Local Fourier Uniformity

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The main question

Question: Is the multiplicative and additive structure of the integers independent?
Conjectural answer: Yes, up to minor local obstructions.
The precise form of this answer is due to Chowla and Elliott.

Conjecture (Chowla-Elliott)

Let $f_1, f_2, \ldots, f_r$ be real-valued multiplicative functions with $|f_i| \leq 1$. Then, for any distinct $h_1, \ldots, h_r$ there exists a constant $C_{h_1,\ldots,h_r}$ such that ,

$$\frac{1}{x}\sum_{n\leq x} f_1(n+h_1)\ldots f_r(n+h_r) \sim C_{h_1,\ldots,h_r}\prod_{i=1}^{r}\Big(\frac{1}{x}\sum_{n\leq x} f_i(n)\Big).$$

Note that $C_{h_1,\ldots,h_r}$ can be zero (this is in fact the most interesting case!) and in that case we interpret the asymptotic as saying that the left-hand side is $o(1)$.

Disclaimer

1. Until recently the Chowla-Elliott conjecture appeared to be completely out of reach
2. Since 2015 there has been a lot of progress.
3. I am (overly?) optimistic that we will see a resolution within the next 10 years.
4. Since there has been rapid progress there are some patchy "gray areas" which in principle ought to be not there, and we don't understand why they are there. I will therefore try to give a quick survey of the state of the art (nonetheless I will omit some results).

Recent progress (Averaged versions)

There has been substantial recent progress on "averaged" versions of this conjecture.

Theorem (Matomäki-Radziwiłł, 2015)

Let $f_1, f_2 : \mathbb{N} \to \mathbb{R}$ *be multiplicative functions with* $|f_1|, |f_2| \leq 1$*. Then, for any* $H \to \infty$ *arbitrarily slowly with* $x \to \infty$*,*

$$\frac{1}{2H} \sum_{|h| \leq H} \frac{1}{x} \sum_{n \leq x} f_1(n) f_2(n+h) \sim C \prod_{i=1}^{2} \Big(\frac{1}{x} \sum_{n \leq x} f_i(n) \Big).$$

with $C > 0$ *a constant.*

Actually we obtained a description of the behavior of short averages,

$$\sum_{x \leq n \leq x+H} f(n)$$

for almost all $x \in [X, 2X]$. This essentially implies the above statement.

Recent progress (Averaged versions)

Theorem (Matomäki-Radziwiłł-Tao, 2015)

Let $f_1, f_2 : \mathbb{N} \to \mathbb{R}$ be multiplicative functions with $|f_1|, |f_2| \leq 1$. Then, for any $H \to \infty$ arbitrarily slowly with $x \to \infty$,

$$\frac{1}{2H} \sum_{|h| \leq H} \Big| \frac{1}{x} \sum_{n \leq x} f_1(n) f_2(n+h) - C_h \prod_{i=1}^{2} \Big(\frac{1}{x} \sum_{n \leq x} f_i(n) \Big) \Big| = o(1)$$

for some constants $C_h = O(1)$.

In the proof one needs control not only over,

$$\sum_{x \leq n \leq x+H} f(n) \text{ but also over } \sum_{x \leq n \leq x+H} f(n) e(n\alpha)$$

for almost all $x \in [X, 2X]$ and with α fixed. Roughly speaking the case of $\alpha \in \mathbb{Q}$ follows from my result with Matomäki, while the case of $\alpha \notin \mathbb{Q}$ from a version of Vinogradov's method due to Daboussi-Delange-Katai-Bourgain-Sarnak-Ziegler.

Recent progress (Logarithmic versions)

Theorem (Tao, 2015)

Let $f_1, f_2 : \mathbb{N} \to \mathbb{R}$ be multiplicative functions. Then, given h, there exists a C_h (possibly zero) such that as $x \to \infty$,

$$\sum_{n \le x} \frac{f_1(n) f_2(n+h)}{n} \sim C_h \log x$$

1. The proof depends on the previous two results and the "entropy decrement argument". The latter is very sensitive to f_1, f_2 being non-lacunary and mostly of size ≈ 1 (in absolute value).
2. The logarithmic averaging allows one to introduce a third variable, which allows to make use of the previous "averaged" results.

