An invitation to Combinatorial Commutative Algebra

PhD Seminar

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Introduction

- Combinatorial commutative algebra is a very lively branch in modern mathematics. It combines the broad abstract methods of algebra, geometry and topology with the more intuitive ones of combinatorics, "the art of counting", which is a common ground for mathematicians coming from many branches.
- In particular, the interplay between commutative algebra, algebraic geometry and combinatorics has been proved to be effective also in seemingly distant areas like statistics (giving rise to the realm of algebraic statistics) and even in different fields like computer science and biology.

Monomial and toric ideals

Combinatorial commutative algebra often deals with squarefree monomials inside a polynomial ring $S = \Bbbk[x_1, \dots, x_n]$. Here "squarefree" just means that no variable can appear twice inside the same monomial: $x_1 x_2$ is squarefree, but $x_1^2 x_2$ is not.

Given a set $m_1, \dots, m_t$ of squarefree monomials, we can construct two different algebraic-combinatorial objects of interest:

- the *squarefree monomial ideal* of S generated by $m_1, \dots, m_t$;
- the $\Bbbk$-algebra $\Bbbk[m_1, \dots, m_t]$ generated by our set. The kernel of the standard presentation

$$\begin{array}{ccc} \Bbbk[y_1, \dots, y_t] & \twoheadrightarrow & \Bbbk[m_1, \dots, m_t] \\ y_i & \mapsto & m_i \end{array}$$

  is the *toric ideal* associated with our set.

During the rest of this talk we will focus on squarefree monomial ideals.

Modules

Let R be a (commutative and unitary) ring. An abelian group $(M, +)$ is said a (left) R-module if there exists an application

$$R \times M \to M$$
$$(r, m) \mapsto rm$$

such that $(r + r')m = rm + r'm$, $r(m + m') = rm + rm'$, $(rr')m = r(r'm)$ and $1_R m = m$ for all $r, r' \in R$, $m, m' \in M$.

- When $R = \mathbb{Z}$, we get abelian groups.
- When $R = \Bbbk$, we get $\Bbbk$-vector spaces.
- If I is an ideal of R, then both I and R/I have a natural R-module structure.
- $R^n = \bigoplus_{i=1}^n R$ is called a *free* R-module; note that free R-modules are the only R-modules endowed with a basis.

Chain complexes and homology

A map of R-modules $\phi\colon M \to N$ is a map of abelian groups such that $\phi(rm) = r\phi(m)$ for all $r \in R$, $m \in M$.
A *chain complex* $(C_\bullet, \partial_\bullet)$ is a collection of R-modules C_i and maps $\partial_i\colon C_i \to C_{i-1}$ (where $i \in \mathbb{Z}$) such that $\partial_i \circ \partial_{i+1} = 0$ for all i.

$$C_\bullet\colon \ \ldots \to C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \to \ldots$$

Note that the condition on $\partial_\bullet$ means that $\mathrm{im}(\partial_{i+1}) \subseteq \ker(\partial_i)$ for all i.
It then makes sense to consider the R-module

$$H_i(C_\bullet) := \ker(\partial_i)/\mathrm{im}(\partial_{i+1}).$$

This is called the i-th *homology module* of the chain complex $C_\bullet$.
When $H_i(C_\bullet) = 0$, i.e. $\mathrm{im}(\partial_{i+1}) = \ker(\partial_i)$, we say that $C_\bullet$ is *exact* in the i-th homological position.

From abelian groups to general modules / 1

Let G be a $\mathbb{Z}$-module, i.e. an abelian group. It is well-known that any group, in particular an abelian one, can be expressed in terms of generators and relations. For simplicity, assume G is finitely generated. Let us consider, for instance,

$$G = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}.$$

G has three generators $g_1 = (1, \bar{0}, \bar{\bar{0}})$, $g_2 = (0, \bar{1}, \bar{\bar{0}})$ and $g_3 = (0, \bar{0}, \bar{\bar{1}})$; by definition, one has that $2(0, \bar{1}, \bar{\bar{0}}) = 9(0, \bar{0}, \bar{\bar{1}}) = 0$. This can be translated by saying that the sequence

$$0 \to \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 9 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} g_1 & g_2 & g_3 \end{pmatrix}} G \to 0$$

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$$0 \to \mathbb{Z}^2 \xrightarrow{\begin{pmatrix}\text{relations on the}\\ \text{generators of } G\end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix}\text{generators of } G\end{pmatrix}} G \to 0$$

is exact.

