Primes in arithmetic progressions to large moduli

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Second Symposium in Analytic Number Theory, Cetraro
July 2019

Introduction

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Theorem (Siegel-Walfisz)

If $q \leq (\log x)^A$ *and* $\gcd(a, q) = 1$ *then*

$$\pi(x; q, a) = (1 + o(1))\frac{\pi(x)}{\phi(q)}.$$

Theorem (GRH Bound)

Assume GRH. If $q \leq x^{1/2-\epsilon}$ *and* $\gcd(a, q) = 1$ *then*

$$\pi(x; q, a) = (1 + o(1))\frac{\pi(x)}{\phi(q)}.$$

Conjecture (Montgomery)

If $q \leq x^{1-\epsilon}$ *and* $\gcd(a, q) = 1$ *then*

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Introduction II

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Theorem (Bombieri-Vinogradov)

Let $Q < x^{1/2-\epsilon}$. Then for any A

$$\sum_{q\sim Q}\sup_{(a,q)=1}\left|\pi(x;q,a)-\frac{\pi(x)}{\phi(q)}\right| \ll_A \frac{x}{(\log x)^A}$$

Corollary

*For **most** $q \le x^{1/2-\epsilon}$, we have*

$$\pi(x;q,a) = (1+o(1))\frac{\pi(x)}{\phi(q)}$$

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Theorem (Bombieri-Vinogradov)

Let $Q < x^{1/2-\epsilon}$. Then for any A

$$\sum_{q\sim Q} \sup_{(a,q)=1} \left| \pi(x;q,a) - \frac{\pi(x)}{\phi(q)} \right| \ll_A \frac{x}{(\log x)^A}$$

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for every a with $\gcd(a,q)=1$.

From the point of view of e.g. sieve methods, this is essentially as good as the Riemann Hypothesis!

Beyond GRH

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Theorem (BFI1)

Fix a. Then we have (uniformly in θ)

$$\sum_{\substack{q\sim x^\theta\\(q,a)=1}} \left|\pi(x;q,a)-\frac{\pi(x)}{\phi(q)}\right| \ll_a (\theta-1/2)^2\frac{x(\log\log x)^{O(1)}}{\log x}+\frac{x}{\log^3 x}.$$

This is non-trivial when θ is very close to $1/2$.

Theorem (BFI2)

Fix a. Let $\lambda(q)$ be 'well-factorable'. Then we have

$$\sum_{\substack{q\sim x^{4/7-\epsilon}\\(q,a)=1}} \lambda(q)\Bigl(\pi(x;q,a)-\frac{\pi(x)}{\phi(q)}\Bigr) \ll_{a,A} \frac{x}{\log^A x}.$$

This is often an adequate substitute for BV with exponent $4/7$!

Beyond GRH II

More recently, Zhang went beyond $x^{1/2}$ for smooth/friable moduli.

Theorem (Zhang,Polymath)

$$\sum_{\substack{q\le x^{1/2+7/300-\epsilon}\\ p|q\Rightarrow p\le x^{\epsilon^2}\\ (q,a)=1}} \left|\pi(x;q,a)-\frac{\pi(x)}{\phi(q)}\right| \ll_A \frac{x}{(\log x)^A}$$

The implied constant is independent of a.

New results

Theorem (M.)

Let $\delta < 1/42$ *and* $\mathcal{Q}_\delta := \{q \sim x^{1/2+\delta} : \exists d | q \text{ s.t. } x^{2\delta+\epsilon} < d < x^{1/14-\delta}\}$.

$$\sum_{\substack{q \in \mathcal{Q}_\delta \\ (q,a)=1}} \left| \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \right| \ll_A \frac{x(\log \log x)^{O(1)}}{\log^5 x}.$$

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Corollary

Let $\delta < 1/42$. *For* $(100 - O(\delta))\%$ *of* $q \sim x^{1/2+\delta}$ *we have*

$$\pi(x;q,a) = (1+o(1))\frac{\pi(x)}{\phi(q)}$$

Corollary

$$\sum_{q_1\sim x^{1/21}} \sum_{\substack{q_2\sim x^{10/21-\epsilon}\\(q_1q_2,a)=1}} \left|\pi(x;q_1q_2,a) - \frac{\pi(x)}{\phi(q_1q_2)}\right| \ll_a \frac{x(\log\log x)^{O(1)}}{\log^5 x}$$

New Results II

Theorem (M.)

Let $\lambda(q)$ be 'very well factorable'. Then we have

$$\sum_{\substack{q\le x^{3/5-\epsilon}\\(q,a)=1}} \lambda(q)\Big(\pi(x;q,a)-\frac{\pi(x)}{\phi(q)}\Big) \ll_{a,A} \frac{x}{(\log x)^A}.$$

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Corollary

Let $\lambda^+(d)$ be sieve weights for the linear sieve. Then

$$\sum_{\substack{q\le x^{7/12-\epsilon}\\ (q,a)=1}} \lambda^+(q)\Big(\pi(x;q,a)-\frac{\pi(x)}{\phi(q)}\Big) \ll \frac{x}{(\log x)^A}.$$

