Integral Factorial Ratios

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The Problem: Classify Integral Factorial Ratios

Find all positive integers $a_1, \ldots, a_K, b_1, \ldots, b_L$ such that for all $n \in \mathbb{N}$

$$\frac{(a_1 n)!(a_2 n)! \cdots (a_K n)!}{(b_1 n)! \cdots (b_L n)!} \text{ is an integer,}$$

with

$$\sum_{i=1}^{K} a_i = \sum_{j=1}^{L} b_j.$$

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Can assume:

$$a_i \neq b_j$$

$$\text{g.c.d.}(a_1, \ldots, a_K, b_1, \ldots, b_L) = 1$$

$$L > K$$

$$L = K + D \qquad D = \text{“height”}$$

The case $L = K + 1$

Three infinite families:

$$K = 1, L = 2: \qquad \frac{((a+b)n)!}{(an)!(bn)!} = \binom{(a+b)n}{an}, \qquad (a,b) = 1$$

$$K = 2, L = 3: \qquad \frac{(2an)!(2bn)!}{(an)!(bn)!((a+b)n)!}, \qquad (a,b) = 1$$

$$K = 2, L = 3: \qquad \frac{(2an)!(bn)!}{(an)!(2bn)!((a-b)n)!}, \qquad a > b, (a,b) = 1$$

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Fifty two sporadic examples! 29 examples with $K = 2, L = 3$; 21 examples with $K = 3, L = 4$; 2 examples with $K = 4, L = 5$.

$$\frac{30,\ 1}{15,\ 10,\ 6}; \qquad \frac{30,\ 9,\ 5}{18,\ 15,\ 10,\ 1}; \qquad \frac{24,\ 9,\ 6,\ 4}{18,\ 12,\ 8,\ 3,\ 2}$$

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Theorem established by Bober (2007), following Rodriguez Villegas (2007) and Beukers & Heckman (1989). Vasyunin (2002).

Landau's criterion

Integrality of

$$\frac{(a_1 n)! \cdots (a_K n)!}{(b_1 n)! \cdots (b_L n)!}$$

for all $n \in \mathbb{N}$ equivalent to:

$$f(x; \mathbf{a}, \mathbf{b}) = \sum_{i=1}^{K} \lfloor a_i x \rfloor - \sum_{j=1}^{L} \lfloor b_j x \rfloor \geq 0$$

for all $x \in \mathbb{R}$.

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Explains why can assume primitive: $\gcd(a_1, \ldots, a_K, b_1 \ldots b_L) = 1$.

Apart from some rational numbers: $(\lfloor -x \rfloor = -\lfloor x \rfloor - 1)$

$$f(-x; \mathbf{a}, \mathbf{b}) = (L - K) - f(x; \mathbf{a}, \mathbf{b}).$$

Can assume $D = L - K > 0$, and must have

$$f(x; \mathbf{a}, \mathbf{b}) \in \{0, \ldots, D\}.$$

Chebyshev's example

$$\frac{(30n)!n!}{(15n)!(10n)!(6n)!} \in \mathbb{N}.$$

Used to prove: for large x

$$0.9212\ldots\frac{x}{\log x} \le \pi(x) \le 1.1055\ldots\frac{x}{\log x}.$$

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Idea: show that the power of any prime dividing ratio is non-negative.
Reduces to needing: for any $x \geq 0$

$$f(x) = \lfloor 30x \rfloor + \lfloor x \rfloor - \lfloor 15x \rfloor - \lfloor 10x \rfloor - \lfloor 6x \rfloor \geq 0$$

Equivalent to:

$$f(x) = \{15x\} + \{10x\} + \{6x\} - \{30x\} - \{x\} \in \{0, 1\}.$$

$$\lfloor mx \rfloor = \lfloor x \rfloor + \lfloor x + 1/m \rfloor + \lfloor x + 2/m \rfloor + \ldots + \lfloor x + (m-1)/m \rfloor$$

$$f(x) = \sum \lfloor x + \text{blue} \rfloor - \sum \lfloor x + \text{red} \rfloor$$

$$0,\ \frac{1}{30},\ \frac{1}{5},\ \frac{7}{30},\ \frac{1}{3},\ \frac{11}{30},\ \frac{2}{5},\ \frac{13}{30},\ \frac{1}{2},\ \frac{17}{30},\ \frac{3}{5},\ \frac{19}{30},\ \frac{2}{3},\ \frac{23}{30},\ \frac{4}{5},\ \frac{29}{30}$$

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$$\frac{(x^{30}-1)(x-1)}{(x^{15}-1)(x^{10}-1)(x^6-1)}$$

Roots of numerator and denominator interlace.

