples :

- (X, T) with X and T trivial $\sim$ PNT.
- (X, T) with $X = \mathbb{T}$ and T a translation
  $\sim$ Vinogragov's estimate on exponential sum over primes
  $\Rightarrow$ Ternary Goldbach.
- (X, T) with X nilmanifold and T a translation $\sim$ Green-Tao.
- Others regular flows . . .

Recent examples :

- A number of results, but mainly for regular flows.
  Regular/irregular : next page.
- See survey paper by Ferenczi/Kulaga-Przymus/Lemanczyk.

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MDC for irregular flows

- Note that there are *irregular flows* for which the Birkhoff average

$$\frac{1}{N}\sum_{n\leq N} f(T^n x)$$

may not exist some $x \in X$.

- Irregular flows are not very rare. KAM theory, *small denominator problem*.
- MDC $\Rightarrow$ For any zero-entropy flow (X, T), any $f \in C(X)$, and any $x \in X$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\leq N}\mu(n)f(T^n x) = 0.$$

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Distal flows and skew products

- Distal flows are typical examples of zero-entropy flows.
- A flow (X, T) with a compatible metric d is called *distal* if

$$\inf_{n\geq 0} d(T^n x, T^n y) > 0$$

whenever $x \neq y$.
- Furstenberg's structure theorem of minimal distal flows (1963) : skew products are building blocks of distal flows. Complicated ; transfinite induction, etc.
- $\{\text{zero-entropy flows}\} \supset \{\text{distal flows}\} \supset \{\text{skew products}\} \supset \{\text{irregular skew products}\}$.

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Irregular skew products on $\mathbb{T}^2$

- Let $\mathbb{T}^2$ be the 2-torus, and

$$T : (x, y) \mapsto (x + \alpha, y + h(x)),$$

where $\alpha \in [0, 1)$ and h a continuous real function of period 1.

- Furstenberg (1961) : $(\mathbb{T}^2, T)$ is distal but irregular. Irregularity comes from non-diophantine α.
- Definition : Fix $B > 0$. A real α is *diophantine* w.r.t B, if

$$\|m\alpha\| \geq m^{-B}$$

for all large positive integers m.

- MDC is expected to hold even for irregular $(\mathbb{T}^2, T)$, i.e. for α non-diophantine.

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Irregular skew products on $\mathbb{T}^2$, II

Theorem 1 (L.-Sarnak, 2015)

MDC holds for $(\mathbb{T}^2, T)$ for all α, if h is analytic with an additional assumption on its Fourier coefficients.

- The point : for all α, as is not common in the KAM theory.
- Wang (2017) : Additional assumption removed.
- Huang-Wang-Ye (2019) : h relaxed to C^∞-smooth.
- Kanigowski-Lemanczyk-Radziwill (arXiv 2019) : h absolutely continuous.

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2. Skew products on $\mathbb{T} \times \Gamma\backslash G$

Skew products on $\mathbb{T} \times \Gamma\backslash G$

- Now let G be the 3-dimensional Heisenberg group with the cocompact discrete subgroup Γ, namely

$$G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}.$$

  Then $\Gamma\backslash G$ is the 3-dimensional Heisenberg nilmanifold.
- Study the MDC for skew products on

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- Goes beyond $\mathbb{T}^2$.

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Skew products on $\mathbb{T} \times \Gamma\backslash G$, II

Theorem 2 (Huang-L.-Wang, 2019 arXiv)

Let $\alpha \in [0, 1)$ and let φ, ψ be C^∞-smooth functions with period 1. Define the skew product T on $\mathbb{T} \times \Gamma\backslash G$ by

$$T : (t, \Gamma g) \mapsto \left(t + \alpha, \Gamma g \begin{pmatrix} 1 & \varphi(t) & \psi(t) \\ 0 & 1 & \varphi(t) \\ 0 & 0 & 1 \end{pmatrix} \right).$$

Then, for any $(t, \Gamma g) \in \mathbb{T} \times \Gamma\backslash G$ and any $f \in C(\mathbb{T} \times \Gamma\backslash G)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) f(T^n(t, \Gamma g)) = 0.$$

Remarks

- Note that the skew product $(\mathbb{T} \times \Gamma\backslash G, T)$ in Theorem 2 is irregular, but Theorem 2 holds for all α.
- The flow $(\mathbb{T} \times \Gamma\backslash G, T)$ is distal ; see next page Proposition 3. Thus Theorem 2 verifies the MDC in this context.

