Periodic twists of GL_3 L-functions

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a joint work with
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Symposium in Analytic Number Theory
Cetraro 2019

For f a fixed modular form, $\chi \pmod q$ a Dirichlet character and

$$L(f.\chi, s) = \sum_{n\geq 1} \frac{\lambda_f(n)\chi(n)}{n^s},\ \Re s > 1$$

the twisteds Hecke L-function. The following subconvex bound was first proven by Duke, Friedlander, Iwaniec:

Subconvex bound

For $\Re s = 1/2$

$$L(f.\chi, s) \ll_{f,s} q^{1/2-\delta+o(1)},\ \delta > 0$$

The bound is substantially equivalent to: for $V \in \mathcal{C}^\infty([1,2])$

$$\sum_n \lambda_f(n)\chi(n)V(\frac{n}{q}) \ll_{f,V} q^{1-\delta+o(1)}.$$

A fews years ago, Fouvry, Kowalski and myself looked to establish similar bounds with $\chi \,(\mathrm{mod}\, q)$ replaced by more general q-periodic arithmetic functions. For instance

- Kloosterman fractions: $n \mapsto e(a\frac{\overline{n}}{q}),\ (n,q)=1$
- Hyper-Kloosterman sums: $n \mapsto \mathrm{Kl}_k(n;q) = \frac{1}{q^{\frac{k-1}{2}}}\sum_{x_1 x \cdots x_k = n} e(\frac{x_1+\cdots+x_k}{q}),\ (n,q)=1.$

These functions (along with Dirichet characters) are examples of *trace functions*.

Trace functions

Given $(\ell, q) = 1$, choose an embedding $\iota : \overline{\mathbb{Q}_\ell} \hookrightarrow \mathbb{C}$.
The basic datum is a Galois representation

$$\rho : \mathrm{Gal}(\overline{\mathbb{F}_q[T]}/\mathbb{F}_q(T)) \to \mathrm{GL}(V)$$

for V a finite dimensional $\overline{\mathbb{Q}_\ell}$-vector space.
We assume that ρ is (ι-)pure of weight 0: the eigenvalues of the Frobenius at any unramified place of $\mathbb{F}_q(T)$ have absolute value 1.
The trace function associated with ρ is the function

$$K_\rho : t \in \mathbb{F}_q \mapsto \mathrm{tr}(\mathrm{Frob}_t | V^{I_t}) \in \overline{\mathbb{Q}_\ell} \hookrightarrow \mathbb{C}.$$

(here "t" denote the place associated with the polynomial $T - t$.)
It follows from purity that

$$\|K_\rho\|_\infty \leq \dim V.$$

To such a trace function, is associated the *conductor* $C(\rho)$ which is a measure the complexity of the geometric representation (the sum of the rank and of the ramification invariants, the drops and the Swan conductors).

Theorem (FKM)

Suppose f cuspidal. For any trace function $K = K_\rho : \mathbb{F}_q \to \mathbb{C}$, *one has*

$$\sum_n \lambda_f(n)K(n)V(\frac{n}{q}) \ll_{f,V,C(\rho)} q^{1-\delta+o(1)},\ \delta = 1/8.$$

Here the dependency in $C(\rho)$ *is polynomial. Moreover this bound holds for f non- cuspidal, if K "is not" an additive character* $n \mapsto e(\frac{an}{q})$ *(ie.* ρ *"is not" an Artin-Schreier representation)*

By a version of Schur's lemma one is essentially reduced to the case where ρ geometrically irreducible.

The proof uses the *amplification method* but in the different way than DFI:

- DFI amplify the character χ within the family of character $\{\chi' \pmod{q}\}$; ie. proceed from the trivial bound

$$|\sum_n \lambda_f(n)\chi(n)V(\frac{n}{q})|^2|M_\chi(\chi)|^2 \leq \sum_{\chi' \pmod{q}} |\sum_n \lambda_f(n)\chi'(n)V(\frac{n}{q})|^2|M_\chi(\chi')|^2$$

  and then bound the second moment on the righthand side by opening the squares and using harmonic analysis; here $\chi' \mapsto M_\chi(\chi')$ is a suitable "amplifier" of χ.

