MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES

Khovanskii bases and how to use them

Max Planck Institute for Mathematics in the Sciences

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Def. A *term order* in the polynomial ring $S = k[x_1, \dots, x_n]$ is a total order $\leq$ on the set of monomials of S such that:

1. $\leq$ is reflexive, antisymmetric and transitive and total: for each m_1, m_2 monomials then $m_1 \leq m_2$ or $m_2 \leq m_1$ (total order).
2. If $m_1 \leq m_2$ then $m \cdot m_1 \leq m \cdot m_2$ for every monomial m (compatible with multiplication).
3. If $m_1 \mid m_2$ then $m_1 \leq m_2$ ($\overset{(2)}{\Longleftrightarrow} 1 \leq m$ for every monomial m).

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3. If $m_1 \mid m_2$ then $m_1 \leq m_2$ ($\overset{(2)}{\iff} 1 \leq m$ for every monomial m).

Condition (3) guarantees that every descending chain of monomials is finite $\Rightarrow$ induction over "the biggest" term of a polynomial.

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Now we consider $k[x_1, \ldots, x_n]$ and we write $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\alpha \in \mathbb{N}^n$. $x_1 > x_2 > \cdots > x_n$

Lexicographic

$x^\alpha \leq_{\text{Lex}} x^\beta$ iff:

- $\alpha_1 < \beta_1$ or
- $\alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$ or
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Initial term

Def. Given $f = \sum c_\alpha x^\alpha \in k[x_1, \ldots, x_n]$, $f \neq 0$, and a term order $\leq$ we define the *initial term* of f as:

$$\mathrm{in}_{\leq}(f) = \max\{x^\alpha \mid x^\alpha \in \mathrm{Supp}(f)\}.$$

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Es. $x > y > z > t$

- $\mathrm{in}_{\mathsf{Lex}}(x^3t + xy^3 - xy^2zt) = x^3t$.
- $\mathrm{in}_{\mathsf{RLex}}(x^3t + xy^3 - xy^2zt) = xy^3$.
- $\mathrm{in}_{\mathsf{DRLex}}(x^3t + xy^3 - xy^2zt) = xy^2zt$.

Initial Algebra

Let $S = k[x_1, \ldots, x_n]$ be the polynomial ring.

Def. A *polynomial algebra* generated by $\mathcal{F} \subseteq S$ is a subset of the polynomial ring $R \subseteq S$

$$R = \{p(f_1, \ldots, f_s) \mid s \in \mathbb{N}, p \in k[t_1, \ldots, t_s], f_1, \ldots, f_s \in \mathcal{F}\}.$$

We write $R = k[\mathcal{F}]$, or $k[f_1, \ldots, f_s]$ if $\mathcal{F}$ is finite and in this case we say that R is *finitely generated*.

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Example

1. $R = k[x + y + z, xy + xz + yz, xyz]$, $\mathrm{in}(R) = k[x, xy, xyz]$.

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1. $R = k[x + y + z, xy + xz + yz, xyz]$, $\mathrm{in}(R) = k[x, xy, xyz]$.
2. $R = k[x + y, xy, xy^2]$, $\mathrm{in}(R) = k[x, xy, xy^2, \dots, xy^n, \dots]$.

What is a Khovanskii basis?

Def. Let R be a finitely generated algebra in $k[x_1, \ldots, x_n]$. A subset $\mathcal{F} \subseteq R$ is a *Khovanskii basis* for a term order $\leq$ if

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Es.

1. F.g. polynomial algebras in 1 variable have a finite Khovanskii basis
2. Polynomial algebras f.g. by monomials have a finite Khovanskii basis.
3. Elementary symmetric polynomials form a Khovanskii basis for the ring of symmetric polynomials:

$$S^{S_n} = k[x_1 + \dots + x_n, x_1x_2 + \dots + x_{n-1}x_n, \dots, x_1 \cdots x_n].$$

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Es. The invariant ring of the alternating group A_n, that is

$$S^{A_n} = k[x_1 + \cdots + x_n, \ldots, x_1 \cdots x_n, \prod_{i<j}(x_j - x_i)],$$

does not admit a finite Khovanskii basis with respect to any term order for every $n \geq 3$ (Göbel).

Algorithm

Subduction algorithm

Input: A Khovanskii basis $\mathcal{F}$ for R and $f \in S$.

