DIMA

Università di Genova

Don't Go Breaking My Cat
Introduction to the Microscopic View Of The Limits Of Reality

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DIMA PhD Seminars, 14.12.2023

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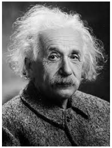
• Quantum mechanics developed in the last century. Is the biggest scientific revolution, especially in terms of its impact in our all day life. The study of its foundation produced more than 70 Nobel Laureates (Nobel prize introduced in 1901):





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• Aim of the talk: present at a very elementary level the essential ideas of the quantum theory in the algebraic approach. The latter approach is the one suited for generalization to field theories (physical theories with uncountably many degrees of freedom)

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Algebraic Viewpoint

• Primary datum: A physical system is characterized by its observable/measurable quantities.
Primary datum: A unital algebra $\mathfrak{A}$ with an associative product and an involution $*$ ($*$-algebra). Additional topological structure can be required: Banach Algebra, C^*-algebra. The elements $a \in \mathfrak{A}$ such that $a^* = a$ are the observables.

• Second datum: The configuration of a system is specified by the outcome of measurements.
Second datum: Linear functional $\omega : \mathfrak{A} \to \mathbb{C}$ that are positive ($\omega(a^* a) \geq 0$) and normalized ($\omega(\mathbb{I}) = 1$). These are called **states**. For $a \in \mathfrak{A}$, the value $\omega(a)$ is the expectation value (probabilistic expected result of the measurement).

• Third datum: Dynamical laws describing the time evolution of the configuration of the system.
Third datum: One-parameter group of $*$-automorphisms on the algebra $\{\tau_t\}_{t\in\mathbb{R}}$.

GNS Theorem: Representation on Hilbert Space

Theorem (Gelfand-Naimark-Segal)

Let $\mathfrak{A}$ be a $*$-algebra with unit $\mathbb{I}$ and let $\omega : \mathfrak{A} \to \mathbb{C}$ be a state. Then there exist a quadruple $(\mathcal{H}_\omega, \mathcal{D}_\omega, \pi_\omega, \Psi_\omega)$ made of a Hilbert space $\mathcal{H}_\omega$, a subspace $\mathcal{D}_\omega \subset \mathcal{H}_\omega$, a linear map $\pi_\omega : \mathfrak{A} \to \mathcal{L}(\mathcal{D}_\omega, \mathcal{H}_\omega)$ and an element $\Psi_\omega \in \mathcal{D}_\omega$ such that:

- i) $\mathcal{D}_\omega = \pi_\omega(\mathfrak{A})\Psi_\omega$
- ii) Ψ_ω is cyclic for π_ω, that is $\mathcal{D}_\omega$ is dense in $\mathcal{H}_\omega$.
- iii) $\pi_\omega : \mathfrak{A} \to \pi_\omega(\mathfrak{A})$ is an algebra $*$-homomorphism. Namely $\pi_\omega(\mathbb{I}) = 1$ and $\pi_\omega(a^*) = \pi_\omega(a)^*\big|_{\mathcal{D}_\omega}$ for $a \in \mathfrak{A}$
- iv) For any $a \in \mathfrak{A}$ holds: $\omega(a) = \langle \Psi_\omega | \pi_\omega(a) \Psi_\omega \rangle$

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Idea of Proof Define a product: $\langle A|B \rangle = \omega(A^* B)$. Then:

$$\mathcal{D}_\omega = \mathfrak{A}/\mathcal{I}_\omega$$

Where $\mathcal{I}_\omega := \{A \in \mathfrak{A} | \omega(A^*A) = 0\}$ is a double-sided ideal (*Gelfand Ideal*). Moreover, $\pi_\omega(A)[B] = [AB]$. Finally $\Psi_\omega = [\mathbb{I}]$ and $\mathcal{H}_\omega = \overline{\mathcal{D}_\omega}$ completed in the norm induced by the scalar product above.

If $\mathfrak{A}$ is finitely generated, then τ_t is represented in $\mathcal{H}_\omega$ by a strongly continuous one-parameter family of unitary U_t.

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Where the configuration space T^*M is the cotangent bundle of a complete Riemannian manifold M (finite dimensional). Describes a classical system with $\dim M$ many degrees of freedom.

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Classical states: probability distributions ρ on T^*M:

$$\int_{T^*M} \rho(q,p) \, \mathrm{dvol}_{T^*M} = 1 \, ,$$

with action as functional:

$$\omega_\rho(f) = \int_{T^*M} \rho(q,p) f(q,p) \, \mathrm{dvol}_{T^*M} .$$

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Dynamics described by a **Hamiltonian** $H \in \mathcal{C}^\infty(T^*M, \mathbb{R})$, via Hamilton's equations:

$$\frac{df}{dt}(q,p) = \left\{f, H\right\}(q,p) := \sum_{k=1}^{m} \left(\frac{\partial f}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial H}{\partial q_k} \frac{\partial f}{\partial p_k} \right)$$

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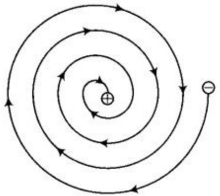
Instability of Atomic Model

By solving the Hamilton's equations for an orbiting electron around a opposite charged particle, the solution describes a continuous emission of electromagnetic radiation (**Bremsstrahlung**). The orbit, in polar coordinates, (r, θ) is:

$$r(t) = \left(r_0^3 - 4\left(\frac{e^2}{m_e c^2}\right)^2 ct\right)^{1/3}, \quad \theta(t) = \theta_0 - \frac{8}{3}\sqrt{\frac{1}{r^7}\left(\frac{e^2}{m_e c^2}\right)^5}.$$

The resulting loss of energy a spiraling orbit.