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The proof uses the *amplification method* but in the different way than DFI:

- FKM (following Bykovski) amplify the Hecke eigenform $f/(q+1)^{1/2}$ within an orthogonormal basis of modular forms of level q, $\mathcal{B}(\Gamma_0(q))$; ie. proceed from the trivial bound

$$\frac{1}{q+1}|\sum_n \lambda_f(n)K(n)V(\frac{n}{q})|^2|M_f(f)|^2 \leq \sum_{f'\in\mathcal{B}(\Gamma_0(q))}|\sum_n \lambda_{f'}(n)K(n)V(\frac{n}{q})|^2|M_f(f')|^2$$

then bound the second moment on the righthand side by opening the squares and using harmonic analysis; here $f' \mapsto M_f(f')$ is a suitable amplifier of f.

After performing harmonic analysis (Petersson-Kuznetsov formula + Poisson) one face some correlation sums

$$C(\widehat{K},\gamma)=\frac{1}{q^{1/2}}\sum_{z\in\mathbb{F}_q}\overline{\widehat{K}(z)}.\widehat{K}(\gamma.z)$$

where

$$\widehat{K}(z)=\frac{1}{q^{1/2}}\sum_{t}K(t)e(\frac{zt}{q})$$

is the Fourier transform of K, and

$$\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{PGL}_2(\mathbb{F}_q),\ \gamma.z=\frac{az+b}{cz+d}.$$

A key fact due to Laumon is that unless ρ is Artin-Schreier (K is an additive character), $\widehat{K}$ is a trace function whose conductor $C(\widehat{\rho})$ is controlled by $C(\rho)$.

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By the work of Deligne and Laumon, the correlation sums $C(\widehat{K},\gamma)$ are typically $\ll_{C(\rho)} 1$ and otherwise they satisfy

$$C(\widehat{K},\gamma) \gg_{C(\rho)} q^{1/2}. \qquad (1)$$

Theorem (Classification of group of automorphisms of sheaves)

The set of γ such that (1) holds is contained in $G_{\widehat{\rho}}(\mathbb{F}_q)$ the set of $\mathbb{F}_q$-points of an algebraic subgroup of PGL_2. Moreover $|G_{\widehat{\rho}}(\mathbb{F}_q)|$ is either "small" (bounded in terms of $C(\rho)$) or has a simple structure.

This show that the correlation sums $C(\widehat{K},\gamma)$ which occur in

$$\sum_{f'\in\mathcal{B}(\Gamma_0(q))} |\sum_n \lambda_{f'}(n)K(n)V(\frac{n}{q})|^2|M_f(f')|^2$$

are of size $\ll_{C(\rho)} 1$ outside a well controlled diagonal set. From there one conclude the proof. □

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More striking examples of this amplification scheme (ie amplifying f inside $\mathcal{B}(\Gamma_0(q), \bullet)$ (instead of χ) are found in the works of Conrey-Iwaniec and Petrow-Young to prove subconvex bounds Weyl type(see the next talk).

Twists of GL_3 L-functions

The subconvexity problem for GL_2 is completely solved (at least qualitatively.)
For GL_3 L-functions, the first break was made by X. Li. Later R. Munshi developed a new set of techniques leading eventually to:

Theorem (Munshi)

Let f be a $SL_3(\mathbb{Z})$-invariant cusp form. For $\Re s = 1/2$,

$$L(f.\chi, s) \ll_{f,s} q^{3/4-\delta+o(1)},\ \delta = 1/308.$$

Munshi's method does not use amplification but an elaborate variant of the δ-symbol method, the Voronoi summation formula and reciprocity for Kloosterman fractions.