Output:

- If $f \in R$: A constant and an expression of f as a polynomial in the elements of $\mathcal{F}$.
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Is $\mathcal{F}$ a K.B.?

$\mathcal{F} = \{f_1, \ldots, f_s\} \subseteq R$.

$$\varphi : k[t_1, \ldots, t_s] \longrightarrow \mathrm{in}(R), \ \varphi(t_i) = \mathrm{in}(f_i).$$

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Theorem. Consider $\ker(\varphi) = (g_1, \dots, g_d)$. Then $\mathcal{F}$ is a Khovanskii basis if and only if the subduction algorithm applied to $g_i(f_1, \dots, f_s)$ gives a constant for each i.

Hilbert function

Def. The *Hilbert function* of a $\mathbb{Z}-$graded $k-$algebra $R = \bigoplus_{d\in\mathbb{Z}} R_d$ is

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Theorem. The Hilbert function can be expressed with a polynomial. There exists a polynomial $\mathrm{HP}_R(t) \in \mathbb{Q}[t]$ such that $\mathrm{HF}_R(d) = \mathrm{HP}_R(d)$ for $d \geq d_0$. The integer d_0 is the *Hilbert regularity* of R.

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Theorem. A polynomial algebra R and its initial algebra have the same Hilbert function:

$$\mathrm{HF}_R(d) = \mathrm{HF}_{\mathrm{in}(R)}(d), \text{ for all } d \in \mathbb{Z}.$$

As a consequence, we get an easy way to compute a $k-$basis for R_d.

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If $R = k[\phi_0, \ldots, \phi_\ell] \subseteq k[x_1, \ldots, x_n]$, where $\{\phi_0, \ldots, \phi_\ell\}$ is a Khovanskii basis w.r.t. $\leq$, we define:

$$A = \{\alpha_i \mid \mathrm{in}_{\leq}(\phi_i) = x^{\alpha_i}\},$$

$$d \cdot A = \{\alpha_{i_1} + \cdots + \alpha_{i_d} \mid \alpha_{i_j} \in A\}.$$

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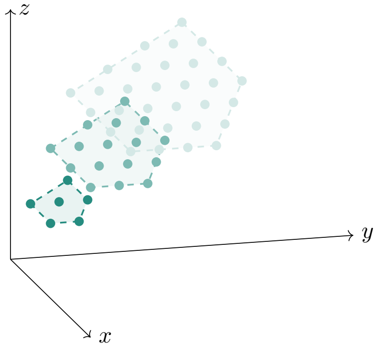
Each element in a basis of R_d corresponds to an element in $d \cdot A$:

$$R_d = \langle b_{d,\beta} | \beta \in d \cdot A \rangle_K,$$

where $b_{d,\beta} = \phi_{i_1} \cdots \phi_{i_d}$ and $0 \leq i_1 \leq \cdots \leq i_d \leq \ell$ are integers such that $\alpha_{i_1} + \cdots + \alpha_{i_d} = \beta$.

Example: del Pezzo surface

- $R = k[x-y, y^2-y, xy-y, x^2-y, xy^2-y, x^2y-y]$
- $\text{in}(R) = k[x, y^2, xy, x^2, xy^2, x^2y]$
- $A = \{(1,0), (0,2), (1,1), (2,0), (1,2), (2,1)\}$
- $2 \cdot A = \{(2,0), (1,2), (2,1), \ldots\}$
- $R_2 = \langle (x-y)^2, (x-y)(y^2-y), (x-y)(xy-y), \ldots \rangle$



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1. Extend Gröbner basis theory to subalgebras.
2. The coordinate ring of a variety is a polynomial algebra.
3. Useful to solve polynomial system using computer algebra.
4. New project computationally possible thanks to Khovansii Basis with M. Panizzut and S. Telen.

Main problem

We consider the problem of finding

$$z \in K^n \quad \text{such that} \quad f_1(z) = \cdots = f_s(z) = 0,$$

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Let $\phi_0, \ldots, \phi_\ell \in K[t_1, \ldots, t_n]$ be a different list of polynomials and let $d_1, \ldots, d_s \in \mathbb{N}^*$ be positive integers such that:

$$f_i(t) = \sum_{|\alpha| = d_i} c_{i,\alpha} \, \phi_0(t)^{\alpha_0} \phi_1(t)^{\alpha_1} \cdots \phi_\ell(t)^{\alpha_\ell}, \quad i = 1, \ldots, s.$$

We are "forcing" f_i to be homogeneous.