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Roots of numerator and denominator interlace.
Rodriguez Villegas: Algebraic hypergeometric function (polynomial of degree 483840)

$$\sum_{n=0}^{\infty} \frac{(30n)!n!}{(15n)!(10n)!(6n)!} x^n = {}_8F_7[\text{blue}; \text{red} \ | x]$$

Interlacing fractions

Problem: Find finite multi-sets (of equal cardinality) Red and Blue of fractions in $[0,1)$ such that (i) Red contains D zeros, (ii) for all $x \in [0,1)$

$$0 \leq \#(\text{Red} \leq x) - \#(\text{Blue} \leq x) \leq D,$$

and (iii) if a fraction appears in one of the sets then all reduced fractions with the same denominator also appear in that set.

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$D = 1$ case: Beukers & Heckman solve more general problem.

Don't require all reduced fractions with the same denominator.

Instead: for all n coprime to denominators of fractions, want n Red and n Blue to interlace.

"Integrality" (apart from finitely many primes in denominator) of

$$\prod_{j=1}^{n} \prod_{i=1}^{k} \frac{(j+\alpha_i)}{(j+\beta_i)}, \quad \text{for all } n \in \mathbb{N}.$$

Reformulating the problem

Saw-tooth function:

$$\psi(x) = 1/2 - \{x\} = \sum_{n \neq 0} \frac{e^{2\pi i n x}}{2\pi i n}.$$

Odd function with mean 0 and periodic with period 1.

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$\mathfrak{a} = [a_1, a_2, \ldots, a_N]$ list of N non-zero integers.

Length: $\ell(\mathfrak{a}) = N$. Sum: $s(\mathfrak{a}) = a_1 + \ldots + a_N$.

Assume non-degenerate: a and $-a$ cannot both be in the list.

Associated 1-periodic function and its L^2–norm:

$$\mathfrak{a}(x) = \sum_{i=1}^{N} \psi(a_i x), \qquad N(\mathfrak{a}) = \int_0^1 \mathfrak{a}(x)^2 dx.$$

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$$\int_0^1 \psi(ax)\psi(bx)dx = \frac{1}{12}\frac{(a,b)^2}{ab}$$

$$N(\mathfrak{a}) = \frac{1}{12}\sum_{i,j=1}^{N} \frac{(a_i,a_j)^2}{a_i a_j} = N(k\mathfrak{a}).$$

Connection with integral factorial ratios

Associate to

$$\frac{(a_1 n)! \cdots (a_K n)!}{(b_1 n)! \cdots (b_L n)!}$$

the list

$$\mathfrak{a} = [a_1, \ldots, a_K, -b_1, \ldots, -b_L], \qquad \text{with } s(\mathfrak{a}) = 0.$$

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Integrality of factorial ratio equivalent to:

$$\mathfrak{a}(x) \in \{-D/2, -D/2 + 1, \ldots, D/2\}.$$

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Especially nice for $D = 1$: $\mathfrak{a}(x) = -1/2$ or $1/2$.
Characterized by the norm: Find lists $\mathfrak{a}$ with

$$\ell(\mathfrak{a}) \text{ odd}, \quad s(\mathfrak{a}) = 0, \text{ and}, \quad N(\mathfrak{a}) = 1/4.$$

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For larger D, the norm alone is not sufficient.
Necessary condition for integrality:

$$N(\mathfrak{a}) \leq D^2/4.$$

Results

Idea: obtain lower bounds for the norms of lists.

$$G(N) = \inf\{N(\mathfrak{a}) : \mathfrak{a} \text{ primitive, non-degenerate list of length } N\}.$$

Theorem:

$$G(N) \sim \frac{\pi^2 e^{-2\gamma}}{72} \frac{N}{(\log\log N)^2}$$

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Explicitly:

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Corollary: When $D = 1$, no factorial ratios with $K + L \geq 11$. When $D = 2$, no factorial ratios with $K + L \geq 82$, and only finitely many factorial ratios with $K + L \geq 76$.

Refines work of Bell & Bober ($D = 1$ bound 112371), and Schmerling ($D = 1$ bound 43; $D = 2$, bound 202).