Recent progress (Logarithmic versions)

Let $\lambda(n)$ denote the Liouville function (i.e $\lambda(n) = (-1)^{\Omega(n)}$ where $\Omega(n)$ is the number of prime factors of n counted with multiplicity).

Theorem (Tao-Teräväinen, 2017)

*Let k be **odd**. Then, for any $h_1, \ldots, h_k$ distinct, as $x \to \infty$,*

$$\sum_{n \le x} \frac{\lambda(n+h_1)\ldots\lambda(n+h_k)}{n} = o(\log x)$$

1. In some sense this is a "parity trick". The proof depends on the fact that k is odd and $\lambda(p) = -1$.
2. The number theoretic content is much lighter (in comparison with the $k = 2$ case). For example for $k = 3$ one only needs cancellations in

   $$\sum_{n \le x} \lambda(n) e(n\alpha).$$

   which was proven by Davenport using ideas of Vinogradov.

Major questions

1. Can we show that for **any** $k \geq 2$, and any distinct $h_1, \ldots, h_k$,

$$\sum_{n \leq x} \frac{\lambda(n+h_1)\ldots\lambda(n+h_k)}{n} = o(\log x) \ ? \qquad (1)$$

   This would, among other things, settle Sarnak's conjecture in logarithmic form.

2. Can we establish (1) without the logarithmic weights? (Even only for $k = 2$?)

The roadmap towards 1. is at the moment much more clear than the one towards 2. (There is some progress on 2. by Tao-Teräväinen, but in my opinion there is still no clear roadmap for 2.)

Fourier uniformity

Tao showed that in order to obtain

$$\sum_{n \le x} \frac{\lambda(n+h_1)\dots\lambda(n+h_k)}{n} = o(\log x) \qquad (2)$$

for every $k \ge 1$ it is necessary to establish the following

Conjecture (Local Fourier Uniformity conjecture)

Let $k \ge 1$ be given. For $H \to \infty$ arbitrarily slowly with $X \to \infty$,

$$\int_1^X \sup_{\substack{P \in \mathbb{R}[X] \\ deg\ P = k}} \Big| \sum_{x \le n \le x+H} \lambda(n) e(P(n)) \Big| \cdot \frac{dx}{x} = o(\log X)$$

1. In reality one also needs to establish the above for nilsequences. This is then also sufficient for (2).
2. As far as I can see the measure $\frac{dx}{x}$ appears to give no real advantage compared to dx. We will therefore consider the measure dx instead.

Previous results

Previous results are rather unsatisfactory:

Theorem (The $k = 0$ case, MRT, 2015)

Let $H \to \infty$ arbitrarily slowly with $X \to \infty$. Then,

$$\sup_{\alpha} \int_1^X \Big| \sum_{x \le n \le x+H} \lambda(n) e(n\alpha) \Big| dx = o(HX).$$

The above is contained in earlier progress on "averaged Chowla".

Theorem (The $k = 1$ case for $H > X^{5/8}$, Zhan, 1991)

Let $\varepsilon > 0$. Let $H > X^{5/8+\varepsilon}$. Then, for $X \to \infty$,

$$\sum_{X \le n \le X+H} \lambda(n) e(n\alpha) = o(H).$$

The above uses Heath-Brown's identity. The method hits a hard limit at $H = \sqrt{X}$ because it is based on Dirichlet polynomial techniques.

New results

Theorem (The $k = 1$ case for $H > X^{\varepsilon}$, MRT, 2018)

Let $\varepsilon > 0$ be given. Then for $H > X^{\varepsilon}$ as $X \to \infty$,

$$\int_1^X \sup_{\alpha} \Big| \sum_{x \le n \le x+H} \lambda(n) e(n\alpha) \Big| dx = o(HX).$$

1. One can prove a variant for general multiplicative functions (even unbounded), as long as they are not $\chi(n)n^{iT}$ pretentious (with $|T| \ll X^2/H$ and χ of condutor $O(1)$).
2. It appears to be possible to lower H to $H > \exp((\log X)^{1/2+\varepsilon})$.