Remarks

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Remarks

Proposition 3 (Distality of $(\mathbb{T} \times \Gamma\backslash G, S)$)

Denote by S the skew product

$$S : (t, \Gamma g) \mapsto \left(t + \alpha, \Gamma g \begin{pmatrix} 1 & \varphi_2(t) & \psi(t) \\ 0 & 1 & \varphi_1(t) \\ 0 & 0 & 1 \end{pmatrix} \right).$$

Then the flow $(\mathbb{T} \times \Gamma\backslash G, S)$ is distal.

- Thus MDC should hold for $(\mathbb{T} \times \Gamma\backslash G, S)$.
- S is more general than T.
- Our method works well for $(\mathbb{T} \times \Gamma\backslash G, T)$, but not directly for $(\mathbb{T} \times \Gamma\backslash G, S)$. It seems interesting to generalize Theorem 2 to $(\mathbb{T} \times \Gamma\backslash G, S)$.

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3. Proof of Theorem 2
An illustration

3.1 Analysis on $C(\mathbb{T} \times \Gamma\backslash G)$

- Let G be the 3-dimensional Heisenberg group with the cocompact discrete subgroup Γ, and $\Gamma\backslash G$ the 3-dimensional Heisenberg nilmanifold.
- Want to construct a subset of $C(\mathbb{T} \times \Gamma\backslash G)$, which spans a $\mathbb{C}$-linear subspace that is dense in $C(\mathbb{T} \times \Gamma\backslash G)$.

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- For integers m, j with $0 \leq j \leq m-1$, define the functions ψ_{mj} and ψ^*_{mj} on G by

$$\psi_{mj}\begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} = e(mz + jx) \sum_{k \in \mathbb{Z}} e^{-\pi(y+k+\frac{j}{m})^2} e(mkx),$$

and

$$\psi^*_{mj}\begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} = ie(mz + jx) \sum_{k \in \mathbb{Z}} e^{-\pi(y+k+\frac{j}{m}+\frac{1}{2})^2} e\left(\frac{1}{2}\left(y + k + \frac{j}{m}\right) + mkx\right).$$

- We check that ψ_{mj} and ψ^*_{mj} are Γ-invariant, that is

$$\psi_{mj}(\gamma g) = \psi_{mj}(g), \quad \psi^*_{mj}(\gamma g) = \psi^*_{mj}(g)$$

for any $g \in G$ and for any $\gamma \in \Gamma$. Thus ψ_{mj} and ψ^*_{mj} can be regarded as functions on the nilmanifold $\Gamma\backslash G$.

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- Let $\mathcal{A}$ be the subset of $f \in C(\mathbb{T} \times \Gamma\backslash G)$ such that

$$f : \left(t, \Gamma \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}\right) \mapsto e(\xi_1 t + \xi_2 x + \xi_3 y)\psi\left(\Gamma \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}\right)$$

where $\xi_1, \xi_2, \xi_3 \in \mathbb{Z}$, and $\psi = \psi_{mj}, \overline{\psi}_{mj}, \psi^*_{mj}$ or $\overline{\psi}^*_{mj}$ for some $0 \le j \le m-1$.

- Let $\mathcal{B}$ be subset of $f \in C(\mathbb{T} \times \Gamma\backslash G)$ satisfying

$$f : (t, \Gamma g) \mapsto f_1(t) f_2(\Gamma g)$$

with $f_1 \in C(\mathbb{T})$ and $f_2 \in C_0(\Gamma\backslash G)$.

Proposition 4 (Structure of $C(\mathbb{T} \times \Gamma\backslash G)$)

The $\mathbb{C}$-linear subspace spanned by $\mathcal{A} \cup \mathcal{B}$ is dense in $C(\mathbb{T} \times \Gamma\backslash G)$.