Recently R. Holowinsky and P. Nelson found a major simplification of Munshi approach leading to a significant improvement:

Theorem (Holowinsky-Nelson)

Let f be a $SL_3(\mathbb{Z})$-invariant cusp form. For $\Re s = 1/2$,

$$L(f.\chi, s) \ll_{f,s} q^{3/4-\delta+o(1)},\ \delta = 1/36.$$

Again this bound is substantially equivalent to the bound

$$\sum_n \lambda_f(1,n)\chi(n)V(\frac{n}{q^{3/2}}) \ll_{f,V} q^{3/2-\delta+o(1)}$$

where $(\lambda_f(m,n))_{m,n}$ denote the Hecke eigenvalues of f. This method is very robust and extends to general trace functions

More generally we define

$$S_V(K,X) := \sum_n \lambda_f(1,n)K(n)V(\frac{n}{X})$$

Theorem (KLMS)

Let K be a trace function of modulus q, and X such that $X \leq q^2$, one has

$$S_V(K,X) \ll_{f,V,C(\rho)} q^{2/9+o(1)}X^{5/6}.$$

- For $X = q^{3/2}$ one obtains $\ll_{f,V,C(\rho)} q^{3/2-1/36+o(1)}$
- the bound is non trivial as long as $X \geq q^{4/3+o(1)}$.

If K is an additive character, S. Miller has proven an analog of Wilton's bound

$$S_V(e(a\frac{\bullet}{q}), X) \ll_f X^{3/4+o(1)}.$$

So wlog wma K is not an additive character.
The first step is to realize the q-periodic function K within a one-parameter family of q-periodic functions. Define

$$\widehat{K}(z,h) := \begin{cases} \widehat{K}(z)e_q(-h\overline{z}) & q \nmid z \\ \widehat{K}(0) & q \mid z \end{cases}$$

for $(z,h) \in \mathbb{Z}^2$ so that

$$K(n,h) := \frac{1}{q^{1/2}} \sum_{z \in \mathbb{F}_q^\times} \widehat{K}(z,h) e_q(-nz).$$

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Taking $h = 0$ in the above

$$K(n,0) = K(n) - \frac{\widehat{K}(0)}{q^{1/2}}.$$

and, more generally, for any probability measure ϖ on $\mathbb{F}_q^\times$, we have

$$K_\varpi(n,0) = K(n) - \frac{\widehat{K}(0)}{q^{1/2}}.$$

where

$$K_\varpi(n,h) := \sum_{u \in \mathbb{F}_q^\times} \varpi(u) K(n, \overline{u}h)$$

It follows that

$$\begin{aligned} S_V(K,X) = & \sum_{u\in\mathbb{F}_q^\times} \varpi(u) \sum_{|h|\leq H} S_V(K(\bullet,\overline{u}h),X) \\ & - \sum_{u\in\mathbb{F}_q^\times} \varpi(u) \sum_{0<|h|\leq H} S_V(K(\bullet,\overline{u}h),X) + Err \\ = & \mathcal{F} - \mathcal{O} + Err. \end{aligned}$$

We take ω to be supported on the classes $u \equiv \overline{p}.l \pmod q$ for pairs of primes $p \sim P,\ l \sim L$ with $P, L < q^{1/2}$.

Bounding $\mathcal{F}$

$$\mathcal{F} = \frac{\log P}{P/2}\frac{\log L}{L/2}\sum_{p,l}\sum_{|h|\leq H}\sum_{n}\lambda_f(1,n)K(n,p\bar{l}h)V(\frac{n}{X}).$$

We apply Poisson on h getting for the h, n sums

$$\frac{H}{q^{1/2}}\sum_{|r|\leq q/H}\sum_{n}\lambda_f(1,n)\widehat{K}(-p\overline{lr})e(\frac{\overline{lr}pn}{q})V(\frac{n}{X})$$

and apply reciprocity

$$e(\frac{\overline{lr}pn}{q}) = e(-\frac{\bar{q}pn}{lr})e(\frac{pn}{qlr}) \approx e(-\frac{\bar{q}pn}{lr}),$$

for $XP = (1/2)q^2L/H$ or $H = q^2L/2XP$.

