General Relativity in a Nutshell
(And Beyond)

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1. Gravity and General Relativity
2. Quantum Mechanics, Quantum Field Theory and All That...
3. An insight into QFT on Curved Backgrounds

Newtonian Gravity

Newton's Law of Gravitation

$$\mathbf{F} = G\frac{m_G M}{r^3}\mathbf{r} \quad G = 6.67 \cdot 10^{-11}\,\frac{N \cdot m^2}{kg^2}$$

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This formulation provides problems (*e.g.* precession of Mercury's orbit perihelion, wrong deviation of light rays, instantaneous propagation)

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$$\begin{cases} x' = x - vt \\ y' = y \\ z' = z \\ t' = t \end{cases}$$

Velocity Transormation Law: $u' = u - v$

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Maxwell Equations are not invariant under Galilean Relativity

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Lorentz Transformations

For two IF R e R' in x-standard configuration, assuming space and time isotropy and homogeneity:

$$\begin{cases} x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \\ y' = y \\ z' = z \\ t' = \frac{t - \frac{v}{c}x}{\sqrt{1 - \frac{v^2}{c^2}}} \end{cases}$$

Velocity Transformation Law: $u' = \dfrac{u - v}{1 - \frac{uv}{c^2}}$

Minkowski Spacetime $\mathbb{M}$

Definition

We call Minkowski Spacetime $\mathbb{M}$ the vector space $\mathbb{R}^4$ endowed with

- *Orientation;*
- *Metric η with signature $(+---)$ (or $(-+++)$);*
- *Time Orientation.*

Causal Structure on $\mathbb{M}$

$$\begin{aligned} \text{Timelike} \quad & \eta(u,u) > 0; \\ \text{Spacelike} \quad & \eta(u,u) < 0; \\ \text{Lightlike} \quad & \eta(u,u) = 0; \\ \text{Causal} \quad & \eta(u,u) \geq 0. \end{aligned}$$

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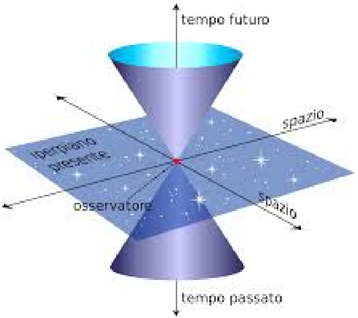

Equivalence Principle

Let us turn on gravity and remember Newton's Second Law

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Weak Equivalence Principle $\Rightarrow$ Existence of Local Inertial Frames

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Weak Equivalence Principle $\Rightarrow$ Existence of Local Inertial Frames

Principle (Equivalence Principle)

In small enough regions of space-time, the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field by means of local experiments.

The Einstein' way to General Relativity

We want to obtain an analogue for the Poisson Equation for the "classic" gravitation potential:

$$\Delta\varphi_G = 4\pi G\rho.$$

We look for:

- 2^{nd} order tensorial equations, linear in the derivatives of greater order;
- We need to reach the Newtonian theory in a suitable limit;
- The equations must grant the condition $\nabla_i T^i_k = 0$ (freely gravitating mass-energy).

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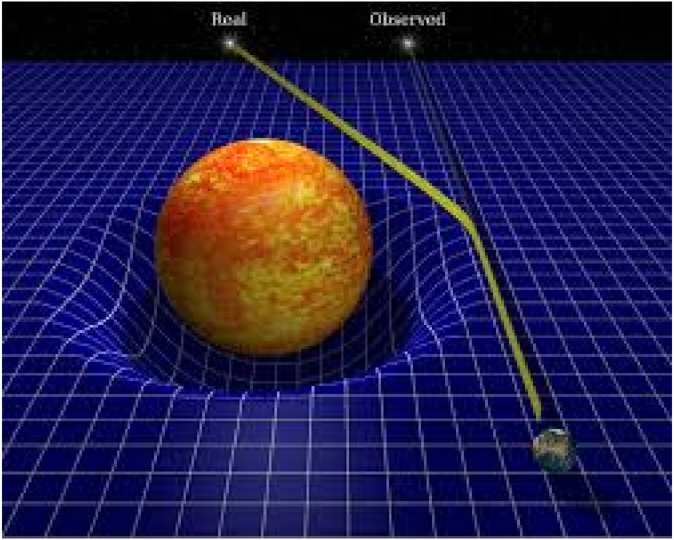
Einstein Equations: $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + (\Lambda g_{\mu\nu}) = -\frac{4\pi G}{c^4}T_{\mu\nu}$

A Pictorial Viewpoint



Some well-known solutions to the Einstein Equations

- Schwarzschild:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

- Friedmann - Lemaitre - Robertson - Walker (FLRW):

$$ds^2 = a^2(t)\left[-\frac{dt^2}{a^2(t)} + dx^2 + dy^2 + dz^2\right]$$

- Kerr
- de Sitter

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Up to now we are far from the solution...