From abelian groups to general modules / 2

In the previous slide we have found an alternative way to express the information in a finitely generated $\mathbb{Z}$-module. What happens when we consider something more complicated?
Let $S = \Bbbk[x, y, z]$ and let I be the ideal generated by x^2, xy and xz.
Then, starting as before, we can construct generators and relations:

$$S^3 \xrightarrow{\begin{pmatrix}\text{relations on the} \\ \text{generators of } I\end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix}\text{generators of } I\end{pmatrix}} I \to 0$$

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This is the idea behind the concept of *free resolution*.

Free resolutions

Definition

Let R be a ring and M be a finitely generated R-module. We say that the complex

$$\mathbb{F}_\bullet : \ \ldots \to F_3 \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \to 0$$

is a *free resolution* for M as an R-module if:

- the sequence is exact in all positions except the zeroth and $H_0(\mathbb{F}_\bullet) \cong M$;
- F_i is a free R-module $\forall i \in \mathbb{N}$.

If R and M are graded (think of the case when R is a polynomial ring and M is a homogeneous ideal), we say the resolution above is *graded* if each F_i is a graded module and each map involved is a homomorphism preserving the degree.

Note that all the maps ϕ_i can be expressed as matrices.

Minimal graded free resolution

Some resolutions are better than others!

Let $S = \mathbb{k}[x_1, \ldots, x_n]$. A graded free resolution where for each F_i we have that

$$\phi_i(F_i) \subseteq \mathfrak{M} F_{i-1},$$

where $\mathfrak{M}$ is the homogeneous maximal ideal of S generated by the variables $x_1, \ldots, x_n$, is called *minimal*. This means precisely that all the nonzero elements appearing in the matrices describing the homomorphisms ϕ_i are homogeneous and belong to $\mathfrak{M}$.

Theorem

Minimal graded free resolutions are unique up to isomorphism of complexes.

Betti numbers

Since there exists only one minimal graded free resolution of M up to isomorphism of complexes, it makes sense to count the copies of the ring that are used in each step of this resolution and associate this set of numbers with M itself. Let us consider our previous example, where M was a homogeneous ideal:

$$0 \to S^{1} \xrightarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} S^{3} \xrightarrow{\begin{pmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix}} S^{3} \to 0$$

There are three copies of S in the zeroth position, another three copies of S in the first position, and one copy of S in the second position.

These numbers are the *Betti numbers* of M over S, denoted by $\beta_i^S(M)$.

In this case, $\beta_0(M) = 3$, $\beta_1(M) = 3$, $\beta_2(M) = 1$, $\beta_i(M) = 0$ for all $i \geq 3$.

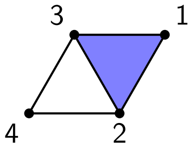
Simplicial complexes: a brief introduction

Let $[n] = \{1, 2, \ldots, n\}$ and let Δ be a subset of $2^{[n]}$. We call Δ a *simplicial complex* if the following property holds:

$$\text{if } F \in \Delta \text{ and } G \subseteq F, \text{ then } G \in \Delta.$$

The elements in Δ are called *faces* of the complex.
There is a nice geometric way to look at these objects (and this is where the name "simplicial complex" comes from): roughly speaking, we can see every face of Δ as a simplex of the "right" dimension.
Simplicial complexes turn out to be collections of simplices glued together by their faces.

$$\Delta = \{\{1,2,3\},\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\},\{1\},\{2\},\{3\},\{4\},\emptyset\}$$



Simplicial complexes and squarefree monomial ideals

How can we do (commutative) algebra with simplicial complexes?