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We use the automorphy of f through Voronoi summation formula:

$$\sum_n \lambda_f(1,n) e(-\frac{\overline{q}pn}{lr}) V(n/X)$$
$$\approx \frac{X}{(Lq/H)^{3/2}} \sum_{n \ll (Lq/H)^3/X} \lambda_f(n,1) Kl_2(\pm \overline{p}qn; lr)$$

We then Cauchy to smooth out n

$$\sum_{p,l,n,r} \sum \cdots \leq (\sum_{n,r} |\lambda_f(n,1)|^2)^{1/2} (\sum_{n,r} | \sum_{p,l} \widehat{K}(-p\overline{lr}) Kl_2(\pm \overline{p}qn; lr)|^2)^{1/2}$$

and apply Poisson on the resulting n-sum

$$\sum_n Kl_2(\pm \overline{p}_1 qn; l_1 r) Kl_2(\pm \overline{p}_2 qn; l_2 r) V_1(\frac{n}{X})$$

and use the expression of the Fourier transform of the product of Kloosterman sums in terms of Ramanujan sums.

We use the automorphy of f through Voronoi summation formula:

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We obtain that for $L \leq P^4$

$$\mathcal{F} \ll q^{o(1)}\left(\frac{X^{3/2}P}{qL^{1/2}} + X^{3/4}(qPL)^{1/4}\right).$$

and to be non-trivial one need at least that $X \geq q^{1+\eta}$.

Remark

At this stage the only information we have used is that K, not being the trace function of an Artin-Schreier representation, satisfies

$$\|\widehat{K}\|_\infty \ll_{C(\rho)} 1.$$

Bounding $\mathcal{O}$

Recall that

$$\mathcal{O} = \frac{\log P}{P/2}\frac{\log L}{L/2}\sum_{p,l}\sum_{\substack{0<|h|\leq H \\ (h,l)=1}}\sum_{n}\lambda_f(1,n)K(n,p\bar{l}h)V(\frac{n}{X}).$$

This time we immediately Cauchy to smooth n and evaluate

$$\sum_{\substack{p_1,h_1,l_1\\p_2,h_2,l_2}}\sum_n K(n,p_1\bar{l}_1h_1)\overline{K(n,p_2\bar{l}_2h_2)}V(\frac{n}{X})$$
$$=\sum_{x_1,x_2\in\mathbb{F}_q^\times}\nu(x_1)\nu(x_2)\sum_n K(n,x_1)\overline{K(n,x_2)}$$

Since $X\geq q^{1+\eta}$, only the zero contribution in the dual variable survives and the sum becomes

$$\frac{X}{q^{1/2}}\sum_{x_1,x_2\in\mathbb{F}_q^\times}\nu(x_1)\nu(x_2)\frac{1}{q^{1/2}}\sum_{u\in\mathbb{F}_q}K(u,x_1)\overline{K(u,x_2)}$$
$$=\frac{X}{q^{1/2}}\sum_{x_1,x_2\in\mathbb{F}_q^\times}\nu(x_1)\nu(x_2)\frac{1}{q^{1/2}}\sum_{u\in\mathbb{F}_q}\widehat{K}(u,x_1)\overline{\widehat{K}(u,x_2)}$$

This time we immediately Cauchy to smooth n and evaluate

$$\sum_{\substack{p_1,h_1,l_1\\p_2,h_2,l_2}} \sum_n K(n, p_1 \bar{l}_1 h_1) \overline{K(n, p_2 \bar{l}_2 h_2)} V(\frac{n}{X})$$

$$= \sum_{x_1, x_2 \in \mathbb{F}_q^\times} \nu(x_1)\nu(x_2) \sum_n K(n, x_1)\overline{K(n, x_2)}$$

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$$= \frac{X}{q^{1/2}} \sum_{x_1, x_2 \in \mathbb{F}_q^\times} \nu(x_1)\nu(x_2) \frac{1}{q^{1/2}} \sum_{u \in \mathbb{F}_q} \widehat{K}(u, x_1)\overline{\widehat{K}(u, x_2)}$$

Moreover

$$\frac{1}{q^{1/2}} \sum_{u \in \mathbb{F}_q} \widehat{K}(u, x_1) \overline{\widehat{K}(u, x_2)} = L(x_1 - x_2)$$

with

$$L(x) = \frac{1}{q^{1/2}} \sum_{u \in \mathbb{F}_q^\times} |\widehat{K}(u)|^2 e(-\frac{\overline{u}x}{q}) + \frac{1}{q^{1/2}} |\widehat{K}(0)|^2.$$

The second term is no problem.
For the first term, observe that if $|\widehat{K}(u)|^2 = 1$ a.e. (which is the case for $K = \chi$ treated by HN) the first term is a Ramanujan sum hence very small.