Corollary: For large D,

$$K + L \leq (1 + o(1)) \frac{18e^{2\gamma}}{\pi^2} D^2 (\log \log D)^2.$$

Refines Bell & Bober (Bombieri & Bourgain, unpublished): $K + L \ll D^2 (\log D)^2$.

Schmerling: $K + L \ll D^2 (\log \log D)^2$.

Truth: Maybe linear in D??

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Theorem: The points $(a_1, \ldots, a_K, b_1, \ldots, b_L) \in \mathbb{R}^{K+L}$ corresponding to integral factorial ratios lie on finitely many vector subspaces of $\mathbb{R}^{K+L}$ of dimension at most $3D^2 - 1$.
At most $3D^2 - 1$ parameter families of integral factorial ratios.
Sharp for $D = 1$. At most 11 parameter families for $D = 2$.
No known non-trivial examples of 4 parameter families with $D = 2$.
Right answer: at most $2D$ parameter families??

Partial progress & Examples for $D \geq 2$

Trivial way to construct factorial ratios with height 2: multiply two height 1 examples.

Definition: Factorial ratio $\mathfrak{a}$ of height D is reducible if $\mathfrak{a} = \mathfrak{b} + \mathfrak{c}$ where $\mathfrak{b}$ and $\mathfrak{c}$ correspond to integral factorial ratios of smaller height. Otherwise, $\mathfrak{a}$ is called irreducible.

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Examples: 1. Multinomial coefficients are reducible

$$[a+b+c, -a, -b, -c] = [a+b+c, -(a+b), -c] + [a+b, -a, -b].$$

2. $b > 7a > 0$

$$[3a, -a, -9a, 2b, -b, 7a-b] \text{ is reducible}$$

$$= a[3, 14, -1, -7, -9] + [7a, 2b, -14a, -b, 7a-b].$$

Can be tricky to see if reducible or not!

3. Wider: Masters thesis ETH

$$[3a, 3b, -a, -b, -(a+b), -(a+b)]$$

is an irreducible integral factorial ratio.

Theorem: $\mathfrak{a}$ primitive list with height 2, $s(\mathfrak{a}) = 0$ and

$$N(\mathfrak{a}) \leq 1/3 + \delta.$$

Apart from finitely many lists, $\mathfrak{a}$ lies in one of 28 families — 2 three parameter families, and the rest 2 parameter.
16 of these families are reducible, and the remaining 12 are irreducible.
All of these families give examples of integral factorial ratios.

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16 of these families are reducible, and the remaining 12 are irreducible.
All of these families give examples of integral factorial ratios.
Significance of 1/3: approximately the norm of a generic multinomial $[a + b + c, -a, -b, -c]$.
Possibly every list with height 2, and norm below 1/3 gives an integral factorial ratio. Explanation?
Examples: $[2a, 2b, 6(a + b), -a, -4a, -b, -4b, -3(a + b)]$,

$$[6a, 2b, 3b, -2a, -3a, -b, -6b, 2b - a].$$

Non-example: $[3a, 18a, -a, -9a, -b, -11a + b]$. Typical norm: $37/108 = 0.34259\ldots$.

Definition: A list $\mathfrak{b} = [b_1, \ldots, b_k]$ is monotone if

$$\sum_{i=1}^{k} \lfloor b_i x \rfloor \text{ is monotone.}$$

Examples: $[1, -k]$, $[1, -2, -3]$, $[1, -2, -k, 2k]$ (for odd k), $[1, -2, k]$ (for even k) are all monotone.

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Theorem: $\mathfrak{a}$ and $\mathfrak{b}$ primitive lists.
$\mathfrak{b}$ monotone. $s(\mathfrak{a})$, $s(\mathfrak{b})$ non-zero and coprime.
Suppose $s(\mathfrak{b})\mathfrak{a} + (-s(\mathfrak{a}))\mathfrak{b}$ is an integral factorial ratio of height D.
Then, lists of height $D+1$ in the family

$$a\mathfrak{a} + b\mathfrak{b} + (-as(\mathfrak{a}) - bs(\mathfrak{b}))[1]$$

are integral factorial ratios.

Example: $\mathfrak{a} = [6, -2, -3]$, $\mathfrak{b} = [6, -1]$.
$s(\mathfrak{a}) = 1$, $s(\mathfrak{b}) = 5$; $\qquad 5s(\mathfrak{a}) - \mathfrak{b} = [30, 1, -6, -10, -15]$.
$[6a, -2a, -3a, b, -6b, -(a-5b)]$ integral factorial ratio height 2.