Fix a field $\mathbb{k}$. Given a complex Δ with n vertices, one can consider the ideal I_Δ of the polynomial ring $\mathbb{k}[x_1, \ldots, x_n]$ generated by the nonfaces of Δ, i.e. the squarefree monomial ideal defined in the following way:

$$I_\Delta := (x_{i_1} x_{i_2} \ldots x_{i_s} \mid \{i_1, i_2, \ldots, i_s\} \notin \Delta).$$

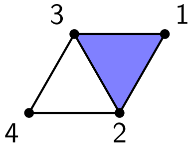
Definition

I_Δ is called the *Stanley-Reisner ideal* of Δ.

Note that we can restrict ourselves to the *minimal* (with respect to inclusion) nonfaces of Δ, since nonfaces that are not minimal will give redundant generators of the ideal.

An example of a Stanley-Reisner ideal

$$\Delta = \{\{1,2,3\},\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\},\{1\},\{2\},\{3\},\{4\},\emptyset\}$$

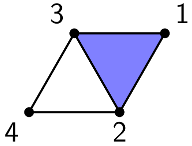


- Nonfaces of Δ: $\{1,4\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}$.
- Stanley-Reisner ideal of Δ:

$$I_\Delta = (x_1x_4, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4, x_1x_2x_3x_4).$$

An example of a Stanley-Reisner ideal

$$\Delta = \{\{1,2,3\},\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\},\{1\},\{2\},\{3\},\{4\},\emptyset\}$$

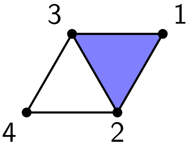


- Minimal nonfaces of Δ: $\{1,4\}, \cancel{\{1,2,4\}}, \cancel{\{1,3,4\}}, \{2,3,4\}, \cancel{\{1,2,3,4\}}$.
- Stanley-Reisner ideal of Δ:

$$I_\Delta = (x_1x_4, \cancel{x_1x_2x_4}, \cancel{x_1x_3x_4}, x_2x_3x_4, \cancel{x_1x_2x_3x_4}).$$

An example of a Stanley-Reisner ideal

$$\Delta = \{\{1,2,3\},\{1,2\},\{1,3\},\{2,3\},\{2,4\},\\ \{3,4\},\{1\},\{2\},\{3\},\{4\},\emptyset\}$$



- Minimal nonfaces of Δ: $\{1,4\},\{2,3,4\}$.
- Stanley-Reisner ideal of Δ:

$$I_\Delta = (x_1x_4, x_2x_3x_4).$$

The Stanley-Reisner correspondence

- We have just seen that, given a simplicial complex Δ, we can associate a squarefree monomial ideal with it.
- On the other hand, given any minimal set of squarefree monomials, we can associate a unique simplicial complex with it.

This is called the Stanley-Reisner correspondence:

$$\boxed{\text{squarefree monomial ideals} \longleftrightarrow \text{simplicial complexes}}$$

The Stanley-Reisner correspondence is not the only way to associate squarefree monomial ideals with simplicial complexes and viceversa, but it is in some sense the most natural. More on this later!

Simplicial homology / 1

Given a simplicial complex Δ and a field $\Bbbk$, one can construct the associated chain complex $\widetilde{C}_\bullet(\Delta, \Bbbk)$: this is done as follows.

- $\widetilde{C}_i(\Delta, \Bbbk)$ is the $\Bbbk$-vector space whose dimension equals the number of faces of dimension i in Δ. The empty set has dimension -1.
- Given a face F and the corresponding basis element e_F, one has that

$$\partial(e_F) = \sum_{v \in F} (-1)^{[v,F]} e_{F \setminus v},$$

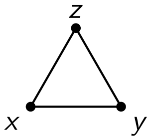
where $(-1)^{[v,F]}$ is a sign depending on the position of v inside the face F (after fixing a total order on the vertices of Δ).