Semi-classical theory of Gravity and QFT over Curved Backgrounds

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Everything can be made rigorous using the Algebraic Formulation!

Canonical Quantisation

In Schrödinger picture the vector states are represented by $\psi(x;t) \in L^2(\mathbb{R}^3)$, which are the components of

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We associate a quantum observable to every classical one via the "promotion to operator" prescription. In Schrödinger picture:

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From the classical energy-dispersion relation we obtain the Schrödinger Equation:

$$E = \frac{p^2}{2m} + V(x) \Rightarrow -i\partial_t\,|\psi\rangle = -\frac{1}{2m}\frac{d^2}{dx^2}\,|\psi\rangle + V(\hat{x}).$$

Quantisation of the Harmonic Oscillator

The Schrödinger equation for a 1-dimensional harmonic oscillator reads

$$-i\partial_t \psi(x;t) = -\frac{1}{2m}\frac{d^2}{dx^2}\psi(x;t) + \frac{1}{2}\omega^2 \hat{x}^2$$

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Its solution is

$$\psi_n(x;t) = e^{-\frac{1}{2}\omega x^2} H_n(x\sqrt{\omega}) e^{-iE_n t}, \quad E_n = \left(n + \frac{1}{2}\right)\omega.$$

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Important features of Quantum Mechanics:

- Discrete spectrum of energy eigenstates;
- $E_0 = \frac{1}{2}\omega$: there is a ground state with non-zero energy ($\rightsquigarrow$Heisenberg Principle)

Second Quantisation

Let us introduce the Lowering and Raising Operators:

$$\hat{a} = \frac{1}{\sqrt{2\omega}}\left(\omega\hat{x} + i\hat{p}\right) \quad \hat{a}^{\dagger} = \frac{1}{\sqrt{2\omega}}\left(\omega\hat{x} - i\hat{p}\right)$$

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Calling $|n\rangle$ the eigenstates of $\hat{N}$ we obtain that a generic state vector is given by:

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t}|n\rangle .$$

Special Relativity and Quantum Mechanics: the Klein-Gordon Equation

Switching to the relativistic energy dispersion relation $E^2 = \mathbf{k}^2 + m^2$ $(c = 1)$ we get the the Klein-Gordon Equation:

$$(\Box + m^2) |\psi\rangle = 0.$$

One can derive it computing the Euler-Lagrange Equations of the Klein-Gordon Lagrangian:

$$S_{KG} = \int d^4x \mathscr{L}_{KG}, \quad \mathscr{L}_{KG} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2$$

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Problems:

- Possible negative-energy solutions (Fourier transform);
- Violation of causality;
- $||\psi\rangle|^2$ can not be interpreted as a probability amplitude.

A World Made of Fields

A solution to these problems can be found considering a system with infinitely many degrees of freedom, *i.e.* a field (Dirac sea).

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$$\mathscr{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \rightsquigarrow \text{Harmonic Oscillator!}$$

$$(x; p) \mapsto (\phi(x^\mu); \pi(x^\mu))$$

$\phi(x^\mu)$ has no more to be read as a wave function, but fixed-time initial value of the KG equation.

Space of Solutions

The KG Equation admits plane-wave solutions:

$$\phi(x^\mu) = \phi_0 e^{ik_\mu x^\mu} = \phi_0 e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}} \quad \omega^2 = \mathbf{k}^2 + m^2.$$

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We look for a complete o.n. set of solutions, hence we need a scalar product on the solutions' space:

$$(\phi_1, \phi_2) = -i \int_{\Sigma_t} \left(\phi_1 \partial_t \phi_2^* - \phi_2^* \partial_t \phi_1\right) d^3x$$

so that we get the set ($k_\mu : \omega^2 = \mathbf{k}^2 + m^2$):

$$f_{\mathbf{k}}(x^\mu) = \frac{e^{ik_\mu x^\mu}}{2\pi\sqrt{2\omega}}, \quad (f_{\mathbf{k}_1}, f_{\mathbf{k}_2}) = \delta^{(3)}(\mathbf{k}_1 - \mathbf{k}_2).$$

Positive- and Negative-Frequencies Solutions

The solution are labelled by the continuous parameter $\mathbf{k}$ and are determined up to the sign of ω. Since energy ($E = h\omega$) is a positive-definite quantity we would like to consider only solutions with positive frequency.