Found 50+ irreducible two parameter families with height 2.
Many more 2 parameter examples found by Rodrigo Angelo.
E.g. All but one of the 52 sporadic examples of height 1 extend to irreducible 2 parameter families of height 2.
Angelo: example of length 18

$$\frac{48, 28, 22, 19, 18, 8, 3, 2}{38, 25, 24, 16, 14, 11, 9, 6, 4, 1}$$

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Askey (1986): Advanced *Monthly* problem:

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Patruno: variations of this "advanced" problem seem to appear endlessly ... The following should put this tired old problem to rest.

Case of Macdonald–Morris constant term conjectures. (Zeilberger)
Number of examples of integral factorial ratios (related also to Selberg's integral).
More examples (with combinatorial proofs) by Gessel.

The key idea: k–separated lists

Theorem (weak form): Apart from the three infinite families, there are only finitely many integral factorial ratios with height 1.

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Key Definition: Primitive list $\mathfrak{a}$ of length N, and integer $k \geq 2$. $\mathfrak{a}$ is k–separated of type (ℓ, m) if there are two primitive lists $\mathfrak{b}$ and $\mathfrak{c}$ with lengths $1 \leq \ell \leq m$ and $\ell + m = N$ such that:

1. There are two coprime integers B and C such that

$$\mathfrak{a} = B\mathfrak{b} + C\mathfrak{c} \qquad \text{(concatenate the two lists).}$$

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3. Say $k|B$ (analogously if $k|C$). Then for any $kb \in B\mathfrak{b}$ and any $c \in C\mathfrak{c}$ one has

$$(kb, c) = (b, c).$$

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Examples: (1) $[30, 1, -15, -10, -6] = 5 \times [6, -3, -2] + [1, -6]$
(2) $[1, -2p, 2p^2] = [1, -2p] + 2p^2[1] = [1] + (-2p)[1, -p]$
(3) Any primitive list containing a multiple of p is p–separated.

Finiteness of lists that are at most k–separated

If $\mathfrak{a}$ (primitive of length N) contains a multiple of $p^{(r-1)(N-1)+1}$ then $\mathfrak{a}$ is p^r–separated.

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By hypothesis & pigeonhole, this sequence of exponents contains a gap $\geq r$.

I.e. $\mathfrak{a}$ contains two elements a_1 and a_2 with $p^{e_i} \| a_i$ and $e_2 \geq e_1 + r$, and no element in $\mathfrak{a}$ contains a power of p between e_1 and e_2.

Split $\mathfrak{a}$ into the multiples of p^{e_2} and the non-multiples of p^{e_2}.

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Given N and k, there are only finitely many primitive lists of length N that are at most k–separated.

For example, if $\mathfrak{a}$ is at most 4–separated, then the elements of $\mathfrak{a}$ are divisors of

$$2^{2(N-1)} \times 3^{N-1}.$$

Computing norms of a k–separated list

Suppose $\mathfrak{a}$ is k–separated: $\mathfrak{a} = B\mathfrak{b} + C\mathfrak{c}$, and put

$$\widetilde{\mathfrak{b}} = \frac{B}{(B,k)}\mathfrak{b}, \quad \widetilde{\mathfrak{c}} = \frac{C}{(C,k)}\mathfrak{c}; \qquad (\text{ so } \mathfrak{a} = k\widetilde{\mathfrak{b}} + \widetilde{\mathfrak{c}}, \text{ or } \widetilde{\mathfrak{b}} + k\widetilde{\mathfrak{c}}).$$

Then

$$N(\mathfrak{a}) = \left(1 - \frac{1}{k}\right)(N(\mathfrak{b}) + N(\mathfrak{c})) + \frac{1}{k}N(\widetilde{\mathfrak{b}} + \widetilde{\mathfrak{c}}).$$

Note: $\widetilde{\mathfrak{b}} + \widetilde{\mathfrak{c}}$ has length $\geq |\ell(\mathfrak{b}) - \ell(\mathfrak{c})|$ and has same parity as N.