In general we have the following elementary:

Lemma

Given $\mu, \nu, L : \mathbb{F}_q \to \mathbb{C}$ *we have*

$$\sum_{x_1, x_2 \in \mathbb{F}_q} \nu(x_1)\nu(x_2)L(x_1 - x_2) \leq q^{1/2}\|\nu\|_2^2 \|\widehat{L}\|_\infty.$$

which is proven by separating x_1, x_2 in $L(x_1 - x_2)$ using the inverse Fourier transform formula and Cauchying.

In the present case we have

$$\widehat{L}(u) = |\widehat{K}(0)|^2 \delta_{u \equiv 0 \,(\mathrm{mod}\, q)} + |\widehat{K}(\bar{u})|^2 \delta_{u \not\equiv 0 \,(\mathrm{mod}\, q)},$$

and (assuming $PHL < q$)

$$\begin{aligned} \|\nu\|_2^2 &= |\{(p_1, h_1, l_1, p_2, h_2, l_2),\ p_1 \bar{l}_1 h_1 \equiv p_2 \bar{l}_2 h_2 \,(\mathrm{mod}\, q)\}| \\ &= |\{(p_1, h_1, l_1, p_2, h_2, l_2),\ p_1 l_2 h_1 = p_2 l_1 h_2\}| = (PHL)^{1+o(1)}. \end{aligned}$$

This yields

$$\mathcal{O} \ll_f q^{o(1)} \|\widehat{K}\|_\infty \frac{qX^{1/2}}{P}.$$

Combining the $\mathcal{F}$ and $\mathcal{O}$ bounds we conclude.

Remark

The only information we have used is that K, not being the trace function of an Artin-Schreier representation, satisfies

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The reason why the proof uses relatively "little" ℓ-adic cohomology ("only" Deligne's Weil II) is because the convexity range is $n \sim q^{3/2}$ while the period of K is q making is possible to apply Poisson to great effect.

Things should become very different if one tries to get X close to or below q.

For instance being able to go below q for $K(n) = Kl_3(n; q)$ would make it possible to evaluate asymptotically the first moment

$$\sum_{\chi \pmod{q}} L(f.\chi, 1/2)$$

and to obtain non-vanishing results for central values of twists: so far this is known only on average over suitable composite moduli $q_1 q_2$ (W. Luo).

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For instance being able to go below q for $K(n) = Kl_3(n;q)$ would make it possible to evaluate asymptotically the first moment

$$\sum_{\chi \pmod{q}} L(f.\chi, 1/2)$$

and to obtain non-vanishing results for central values of twists: so far this is known only on average over suitable composite moduli $q_1 q_2$ (W. Luo).

We expect going below q to be quite challenging:

- for f the $1 \oplus 1 \oplus 1$ Eisenstein series and $K = Kl_3$ this amount to the groundbreaking paper of FI on d_3 in large arithmetic progressions.
- for f the $g \oplus 1$ for g a GL_2 cusp form and $K = Kl_3$ this was worked out by R. Zacharias and this uses crucially bounds for bilinear sums of Kloosterman sums proven by KMS:

$$\sum\sum_{m,n \sim q^{1/2}} \alpha_m \beta_n Kl_k(mn; q) \ll q^{o(1)} \|\alpha\|_2 \|\beta\|_2 q^{1/2-\delta}, \ \delta > 0.$$

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Subconvexity for twists of $\mathrm{GL}_3 \times \mathrm{GL}_2$

Very recently (June 22 2019) P. Sharma posted a detailed draft of a subconvex bound for twists of $\mathrm{GL}_2 \times \mathrm{GL}_3$ L-function:

Theorem

Let φ be a $\mathrm{GL}_3(\mathbb{Z})$ cup form and f be a GL_2-cusp form. One has for $\Re s = 1/2$

$$L(\varphi \times f.\chi, s) \ll q^{3/2-\delta+o(1)},\ \delta > 0.$$

The bound is essentially equivalent to

$$\sum_n \lambda(1,n)\lambda_f(n)\chi(n)V(\frac{n}{q^3}) \ll_{\varphi,f,V} q^{3-\delta+o(1)}.$$

The proof uses

- δ-symbol methods.
- Conductor decreasing trick.
- GL_2 and GL_3-Voronoi.
- Cauchy.
- Poisson (aka GL_1-Voronoi). Some non-zero frequencies contribute here.
- Squareroot cancellation in multivariable exponential sums by summoning the Adolphson-Sperber non-degeneracy criterion.

Excepted for the very last step, the proof does not use that χ is a Dirichlet character (in particular does not use multiplicativity). One can therefore redo the proof with K replaced by a general trace function.

- In the end the most complicated exponential sum one need to face is: for $(l, m, p) \in \mathbb{F}_q^\times$ some parameters (arising from amplification and δ-symbol methods)

$$Z_{\ell,m,p}(v) := \frac{1}{q^{1/2}} \sum_{a \,(\mathrm{mod}\, q)} K(a)\mathrm{Kl}_2(\overline{p}^2 ma; q)\mathrm{Kl}_2(\overline{p}^3 \ell \overline{v} a; q)$$

$$\mathcal{C}_{\ell,m,p,\ell',m',p'}(h; q) := \frac{1}{q^{1/2}} \sum_{v \in \mathbb{F}_q^\times} Z_{l,m,p}(v)\overline{Z_{l',m',p'}(v + \overline{pp'}h)}.$$

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Theorem (KLMS)

If the sheaf $\mathcal{F}$ associated to K does not satisfy any of these conditions

- *For $\lambda \in \mathbb{F}_q^\times - \{1\}$ the geometric monodromy of $\mathcal{F}$ has some quotient isomorphic to $[\times\lambda]^*\mathcal{K}\mathcal{L}_2$.*
- *For some $\lambda \in \mathbb{F}_q^\times - \{1\}$, $\mathcal{F}$ and $[\times\lambda]^*\mathcal{F}$ are geometrically isomorphic.*
- *The local monodromy of $\mathcal{F}$ at ∞ has a slope equal to $1/2$.*

then whenever $h \neq 0 \pmod q$ or $(l, m, p) \neq (l', m', p')$ one has

$$\mathcal{C}_{\ell,m,p,\ell',m',p'}(h; q) \ll 1$$

- The Z function can be obtained from K by a sequence of simple transformations (we assume $\ell = m = p = 1$ for simplicity)

$$K(x) \xrightarrow{\times \mathrm{Kl}_2} L(x) = K(x)\mathrm{Kl}_2(x) \xrightarrow{FT} \widehat{L}(y)$$

$$\xrightarrow{\mathrm{inv}} M(y) := \widehat{L}(y^{-1}) \xrightarrow{FT} \widehat{M}(u) \xrightarrow{\mathrm{inv}} M(u^{-1})$$

  where FT denote the Fourier transform and $\mathrm{inv} : x \to x^{-1}$ the inversion.
- These transformations have geometric analog at the level of sheaves and one can track how the singularities of $\mathcal{F}$ evolve when applying these (the deep but explicit work of Laumon on the local Fourier transform is used there) to see when the two copies of Z correlate.

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Corollary

Whenever K does not satisfy any of the above conditions, one has

$$\sum_n \lambda(1,n)\lambda_f(n)K(n)V(\frac{n}{q^3}) \ll_{\varphi,f,V} q^{3-\delta+o(1)},\ \delta > 0.$$

Remark

The second condition excludes a priori $K=\chi$ however, in that specific case, the conclusion holds for $h \neq 0$ and when $h=0$ the failure is localized along an explicit and small diagonal set and the bound remains valid.

Thank you !