Comparison

Result	Size of q	Type of q	Proportion of q
BFI1	$x^{1/2+o(1)}$	All	$(100-\delta)\%$
BFI2	$x^{4/7-\epsilon}$	Factorable	$\delta\%$
Zhang	$x^{1/2+7/300-\epsilon}$	Factorable	$\delta\%$
M1	$x^{11/21-\epsilon}$	Partially Factorable	$(100-\delta)\%$
M2	$x^{3/5-\epsilon}$	Factorable	$\delta\%$

Result	Coefficients	Residue class	Cancellation
BFI1	Absolute values	Fixed	$o(1)$
BFI2	Factorable weights	Fixed	$\log^A x$
Zhang	Absolute values	Uniform	$\log^A x$
M1	Absolute values	Fixed	$\log^{5-\epsilon} x$
M2	Factorable weights	Fixed	$\log^A x$

Note that $3/5 > 4/7 > 11/21 > 1/2 + 7/300$.

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Combine Zhang-style estimates with Kloostermania

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Let us recall the situation when $q \sim x^{1/2+\delta}$ where $\delta > 0$ is fixed but small. Using BFI proof ideas:

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2. Working through the BFI argument their proof can essentially handle all such numbers except for
   - Products $p_1 p_2 p_3 p_4 p_5$ of 5 primes with $p_i = x^{1/5+O(\delta)}$
   - Products $p_1 p_2 p_3 p_4$ of 4 primes with $p_i = x^{1/4+O(\delta)}$

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BFI result follows on noting that these terms are only a $O(\delta)$ proportion of the terms.

We can concentrate on these 'bad products'.

Products of 5 Primes

Consider terms $p_1 p_2 p_3 p_4 p_5$ with $p_i \in [x^{1/5-\delta}, x^{1/5+\delta}]$

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- Refinement of BFI can handle $p_1 p_2 p_3 p_4 p_5$ with $q < x^{4/7-\epsilon}$ when $p_i \approx x^{1/5}$ **except** when $p_i \in [x^{1/5} \log^{-A} x, x^{1/5} \log^{A} x]$

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- I still can't handle these terms, but they now contribute $O((\log\log x)^{O(1)}/\log^4 x)$ proportion for a wide range of q. (This is why I only save $4 - \epsilon \log x$ factors.)

Algebraic Geometry doesn't help much, but we can refine Kuznetsov-based estimates to handle these terms

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- Note: In this case there is a factor $p_1 p_4 = x^{1/2+O(\delta)}$ very close to $1/2$. This is the situation when Zhang-style arguments are most effective!
- Provided q has a suitable factor close to $x^{1/2}$, we can handle these terms using the Weil bound.

The technical parts which spectral theory estimates can't handle are precisely parts that the algebraic geometry estimates are best at *when there is a suitable factor*

Numerics

As stated these ideas combine to give a result for $q \sim x^{1/2+\delta}$ for some small $\delta > 0$.

To get good numerics, need to refine estimates for other parts of prime decomposition

- Generalize ideas based on Deligne's work (Fouvry, Kowalski,Michel) to handle products of 3 primes when the modulus has a convenient small factor.

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- Generalize ideas of Fouvry for products of 7 primes when the modulus has a convenient small factor.

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- Generalize ideas of Fouvry for products of 7 primes when the modulus has a convenient small factor.
- Auxilliary estimate when there is a very small factor

Together these improve all terms in the decomposition, with a reasonable range of q!

Overview

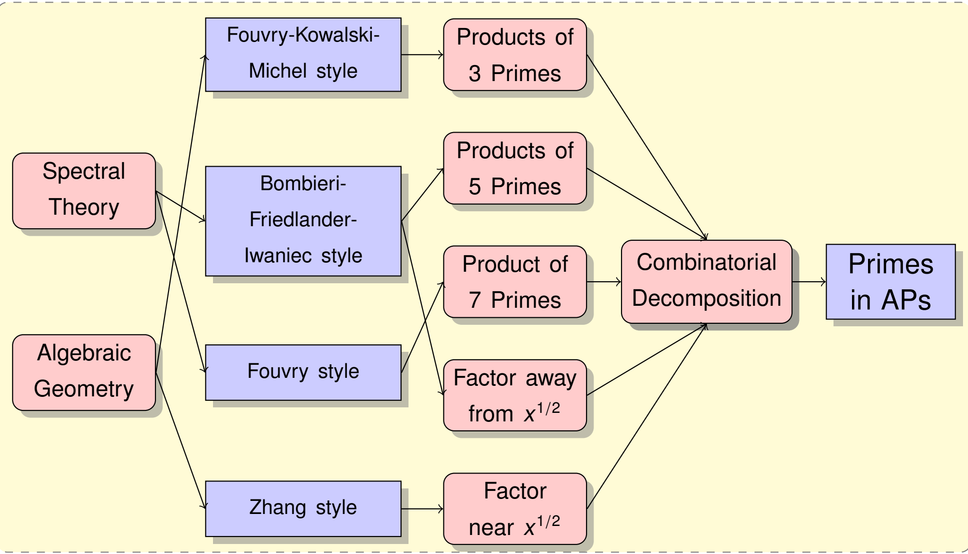

Figure: Outline of steps to prove primes in arithmetic progressions

Questions

Thank you for listening.