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Proof: Say $\mathfrak{a} = k\widetilde{\mathfrak{b}} + \widetilde{\mathfrak{c}}$. Note $N(k\widetilde{\mathfrak{b}}) = N(\widetilde{\mathfrak{b}}) = N(\mathfrak{b})$.

$$N(\mathfrak{a}) = N(\mathfrak{b}) + N(\mathfrak{c}) + 2\sum_{\substack{kb \in k\widetilde{\mathfrak{b}} \\ c \in \widetilde{\mathfrak{c}}}} \frac{(kb,c)^2}{12kbc} = N(\mathfrak{b}) + N(\mathfrak{c}) + \frac{2}{k}\sum_{\substack{b \in \widetilde{\mathfrak{b}} \\ c \in \widetilde{\mathfrak{c}}}} \frac{(b,c)^2}{12bc}$$

$$= N(\mathfrak{b}) + N(\mathfrak{c}) + \frac{2}{k}\int_0^1 \widetilde{\mathfrak{b}}(x)\widetilde{\mathfrak{c}}(x)\,dx.$$

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$N([1,-2]) = 1/12, \quad N([1,-3]) = 1/9, \quad N([1,-4]) = 1/8, \quad \dots$
Limit point 1/6.

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Length 3: Family $[a,-2a,b]$ with $(a,b) = 1$.
Smallest norm 1/8 for $[1,-2,4]$; next smallest

$$N([1,-2,-3]) = N([2,3,-6]) = N([1,-2,6]) = N([1,-3,6]) = 5/36$$

Limit point in this family $1/6 = 1/12 + 1/12$.
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Length 4: Smallest norm $1/9$ attained for $[1,-2,-3,6]$.
Lists $[a,-2a,b,-2b]$ give limit point $1/6$.
Next limit point: $1/9 + 1/12 = 7/36$ (family $[a,-2a,b,-3b]$)

Summary of lists with small norm

$\mathfrak{a}$ primitive list.
$N(\mathfrak{a}) \geq 31/180 = 1/6 + 1/180$ except for following cases:

$$\mathfrak{a} = [1], \quad [a, b], \quad [a, -2a, b], \quad [a, -2a, b, -2b]$$

$$\text{Norm } \tfrac{17}{108}: \quad [1, -3, 9], \quad [1, -2, -3, 6, 9, -18]$$

$$\text{Norm } \tfrac{1}{6}: \quad [1, -3, -4], \quad [3, 4, -12], \quad [1, -3, 6, -12], \quad [1, -3, -4, 6],$$

$$[1, -3, -4, 12], \quad [1, -2, 4, -12], \quad [1, -2, -3, 4], \quad [1, -2, -3, 12],$$

$$[1, -4, -6, 12], \quad [2, -3, -4, 12], \quad [3, -4, -6, 12], \quad [1, -2, -3, 4, 6],$$

$$[1, -2, -3, 6, -12], \quad [2, 3, -4, -6, 12], \quad [1, -2, 4, 6, -12],$$

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Finiteness of integral factorial ratios of height 1

Goal: Primitive $\mathfrak{a}$ with $s(\mathfrak{a}) = 0$, $\ell(\mathfrak{a})$ odd, and $N(\mathfrak{a}) = 1/4$.
From bounds for $G(N)$ reduced to $\ell(\mathfrak{a}) = 3, 5, 7, 9$.
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$$s(\mathfrak{a}) = Bs(\mathfrak{b}) + Cs(\mathfrak{c}) = 0.$$

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Either $\mathfrak{b}$ or $\mathfrak{c}$ has odd length, and that list has norm $\geq 1/4$.
Conclusion: Can assume $\mathfrak{a}$ is k–separated with k large.
If it decomposes into the primitive lists $\mathfrak{b}$, $\mathfrak{c}$, then $\mathfrak{a}$ is uniquely determined by these lists.

$$N(\mathfrak{a}) \geq \left(1 - \frac{1}{k}\right)(N(\mathfrak{b}) + N(\mathfrak{c})).$$

Recall: $[1]$, $[1,-2]$ have norm $1/12$.
All other lists have norm $\geq 1/9$.
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Back to the proof: $N(\mathfrak{a}) \geq (1-1/k)(N(\mathfrak{b})+N(\mathfrak{c}))$.
Assume $N(\mathfrak{b}) \leq N(\mathfrak{c})$.
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Must have $N(\mathfrak{c}) \leq 1/6 + \epsilon$.
Apart from finitely many cases, must have

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$$\mathfrak{b} = [1], \quad \mathfrak{c} = [a, b] \text{ — binomial coefficient}$$

$$\mathfrak{b} = [1, -2], \quad \mathfrak{c} = [a, -2a, b]; \qquad \mathfrak{b} = [1], \quad \mathfrak{c} = [a, -2a, b, -2b]$$

both lead to the family $[2a, -a, 2b, -b, -(a + b)]$.