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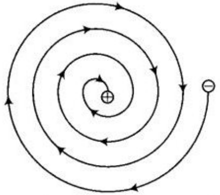
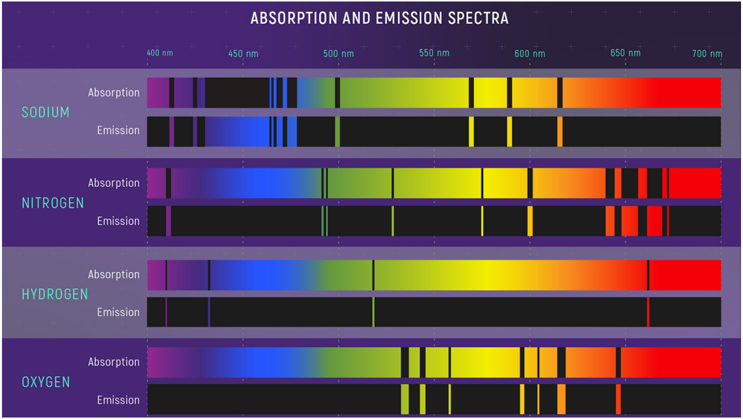


Continuous loss of energy by a revolving electron

Instability of the atomic model from the perspective of classical physics: an electron should fall on the nucleus in roughly $10^{-8}s$

Spectra of Emission of Atoms

Life jacket: Observed spectra of emission of radiation by atoms are not continuous as predicted by electrodynamics



Light emitted/absorbed just at certain discretized frequencies.
Problem: Absence of a classical explanation to this phenomenon

Stern-Gerlach Experiment

Consider a Silver atom (Ag^{47} has a valence electron) moving in a region with a varying magnetic field.

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Extension: To understand this property consider instead a stream of an ensemble of unpolarized Silver atoms. If the electron is a classical spinning object, S_z should be a function of position and momenta (variables of configuration space) varying continuously between $|\mathbf{S}|$ and $-|\mathbf{S}|$.

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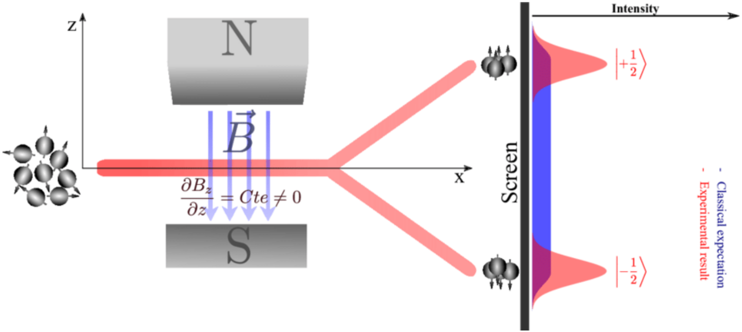

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Quantum Observables

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Can be obtained from the classical one (when it exist) via a quantization procedure deforming the classical commutative into a non-commutative product. For $A, B \in \mathcal{A}$ denoting by $\hat{A}, \hat{B} \in \mathfrak{A}$:

$$[\hat{A}, \hat{B}] = i\hbar\{A, B\} \quad \Rightarrow \quad [\hat{q}_x, \hat{p}_x] = i\hbar$$

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A finitely generated algebra, is represented on a finite dimensional Hilbert space $\mathcal{H}$ (GNS) as:

$$\mathfrak{A} := \left[\hat{A} \in \mathrm{End}(\mathcal{H}) \, , \, \circ \, , \, \hat{A}^* = \hat{A}^\dagger \right]$$

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Quantum State: A density matrix $\rho \in \mathcal{B}(\mathcal{H})$ positive, symmetric and normalized:

$$\rho \geq 0 \,, \quad \rho^\dagger = \rho \,, \quad \mathrm{Tr}(\rho) = 1$$

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Hamilton equations become the Heisenberg equations describing dynamics:

$$\frac{d\hat{A}}{dt} = \frac{1}{i\hbar}[\hat{A}, \hat{H}]$$

Copenhagen (Probabilistic) Interpretation of Quantum Mechanics

Born's rule: The expectation value of an observable A on a state ω is interpreted as the averaged distribution associated to each possible outcome of the measurement A on the state ω. In particular, for $\mathbb{I}$, the normalization condition is the normalization of the probability. ω is in a superposition of the outcomes of A each with a certain associated probability.
Moreover, once a measurement is performed, the state becomes (collapses) to the eigenstate associated to the outcome of the measurement. Namely, the result on the state of a measurement is represented by the action of a projection P_{a_i} outcome eigenvalue a_i.