- Let $\mathcal{A}$ be the subset of $f \in C(\mathbb{T} \times \Gamma\backslash G)$ such that

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Proposition 4 (Structure of $C(\mathbb{T} \times \Gamma\backslash G)$)

The $\mathbb{C}$-linear subspace spanned by $\mathcal{A} \cup \mathcal{B}$ is dense in $C(\mathbb{T} \times \Gamma\backslash G)$.

- Let $\mathcal{A}$ be the subset of $f \in C(\mathbb{T} \times \Gamma\backslash G)$ such that

$$f : \left(t, \Gamma \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}\right) \mapsto e(\xi_1 t + \xi_2 x + \xi_3 y)\psi\left(\Gamma \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}\right)$$

  where $\xi_1, \xi_2, \xi_3 \in \mathbb{Z}$, and $\psi = \psi_{mj}, \overline{\psi}_{mj}, \psi^*_{mj}$ or $\overline{\psi}^*_{mj}$ for some $0 \le j \le m-1$.
- Let $\mathcal{B}$ be subset of $f \in C(\mathbb{T} \times \Gamma\backslash G)$ satisfying

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3.2 Theorem 2 for rational α

The case $f \in \mathcal{A}$, I

By a straightforward calculation,

$$T^n : (t_0, \Gamma g_0) \mapsto (t_0 + n\alpha, \Gamma g_n),$$

where, on writing

$$g_0 = \begin{pmatrix} 1 & y_0 & z_0 \\ 0 & 1 & x_0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_n = \begin{pmatrix} 1 & y_n & z_n \\ 0 & 1 & x_n \\ 0 & 0 & 1 \end{pmatrix},$$

we have

$$\begin{cases} x_n = x_0 + S_1(n; t_0), \\ y_n = y_0 + S_1(n; t_0), \\ z_n = z_0 + \frac{1}{2}(S_1(n; t_0))^2 - \frac{1}{2}S_3(n; t_0) + S_2(n; t_0) + y_0 S_1(n; t_0), \end{cases}$$

and

$$S_1(n; t) = \sum_{l=0}^{n-1} \varphi(\alpha l + t), \quad S_2(n; t) \ldots \psi, \quad S_3(n; t) \ldots \varphi^2.$$

The case $f \in \mathcal{A}$, II

- Recall for $f \in \mathcal{A}$,

$$f\left(t, \Gamma\begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}\right) = e(t+x+y+z)\sum_{k\in\mathbb{Z}} e^{-\pi(y+k)^2} e(kx).$$

- Compute

$$\begin{aligned} & f(T^n(t_0, \Gamma g_0)) \\ &= f\left(t_0 + n\alpha, \Gamma\begin{pmatrix} 1 & y_n & z_n \\ 0 & 1 & x_n \\ 0 & 0 & 1 \end{pmatrix}\right) \\ &= e(t_0 + n\alpha + x_n + y_n + z_n)\sum_{k\in\mathbb{Z}} e^{-\pi(y_n+k)^2} e(kx_n). \end{aligned}$$

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Rational α reduces to Hua

- For rational $\alpha = a/q$, one rearranges n into arithmetic progressions modulo q :

$$\sum_{n\le N} \mu(n) f(T^n(t_0, \Gamma g_0)) \ll \left| \sum_{m\in\mathbb{Z}} \widehat{w}(m) \sum_{b=0}^{q-1} \sum_{\substack{n\le N \\ n\equiv b \bmod q}} \mu(n) e(P(n;b)) \right|.$$

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- Hua (1938) : Let $f(x) \in \mathbb{R}[x]$. Let $0 \le a < q$. Then, for arbitrary $A > 0$,

$$\sum_{\substack{n\le N \\ n\equiv a \bmod q}} \mu(n) e(f(n)) \ll \frac{N}{\log^A N},$$