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This is done by introducing, for all $\boldsymbol{\omega} > \mathbf{0}$, positive-frequency solutions

$$\partial_t f_{\mathbf{k}} = -i\omega f_{\mathbf{k}}$$

and negative-frequency solutions

$$\partial_t f^*_{\mathbf{k}} = i\omega f^*_{\mathbf{k}}.$$

Canonical Quantisation

In total analogy with the harmonic oscillator in QM we promote ϕ and π to operator, imposing the equal-time commutation relations (Heisenberg Picture)

$$\left[\phi(\mathbf{x}), \phi(\mathbf{x}')\right]_t = i\delta^{(3)}(\mathbf{x} - \mathbf{x}').$$

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Expanding as a function of the modes

$$\phi(t;\mathbf{x}) = \int d^3\mathbf{x}\left[\hat{a}_\mathbf{k} f_\mathbf{k}(t;\mathbf{x}) + \hat{a}^\dagger_\mathbf{k} f^*_\mathbf{k}(t;\mathbf{x})\right]$$

which leads to the Canonical Commutation Relations

$$\left[\hat{a}_\mathbf{k}, \hat{a}^\dagger_{\mathbf{k}'}\right] = \delta^{(3)}(\mathbf{k}-\mathbf{k}').$$

Particles and Anti-Particles

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$$|n_{\mathbf{k}}\rangle = \frac{1}{\sqrt{n_{\mathbf{k}}!}} (\hat{a}^{\dagger}_{\mathbf{k}})^{n_{\mathbf{k}}} |0\rangle$$

We are creating n particles with momentum $\mathbf{k}$. $|0\rangle$ is the Vacuum State.

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We are creating n particles with momentum $\mathbf{k}$. $|0\rangle$ is the Vacuum State. Likewise in QM, we can introduce the Number Operator $\hat{N}_{\mathbf{k}} = \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}$, whose eigenvectors constitute the Fock Basis.

What's Missing?

- Renormalization Subtraction of (infinite) the point-zero energy
- Interactions and Scattering Theory
- Other kind of Fields (QED, Gauge Theories)
- ....

Fundamental Ideas

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- The interactions do not influence the fixed background;
- Investigation of the back-reaction;
- Black Holes Physics;
- States (Vacua, Thermal...) and Renormalization;
- Making the theory mathematically rigorous.

Choice of the background

$$\mathscr{L} = \sqrt{-g}\left(-\frac{1}{2}\nabla^{\mu}\phi\nabla_{\mu}\phi - \frac{1}{2}m^2\phi^2 - \xi R\phi^2\right) \Rightarrow \Box_g\phi - m^2\phi - \xi R\phi = 0.$$

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Theorem (Bernal - Sanchez)

Let $(\mathcal{M}, g)$ be a 4-dimensional, time-oriented space-time. Then the following statements are equivalent:

1. *$(\mathcal{M}, g)$ is globally hyperbolic;*
2. *$(\mathcal{M}, g)$ is hysometric to $\mathbb{R} \times \Sigma$ with $ds^2 = +\beta dt^2 - h_{ij}dx^i dx^j$. Here $(t, x_i)_{i=1}^3$ is a suitable coordinate system s.t. $\beta \in C^\infty(\mathcal{M}; (0, \infty))$, h is a Riemannian metric on Σ depending smoothly on t and each locus $\{t = const\} \times \Sigma$ is a smooth spacelike Cauchy hypersurface embedded in $\mathcal{M}$.*

What about particles?

We define the conjugate momentum $\pi = \sqrt{-g}\nabla_0\phi$ and define the scalar product on a spacelike 3-surface Σ and take over the quantisation imposing impose the CCR:

$$(\phi_1, \phi_2) \doteq -i \int_\Sigma (\phi_1 \nabla_\mu \phi_2^* - \phi_2^* \nabla_\mu \phi_1)\, n^\mu \sqrt{\gamma} d^3x$$

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Problem

In a general space-time it is impossible to define positive-frequency solutions because there is no unique notion of time. Hence the concept of particle is ill-defined!

Hadamard States

We are dealing with semiclassical Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi < T_{\mu\nu} >, \quad < T_{\mu\nu} > \doteq \langle 0 | \, T_{\mu\nu} \, | 0 \rangle = ?$$

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The lack of the modes expansion implies that there is no definite notion of vacuum state.
This lead to the introduction of Hadamard States:

$$H(x, x') = \frac{U(x, x')}{(2\pi)^2 \sigma_\epsilon} + V(x, x') \log(\sigma_\epsilon) + W(x, x'), \; \sigma_\epsilon \doteq \sigma + 2i\epsilon(t - t') + \epsilon^2$$

- $< T_{\mu\nu} >$ well-defined
- Expectation values with finite fluctuations
- Compatible with the unique Minkowski vacuum state