Work in progress

Progress at the American Institute for Mathematics :

1. After conversations with Teräväinen we understood that our proof extends fairly easily to polynomials. Thus we can get, for any $k \in \mathbb{N}$,

$$\int_1^X \sup_{\substack{P \in \mathbb{R}[X] \\ \deg P = k}} \Big| \sum_{x \leq n \leq x+H} \lambda(n) e(P(n)) \Big| dx = o(HX)$$

   for $H > \exp((\log X)^{1/2+\varepsilon})$.

2. As part of an on-going large AIM collaboration it appears that we can also handle the case of nilsequences for $H > \exp((\log X)^{1/2+\varepsilon})$.
3. So the current bottle-neck towards a full resolution of Chowla-Elliott in logarithmic form is the reduction of H from $\exp(\sqrt{\log X})$ to H growing arbitrarily slowly.

Consequences of Fourier Uniformity

A good illustration of the content of Fourier uniformity for $k = 1$ is:

Corollary (MRT, 2018)

Let $\varepsilon > 0$. Let $\alpha(n)$ and $\beta(n)$ be two **arbitrary** *sequences of complex numbers with $|\alpha(i)| \leq 1$ and $|\beta(i)| \leq 1$ for every $i \geq 1$. Then, for $H > X^{\varepsilon}$ as $X \to \infty$,*

$$\sum_{|h| \leq H} \sum_{n \leq X} \lambda(n)\alpha(n+h)\beta(n+2h) = o(HX).$$

Proof sketch: The above is roughly, H^{-1} times

$$\int_X^{2X} \int_0^1 \Big(\sum_{x \leq n \leq x+H} \lambda(n) e(n\alpha) \Big) \Big(\sum_{x \leq n \leq x+H} \alpha(n) e(n\alpha) \Big) \Big(\sum_{x \leq n \leq x+H} \beta(n) e(-2n\alpha) \Big)$$

Taking the supremum over α in the sum over $\lambda(n)$ and using Cauchy-Schwarz and Plancherel on the remaining two trigonometric polynomials we can bound the inner integral in absolute value by

$\sup_{\alpha} \Big| \sum_{x \leq n \leq x+H} \lambda(n) e(n\alpha) \Big| \cdot H$ and the result follows from Fourier uniformity

Consequences of Fourier uniformity

If the sequences $\alpha(n)$, $\beta(n)$ admit tight sieve majorants then we can allow them to be unbounded.

Corollary (MRT, 2018)

Let $\varepsilon > 0$ and $H > X^{\varepsilon}$. Then, as $X \to \infty$,

$$\sum_{|h| \leq H} \sum_{n \leq X} \lambda(n)\Lambda(n+h)\Lambda(n+2h) = o(HX). \tag{3}$$

1. We are not able to establish that,

$$\sum_{|h| \leq H} \sum_{n \leq X} \Lambda(n+h)\Lambda(n+2h) \sim HX \tag{4}$$

2. (4) is equivalent to a prime number theorem in almost all intervals of length H.
3. The best known result for (4) remains $H > X^{1/6+\varepsilon}$ due to Huxley (Zaccagnini showed using ideas of Heath-Brown that ε can tend to zero from the negative side).
4. In (3) all the heavy-lifting is done by the Liouville function

Consequences of Fourier uniformity : Work in progress

It is possible to generalize the result on Fourier uniformity to unbounded functions. This then has consequences for triple correlations of divisor functions (or more general coefficients of high-rank automorphic forms)

Corollary (MRT, 2019+)

Let $\varepsilon > 0$. Let $k, \ell, m \geq 1$. Let $H > X^{\varepsilon}$. Then, as $X \to \infty$,

$$\sum_{|h| \leq H} \sum_{n \leq X} d_k(n) d_\ell(n+h) d_m(n+2h) \sim CHX(\log X)^{k+\ell+m-3}.$$

with $C > 0$ a constant.

1. For $H = X$ this follows from work of Mathiessen.
2. In the case $k = \ell = m = 2$ Blomer obtained an asymptotic for $H > X^{1/3+\varepsilon}$ using spectral methods of automorphic forms.
3. It is striking that one can go further by using only multiplicativity (and the Littlewood zero-free region).
4. In many applications of spectral methods multiplicativity is never used (this is especially clear when variants apply to half-integral weight forms).