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3.2 Measure complexity

Measure complexity

- Let (X, T) be a flow. For a compatible metric d, define

$$\overline{d}_n(x,y) = \frac{1}{n}\sum_{j=0}^{n-1} d(T^j x, T^j y)$$

for $x, y \in X$, and let

$$B_{\overline{d}_n}(x,\varepsilon) = \{y \in X : \overline{d}_n(x,y) < \varepsilon\}.$$

- Let $M(X, T)$ be the set of all T-invariant Borel probability measures on X. For $\rho \in M(X, T)$, write

$$s_n(X, T, d, \rho, \varepsilon) = \min\left\{m \in \mathbb{N} : \exists x_1, \ldots, x_m \in X \text{ s.t. } \rho\Big(\bigcup_{j=1}^{m} B_{\overline{d}_n}(x_j,\varepsilon)\Big) > 1-\varepsilon\right\}.$$

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Measure complexity

- The measure complexity of (X, T, ρ) is *sub-polynomial* if

$$\liminf_{n\to\infty} \frac{s_n(X, T, d, \rho, \varepsilon)}{n^\tau} = 0$$

for any $\tau > 0$.

- Huang-Wang-Ye (2019) : If the measure complexity of (X, T, ρ) is sub-polynomial for any $\rho \in M(X, T)$, then MDC holds for (X, T).
- Number theory behind HWY : Matomäki-Radziwill-Tao, averaged form of Chowla. Chowla $\Rightarrow$ MDC. The measure complexity defined above can be viewed as an averaged form of entropy.

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3.3 Theorem 2 for irrational α

Theorem 2 for irrational α

Proposition 4

For irrational α, the measure complexity of $(\mathbb{T} \times \Gamma\backslash G, T, \rho)$ is sub-polynomial for any $\rho \in M(\mathbb{T} \times \Gamma\backslash G, T)$.

- The continued fraction expansion :

$$\alpha = [0; a_1, a_2, \ldots, a_k, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$$

  This expansion is infinite since α is irrational. The k-th convergent of α is

$$\frac{l_k}{q_k} = [0; a_1, a_2, \ldots, a_k].$$

- Let $\mathcal{Q} = \{q_k : k \geq 1\}$. For $B > 2$, define

$$\begin{aligned} \mathcal{Q}^\flat &= \{q_k \in \mathcal{Q} : q_{k+1} \leq q_k^B\} \cup \{1\}, \\ \mathcal{Q}^\sharp &= \{q_k \in \mathcal{Q} : q_{k+1} > q_k^B > 1\}. \end{aligned}$$

  The main difficulty comes from $\mathcal{Q}^\sharp$, which includes the irregular case.

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The main difficulty comes from $\mathcal{Q}^\sharp$, which includes the irregular case.

Complicated argument $\rightarrow$

Write $n_k = q_k^{B-1}$. Then $\mathbb{T} \times \Gamma\backslash G$ can be covered by $\varepsilon^{-1} q_k^7$ balls of radius 20ε under the metric $\overline{d}_{n_k}$. It follows that

$$s_{n_k}(\mathbb{T} \times \Gamma\backslash G, T, d, 20\varepsilon) \leq \varepsilon^{-1} q_k^7.$$

$\mathcal{Q}^\sharp$ infinite

Since $\mathcal{Q}^\sharp$ is infinite, we can let q_k tend to infinity along $\mathcal{Q}^\sharp$, getting

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{s_n(\mathbb{T} \times \Gamma\backslash G, T, d, 20\varepsilon)}{n^\tau} \\
&\leq \liminf_{\substack{k\to\infty \\ q_k\in\mathcal{Q}^\sharp}} \frac{s_{n_k}(\mathbb{T} \times \Gamma\backslash G, T, d, 20\varepsilon)}{n_k^\tau} \\
&\leq \liminf_{\substack{k\to\infty \\ q_k\in\mathcal{Q}^\sharp}} \frac{\varepsilon^{-1} q_k^7}{q_k^{8+\tau}} \\
&= 0.
\end{aligned}$$

Since ε can be arbitrarily small, this means that the measure complexity of $(\mathbb{T} \times \Gamma\backslash G, T, \rho)$ is weaker that n^τ when $\mathcal{Q}^\sharp$ is infinite.

Thank you !