Quantum Mechanics is intrinsically indeterministic:
In general, the configuration of a system is known only with a certain probability

Stern-Gerlach Experiment Revisited

Algebra of observables $\mathfrak{A} = \mathrm{Mat}_{2\times 2}(\mathbb{C})$ represented over the Hilbert space $\mathbb{C}^2$ completed with respect to the standard inner product. Generators are:

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The algebra describes a system with two single outcomes for the measurement $|+\rangle$ or $|-\rangle$, eigenstates of S_z with associated projections P_+, P_-, like the outcome of a Stern-Gerlach experiment.
The state of the Silver beam is described by a density matrix. In particular, considering the density matrix:

$$\rho_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

describing an unpolarized beam. This follows noticing that both $\omega_{\rho_1}(P_+) = \omega_{\rho_1}(P_-) = 1/2$ and:

$$\omega_{\rho_1}(S_z) = \omega_{\rho_1}(S_y) = \omega_{\rho_1}(S_x) = 0$$

Heisenberg Uncertainty Theorem

Uncertainty of a symmetric operator A on $\mathcal{H}$ with respect to $\psi \in \mathcal{H}$:

$$(\Delta_\psi A)^2 := \big\langle (A - \langle A \rangle_\psi 1)\psi | (A - \langle A \rangle_\psi 1)\psi \big\rangle = \langle A\psi | A\psi \rangle - \langle A \rangle_\psi^2$$

Theorem (Heisenberg Uncertainty)

Suppose A and B are symmetric operators and ψ a unit vector belonging to $\mathrm{Dom}(AB) \cap \mathrm{Dom}(BA)$. Then:

$$(\Delta_\psi A)^2 (\Delta_\psi B)^2 \geq \frac{1}{4} \big| \langle [A, B] \rangle_\psi \big|^2$$

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Idea Proof: $A' := A - \langle A\rangle_\psi 1$. Cauchy-Schwartz:

$$\langle A'\psi | A'\psi\rangle \langle B'\psi | B'\psi\rangle \geq \big|\langle A'\psi | B'\psi\rangle\big|^2 \geq \big|\mathrm{Im}\langle A'\psi | B'\psi\rangle\big|^2$$
$$= \frac{1}{4}\big|\langle A'\psi | B'\psi\rangle - \langle B'\psi | A'\psi\rangle\big|^2 = \frac{1}{4}\big|\langle \psi | [A', B']\psi\rangle\big|^2$$

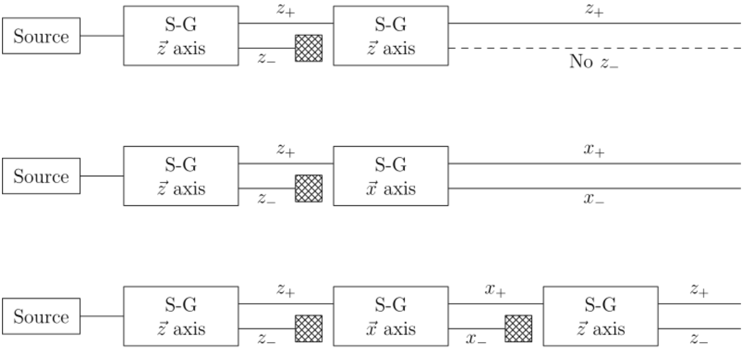
But this is:

$$(\Delta_\psi A)^2 (\Delta_\psi B)^2 \geq \frac{1}{4} \big| \langle [A, B] \rangle_\psi \big|^2$$

Incompatible Observables if they do not commute: we cannot simultaneously know them.

Stern-Gerlach: Different Orientations

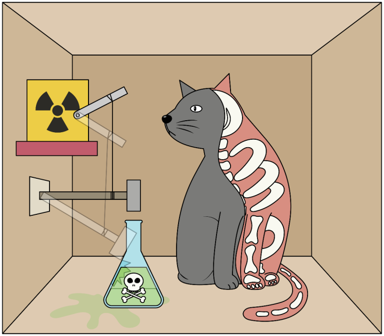
Example of application of the above theorem is seen when considering again a Stern-Gerlach apparatus, but with different orientations. In this way, Spin manifests how it is a purely quantum property of the system



Consistency with Heisenberg uncertainty theorem noticing that:

$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$$

Gedankenexperiment: Schrödinger's cat



Nature is not deterministic at the microscopic scale. Given a system, we cannot predict the outcome of future measurements in general. Only the probability associated to each outcome.

"God does not play with dices" (A. Einstein)
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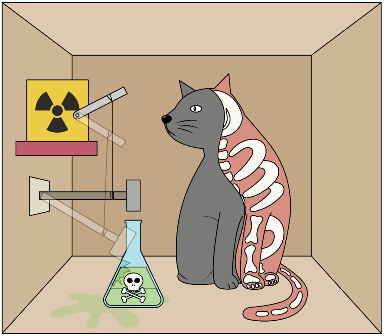

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