The fourth moment of Dirichlet L-functions along a coset and the Weyl bound

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Joint work with Matthew P. Young

The subconvexity problem

Given π an automorphic form, let $C(\pi)$ be its analytic conductor.
Example: χ a Dirichlet character modulo q and $|\cdot|^{it} : n \mapsto n^{it}$

$$C(\chi.|\cdot|^{it}) = (1 + |t|)q.$$

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Michel-Venkatesh (2010): π on GL_1 or GL_2 with unspecified $\delta > 0$.

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The exponent 3/16 re-occurs often in modern incarnations of these problems (Blomer-Harcos-Michel, Blomer-Harcos, Han Wu).

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$$\sum_{|t_j| \le T} \sum_{m|q} \sum_{\pi \in H_{it_j}(m,1)} L(1/2, \pi \otimes \chi)^3 + \int_{-T}^{T} |L(1/2 + it, \chi)|^6 \ell(t)\, dt \ll T^B q^{1+\varepsilon}.$$

$B < \infty$ unspecified, $\ell(t) = t^2(4 + t^2)^{-1}$. $L(1/2, \pi \otimes \chi) \ge 0$ by Guo.

Fact (Atkin-Li 1978, or use local Langands for GL_2):
If $m \mid q$, χ conductor q, $\pi \in H_{it}(m, \chi^2)$, then $\pi \otimes \overline{\chi} \in H_{it}(q^2, 1)$.

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Theorem (P.-Young (2018))

Let χ be primitive of conductor q cube-free and not quadratic.

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Why did the cube-free hypothesis come up, and how to remove it?

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Result is a reciprocal "dual moment" (P. 2014 in quadratic case)

$$\sum_{t_j} \sum_{m|q} \sum_{\pi \in H_{it_j}(m,\chi^2)} L(1/2, \pi \otimes \overline{\chi})^3 \leftrightarrow \sideset{}{^*}\sum_{\psi \pmod q} |L(1/2, \psi)|^4 g(\chi, \psi)$$

$$g(\chi, \psi) := \sum_{u,v \pmod q} \chi(u) \overline{\chi(u+1)\chi(v)} \chi(v+1) \psi(uv-1).$$

Other examples of dual moments

Motohashi (c. 1995):

$$\int w(t)|\zeta(1/2+it)|^4\,dt \leftrightarrow \sum_{t_j} \sum_{\pi \in H_{it_j}(1,1)} \check{w}(t_j) L(1/2,\pi)^3$$

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See also recent work of Blomer-Khan (2017) and Zacharias (2018).

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In these cases we have by a standard large-sieve type inequality:

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Finishes the proof if q is cube-free.

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So, for α primitive modulo $q = p^3$ need to bound

$$\sideset{}{^*}\sum_{\psi \pmod p} |L(1/2,\psi.\alpha)|^4 g(\chi,\psi) \leq p^{\frac{1}{2}} q \sideset{}{^*}\sum_{\psi \pmod p} |L(1/2,\psi.\alpha)|^4$$

$$\ll \begin{cases} qp^{3+\frac{1}{4}+\varepsilon} & \text{Burgess} \\ q^{2+\varepsilon} p^{\frac{1}{2}} & \text{large sieve} \end{cases}$$

Need: $$\sideset{}{^*}\sum_{\psi \pmod p} |L(1/2,\psi.\alpha)|^4 \ll p^{2.5+\varepsilon}.$$

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Theorem (P.-Young 2019)

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i.e. q^* is the least positive integer such that $q^2 \mid (q^*)^3$.
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By itself, the 4th moment along cosets recovers a Weyl-subconvex result of Heath-Brown (1978) for certain special moduli q.

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Analogous to a result of Iwaniec (1980):

$$\int_T^{T+\Delta} |\zeta(1/2+it)|^4 \, dt \ll \max(\Delta, T^{2/3}) T^{\varepsilon}.$$

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and if $q = p^\beta$ with p odd and $\beta = 2\alpha+1$, $\alpha \geq 1$, then

$$|g(\chi,\psi)| \leq \begin{cases} 2q, & p \nmid \Delta, \\ 0, & p \| \Delta, \\ qp^{1/2}\rho(\frac{\Delta}{p^2}, p^{\alpha-1}), & p^2 | \Delta. \end{cases}$$

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$$\sum_{\substack{h \equiv 0 \pmod{d} \\ h \asymp H}} \left(\sum_{n \asymp N} \tau(n+h)\chi(n+h)\tau(n)\overline{\chi(n)} \right) \ll N(1 + \frac{H}{q})(Nq)^{\varepsilon}$$

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E.g. if $d = p^2$, $q = p^3$, and $h = p^2 k$ then

$$\chi(n+h)\overline{\chi(n)} = \chi(1 + h\overline{n}) = e\left(\frac{\ell_\chi k \overline{n}}{p}\right).$$

Dual moment for 4th moment along cosets

Solve the shifted convolution problem with the Bruggeman-Kuznetsov formula with character $\overline{\eta}^2$ at cusps $0, \infty$ and Poisson summation:

$$\sum_{\psi \pmod{p^2}} |L(1/2, \psi.\alpha)|^4 \qquad \leftrightarrow$$

$$\sum_{\eta \pmod{p}} \eta(\ell_\alpha)\tau(\eta)^3 \sum_{t_j} \sum_{\pi \in H_{it_j}(p,\eta^2)} \lambda_\pi(p) L(1/2, \pi \otimes \overline{\eta})^3.$$

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Apply Hölder with exponents $(4,4,4,4)$ and use a (new) spectral large sieve inequality:

Dual moment for 4th moment along cosets

Solve the shifted convolution problem with the Bruggeman-Kuznetsov formula with character $\overline{\eta}^2$ at cusps $0, \infty$ and Poisson summation:

$$\sum_{\psi \pmod{p^2}} |L(1/2, \psi.\alpha)|^4 \quad \leftrightarrow$$

$$\sum_{\eta \pmod{p}} \eta(\ell_\alpha)\tau(\eta)^3 \sum_{t_j} \sum_{\pi \in H_{it_j}(p,\eta^2)} \lambda_\pi(p) L(1/2, \pi \otimes \overline{\eta})^3.$$

Apply Hölder with exponents $(4,4,4,4)$ and use a (new) spectral large sieve inequality:

$$\sum_{\eta \pmod{q}} \sum_{|t_j| \leq T} \sum_{m|q} \sum_{\pi \in H_{it_j}(m,\eta^2)} |L(1/2, \pi \otimes \overline{\eta})|^4 \ll q^2 T^2 (qT)^\varepsilon.$$