Sketch of proof of Fourier Uniformity for $k = 1$

1. I will now sketch the ideas going into the proof of the $k = 1$ case, with $H > X^{\varepsilon}$,

$$\int_1^X \sup_{\alpha} \Big| \sum_{x \le n \le x+H} \lambda(n) e(n\alpha) \Big| dx = o(HX).$$

2. We will suppose that Fourier Uniformity fails and reach a contradiction.
3. Arguing by contradiction we assume that there exists an $\eta > 0$ and a collection $\mathcal{I}$ of $\gg X/H$ disjoint intervals of length H, contained in $[1, X]$, such that, for each $I \in \mathcal{I}$,

$$\Big| \sum_{n \in I} \lambda(n) e(n\alpha_I) \Big| \ge \eta H$$

for some $\alpha_I \in \mathbb{R}$.

Overall strategy

1. We will gain increasing control over the frequencies α_I.
2. We will first show, that the multiplicativity of λ forces that for a positive proportion of I, $e(n\alpha_I) \approx n^{iT_I}$ for some $|T_I| \ll X^2/H$ depending on I (in reality $e(n\alpha_I) \approx e(na/q)n^{iT_I}$ with $q \ll X^{\varepsilon}$ but for simplicity let's ignore the q-aspect).
3. Importantly the values T_I and T_J are sometimes related. If $\frac{I}{p} \cap \frac{J}{q} \neq \emptyset$ for some primes $p, q \in [P, 2P]$ then $T_I = T_J$. P is roughly of size H^{ε}. This is also a consequence of multiplicativity.
4. We then show that the above relationships between the T_I's imply that there is a global T such that $T_I = T$ for a positive proportion of the interval I's. This step uses cancellations in $\sum_{H \leq p \leq 2H} p^{it}$.
5. Therefore for a positive proportion of intervals I,

$$\Big|\sum_{n \in I} \lambda(n) e(n\alpha_I)\Big| \approx \Big|\sum_{n \in I} \lambda(n) n^{iT}\Big| \gg H$$

and this is ruled out by my result with Matomäki.

Overall strategy

1. The first two steps are "local" : they use the large sieve, the Turan-Kubilius inequality and multiplicativity at the primes $p \leq H$.
2. The second step is "global" : it uses cancellations in $\sum_{H \leq p \leq 2H} p^{it}$.

I will start with the second step which is easier to explain and maybe also a bit more surprising.

The "global" step

1. We assume that there is a P, of size roughly H^{ε} and a set of $\gg X/H$ disjoint intervals I, J and real numbers T_I, T_J, such that whenever

$$\frac{I}{p} \cap \frac{J}{q} \neq \emptyset$$

for some $p, q \in [P, 2P]$ then we have $T_I = T_J$. We want to show that there exists a T such that for a positive proportion of intervals I we have $T_I = T + O(X/H)$ (this is the morally the same as $T_I = T$ because if $T_I = T + O(X/H)$ then $\sum_{n \in I} \lambda(n) n^{iT_I} \approx \sum_{n \in I} \lambda(n) n^{iT}$)
2. Given V let $\mathcal{A}(V) = \#\{I : T_I = V + O(X/H)\}$. Then,

$$\frac{X}{H}\Big(\frac{P}{\log P}\Big)^2 \ll \sum_{V} \sum_{\substack{I, J \in \mathcal{A}(V) \\ p, q \in [P, 2P] \\ \frac{I}{p} \cap \frac{J}{q} \neq \emptyset}} 1$$

The "global" step

1. $\frac{I}{p} \cap \frac{J}{q} \neq \emptyset$ implies $|\log x_I - \log x_J + \log p - \log q| \ll \frac{H}{X}$ with x_I , x_J are end-points of I , J.
2. Therefore, we can re-write the previous expression as

$$\frac{X}{H}\Big(\frac{P}{\log P}\Big)^2 \ll \sum_{V} \sum_{\substack{I,J\in\mathcal{A}(V) \\ p,q\in[P,2P]}} \psi\Big(\frac{X}{H}\Big(\log x_I - \log x_J + \log p - \log q\Big)\Big)$$

   for some smooth function ψ and with x_I, x_J the end-points of I, J.
3. Opening ψ as a Fourier transform we then get,

$$\frac{X}{H}\cdot\Big(\frac{P}{\log P}\Big)^2 \ll \frac{H}{X}\sum_{V}\int_{|t|\leq X/H}\Big|\sum_{I\in\mathcal{A}(V)} x_I^{it}\Big|\Big|\sum_{J\in\mathcal{A}(V)} x_J^{it}\Big|\Big|\sum_{p\in[P,2P]} p^{it}\Big|^2 dt.$$

   and where the x_I are H-spaced.