Definition

The homology of $\widetilde{C}_\bullet(\Delta, \Bbbk)$ is called *reduced homology* of Δ and is denoted by $\widetilde{H}_\bullet(\Delta, \Bbbk)$.

- Homology is actually a topological feature of the simplicial complex, i.e. it depends on the "shape" of the object. It is invariant under homeomorphism and even homotopy equivalence.

Let us consider an example:



$$\Delta = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}\}$$

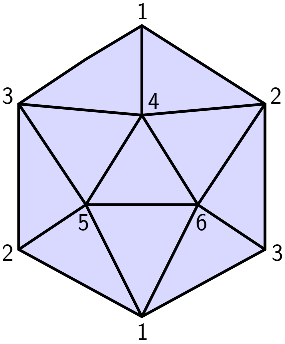
$$\widetilde{C}_\bullet(\Delta, \Bbbk)\colon 0 \leftarrow \Bbbk\{\emptyset\} \xleftarrow{\begin{pmatrix}1 & 1 & 1\end{pmatrix}} \begin{matrix}\Bbbk\{x\} \\ \oplus \\ \Bbbk\{y\} \\ \oplus \\ \Bbbk\{z\}\end{matrix} \xleftarrow{\begin{pmatrix}1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1\end{pmatrix}} \begin{matrix}\Bbbk\{xy\} \\ \oplus \\ \Bbbk\{xz\} \\ \oplus \\ \Bbbk\{yz\}\end{matrix} \leftarrow 0$$

$$\widetilde{H}_{-1}(\Delta, \Bbbk) = \widetilde{H}_0(\Delta, \Bbbk) = 0, \quad \widetilde{H}_1(\Delta, \Bbbk) \cong \Bbbk.$$

Combinatorics is not enough

Attention!

The homology of simplicial complexes *does* depend (in general) on the choice of the field $\Bbbk$!



Standard triangulation Δ of the real projective plane $\mathbb{RP}^2$.
$\widetilde{H}_1(\Delta, \Bbbk) \cong \Bbbk$ if $\Bbbk$ has characteristic 2; otherwise, $\widetilde{H}_1(\Delta, \Bbbk) = 0$.

Recap

- Commutative algebra: *free resolutions* of modules are important because they encode structure information. Betti numbers are numerical invariants tied to minimal free resolutions.
- Combinatorics: *simplicial complexes* are collections of simplices glued together by their faces. They are in one-to-one correspondence with squarefree monomial ideals (Stanley-Reisner correspondence).
- Algebraic topology: *homology* is a set of invariants attached to a simplicial complex (and, more generally, to a topological space). Homotopy equivalent spaces have the same homology.

- In general, computing Betti numbers is possible but costly (since there are Gröbner bases involved). On the other hand, computing homology can be reduced to linear algebra procedures and is hence easier.
- The Stanley-Reisner correspondence is not just formal, but allows us to study algebraic properties of I_Δ via an analysis of the corresponding simplicial complex Δ. The crucial instance of this phenomenon is Hochster's formula, which allows a fruitful interplay between commutative algebra, combinatorics and algebraic topology.

Where the worlds meet: Hochster's formula / 2

Given a simplicial complex Δ, pick a subset R of its vertices. The subcomplex of Δ *induced by* R (denoted by $\Delta|_R$) is the simplicial complex whose faces are exactly the faces of Δ contained in R. We will denote by $|R|$ the cardinality of R.

Hochster's formula (1977)

$$\beta_{i,R}(I_\Delta) = \dim_{\Bbbk} \widetilde{H}_{|R|-i-2}(\Delta|_R; \Bbbk).$$

- Since homology depends on the choice of $\Bbbk$, it turns out that Betti numbers (even of monomial ideals!) are characteristic-dependent in general.
- On the other hand, there are lucky situations where one can really use only the combinatorial structure of the simplicial complex to compute Betti numbers. This has been a very active research topic in the last decades.

Thank you for your attention!