The "global" step

We now analyze,

$$\frac{H}{X}\int_{|t|\leq X/H}\Big|\sum_{I\in\mathcal{A}(V)} x_I^{it}\Big|\Big|\sum_{J\in\mathcal{A}(V)} x_J^{it}\Big|\Big|\sum_{p\in[P,2P]} p^{it}\Big|^2 dt$$

1. On the one hand, the small values of t (say $t = O(1)$) contribute

$$\frac{H}{X}\sum_{V}\#\mathcal{A}(V)^2\cdot\Big(\frac{P}{\log P}\Big)^2 \ll \sup_{V}\#\mathcal{A}(V)\cdot\Big(\frac{P}{\log P}\Big)^2$$

2. On the other hand when t is large, we have cancellations in the Dirichlet polynomial over primes. Using this and the mean-value theorem on the Dirichlet polynomials over x_I we see that the contribution of the large values of t is

$$\ll_A \frac{H}{X}\sum_{V}\frac{X}{H}\#\mathcal{A}(V)\cdot\frac{P^2}{\log^A P} \ll \frac{X}{H}\cdot\frac{P^2}{\log^A P}.$$

for any given $A > 0$.

The "global" step

1. Combining the above two bounds we obtain,

$$\frac{X}{H}\Big(\frac{P}{\log P}\Big)^2 \ll \sup_V \#\mathcal{A}(V) \cdot \Big(\frac{P}{\log P}\Big)^2.$$

   Therefore there exists a V such that for a positive proportion of I's we have $T_I = V + O(X/H)$, which is what we wanted to prove.
2. This argument has an ergodic flavor
3. Cancellations in the prime sum are not strictly necessary; one can instead take high moments of the sum over primes and consider level sets. This allows us to go to $H > \exp(\sqrt{\log X})$.

The "local" step

I will now explain the ideas in the "local" step. We are trying to show that if

$$\Big|\sum_{n\in I}\lambda(n)e(n\alpha_I)\Big| \gg H$$

for a positive proportion of intervals I, then, for a positive proportion of I,

$$\Big|\sum_{n\in I}\lambda(n)e(n\alpha_I)\Big| \approx \Big|\sum_{n\in I}\lambda(n)n^{iT_I}\Big|$$

and moreover

1. We have $|T_I| \ll X^2$
2. If $\frac{I}{p}\cap\frac{J}{q}\neq\emptyset$ for $p, q\in[P, 2P]$ then $T_I = T_J$.

(In reality there is also a twist by $e(na/q)$ that we are ignoring for simplicity, and 2. won't hold for all $p, q\in[P, 2P]$ but for a positive proportion subset)

The local step : The main tools

Our two main tools are :

1. (The large sieve) If

$$\sup_{J \cap I \neq \emptyset} \left| \sum_{n \in J} \lambda(n) e(n\alpha) \right| \geq \eta H \tag{5}$$

   then α is within $O(1/H)$ of a set of at most $O_\eta(1)$ elements. This morally allows us to assume that (up to perturbation by $O(1/H)$ and modulo 1) α is uniquely determined, and $\alpha_J = \alpha_I$ for any J such that $J \cap I \neq \emptyset$.

2. (Turan-Kubilius) For "most" primes p

$$\left| \sum_{n \in I} \lambda(n) e(n\alpha_I) \right| \approx p \left| \sum_{n \in I/p} \lambda(np) e(np\alpha_I) \right| = p \left| \sum_{n \in I/p} \lambda(n) e(np\alpha_I) \right|.$$

   Let us pretend this holds for all primes $p \in [P, 2P]$.

Consequently if I and J are such that $\frac{I}{p} \cap \frac{J}{q} \neq \emptyset$ then $p\alpha_I \equiv q\alpha_J + O(P/H) \pmod 1$. This is the main relation that we will use.

The "local step"

1. The fact that we only know $p\alpha_I \equiv q\alpha_J + O(P/H)$ modulo 1 is very restrictive.
2. It would be great if we could divide by q as having $(p/q)\alpha_I \equiv \alpha_J + O(1/H) \pmod 1$ is equivalent to $p\alpha_I \equiv q\alpha_J + O(P/H) \pmod q$ and thus contains more information.
3. One natural idea is that we can shift the α_J's in whatever way we wish by integers. Perhaps we can show that there exists $\widetilde{\alpha}_J = \alpha_J + k_J$ with $k_J \in \mathbb{Z}$ such that

$$\frac{p}{q}\widetilde{\alpha}_I = \widetilde{\alpha}_J + O(1/H) \pmod 1$$

whenever $\frac{I}{p} \cap \frac{J}{q} \neq \emptyset$ and $p, q \in [P, 2P]$.

4. We were not quite able to prove that (only something weaker). Nonetheless let us pretend that we have the above relationship as it allows to succintly give the flavor of the rest of the proof, eliminating a lot of somewhat overwhelming technicalities but keeping the overall structure.

The "local" step

Let's assume that we have $\frac{p}{q}\alpha_I \equiv \alpha_J + O(1/H) \pmod{1}$ whenever $\frac{I}{p} \cap \frac{J}{q} \neq \emptyset$ for $p, q \in [P, 2P]$. Pick $k \in 2\mathbb{N}$ such that $P^k \approx X$.

1. An application of Hölder (actually Sidorenko's conjecture) shows that for a positive proportion of I there exists many (p_i, q_i) with $i = 1, \ldots, k$ and intervals $I = I_1, I_2, \ldots, I_{k+1} = I$ such that,

$$\frac{p_i}{q_i} \cdot \alpha_{I_{i+1}} \equiv I_i + O(1/H) \pmod{1} \ , \ \frac{q_i}{p_i} I_{i+1} \cap I_i \neq \emptyset.$$

2. Iterating along this cycle we obtain,

$$\frac{p_1}{q_1} \cdots \frac{p_k}{q_k} \alpha_I \equiv \alpha_I + \left(\frac{1}{H}\right) \pmod{1} \ , \ \frac{q_1 \ldots q_k}{p_1 \ldots p_k} I \cap I \neq \emptyset. \qquad (6)$$

3. $\frac{q_1 \ldots q_k}{p_1 \ldots p_k} I \cap I \neq \emptyset$ implies that $|p_1 \ldots p_k - q_1 \ldots q_k| \ll X^{\varepsilon}$ and so multiplying (6) by $p_1 \ldots p_k$ we get

$$(p_1 \ldots p_k - q_1 \ldots q_k)\alpha_I \equiv O\Big(\frac{P^k}{H}\Big) \pmod{Q} \ , \ Q = p_1 \ldots p_k$$

with Q large ($\gg X$). Since there are many choices for p_i, q_i we can also ensure that $p_1 \ldots p_k - q_1 \ldots q_k \neq 0$.

The “local” step

1. Since $P^k/H = o(Q)$ we conclude that α_I can be written as,

$$\alpha_I = \frac{a}{q} \cdot Q + \frac{T_I}{x_I} \tag{7}$$

   for some $T_I \ll X \cdot P^k/H \ll X^2/H$, with x_I the end-point of I and $q \ll X^{\varepsilon}$. In particular for $n \in I$, $e(n\alpha_I) \approx e(na/q)n^{iT_I}$.
2. For simplicity let us suppose that $q = 1$.
3. Plugging (7) into $p\alpha_I \equiv q\alpha_J + O(P/H) \pmod 1$ for $\frac{I}{p} \cap \frac{J}{q} \neq \emptyset$, shows that whenever $\frac{I}{p} \cap \frac{J}{q} \neq \emptyset$ we have $T_I = T_J + O(X/H)$.
4. Having $T_I = T_J + O(X/H)$ is for us morally the same as $T_I = T_J$ since if $T_I = T_J + O(X/H)$ then,

$$\Big| \sum_{n \in I} \lambda(n) n^{iT_I} \Big| \approx \Big| \sum_{n \in I} \lambda(n) n^{iT_J} \Big|$$

5. This concludes the “local” step (although we lied a lot to make the presentation simpler!)

Thank you

Thank you!