Our point of view

We reformulate the problem considering the *unirational variety* X obtained by the closed image of the map

$$\phi : K^n \dashrightarrow \mathbb{P}^\ell_K, \quad t \mapsto (\phi_0(t) : \cdots : \phi_\ell(t))$$

$$X := \mathrm{Cl}\{(\phi_0(t) : \cdots : \phi_\ell(t)) \in \mathbb{P}^\ell_K | \; t \in K^n \setminus \mathbb{V}(\phi_0, \ldots, \phi_\ell)\}.$$

Now we look for the *parameterized solutions* :

$$x \in X \subset \mathbb{P}^\ell_K, \quad \text{s.t.} \quad F_1(x) = \cdots = F_s(x) = 0, \quad \text{with } F_i = \sum_{|\alpha|=d_i} c_{i,\alpha}\, x^\alpha.$$

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We are "replacing" the polynomials ϕ_i with new variables x_i.

Example: oscillators

The following system of two equations in two unknowns arises from the Duffing equation modelling damped and driven oscillators (Breiding, Michałek, Monin, Telen).

$$f_1 = 1 + 3t_1 + 5t_2 + 7t_1(t_1^2 + t_2^2), \quad f_2 = 11 + 13t_1 + 17t_2 + 19t_2(t_1^2 + t_2^2).$$

In this case we have $\phi_0 = 1,\ \phi_1 = t_1,\ \phi_2 = t_2,\ \phi_3 = t_1(t_1^2 + t_2^2),\ \phi_4 = t_2(t_1^2 + t_2^2)$ and

$$f_1 = 1 \cdot \phi_0 + 3 \cdot \phi_1 + 5 \cdot \phi_2 + 7 \cdot \phi_3$$
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The surface X is defined by 3 polynomials:

$$x_1x_4 - x_2x_3 = x_1^2x_2 + x_2^3 - x_4x_1^2 = x_1^3 + x_1x_2^2 - x_3x_0^2 = 0 \quad \text{in } \mathbb{P}^4.$$

The polynomials F_i are defined as follows:

$$F_1 = x_0 + 3\,x_1 + 5\,x_2 + 7\,x_3, \quad F_2 = 11x_0 + 13\,x_1 + 17\,x_2 + 19\,x_4.$$

Khovanskii-Macaulay matrix

We use some matrices to compute solutions of $F_1 = \cdots = F_s = 0$ working directly in $K[X]$. We call them *Khovanskii-Macaulay matrices* $M_X(d)$. In degree $d = 2$ we get the matrix $M_X(2)$

	x_0^2	x_0x_1	x_1^2	x_0x_2	x_1x_2	x_2^2	x_0x_3	x_1x_3	x_2x_3	x_3^2	x_0x_4	x_2x_4	x_3x_4	x_4^2
$x_0 \cdot F_1$	1	3	0	5	0	0	7	0	0	0	0	0	0	0
$x_1 \cdot F_1$	0	1	3	0	5	0	0	7	0	0	0	0	0	0
$x_2 \cdot F_1$	0	0	0	1	3	5	0	0	7	0	0	0	0	0
$x_3 \cdot F_1$	0	0	0	0	0	0	1	3	5	7	0	0	0	0
$x_4 \cdot F_1$	0	0	0	0	0	0	0	0	3	0	1	5	7	0
$x_0 \cdot F_2$	11	13	0	17	0	0	0	0	0	0	19	0	0	0
$x_1 \cdot F_2$	0	11	13	0	17	0	0	0	19	0	0	0	0	0
$x_2 \cdot F_2$	0	0	0	11	13	17	0	0	0	0	0	19	0	0
$x_3 \cdot F_2$	0	0	0	0	0	0	11	13	17	0	0	0	19	0
$x_4 \cdot F_2$	0	0	0	0	0	0	0	0	13	0	11	17	0	19

For general d, the rows of $M_X(d)$ are indexed by all multiples $x^\alpha \cdot F_i$, where x^α runs over a basis of $K[X]_{d-\deg(F_i)}$. The columns are indexed by a monomial basis of $K[X]_d$.

Main Theorem

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Theorem. Let X be an arithmetically Cohen-Macaulay variety of dimension n and $I = \langle F_1, \ldots, F_n \rangle \subset K[X]$ be a homogeneous ideal with $\deg(F_i) = d_i$, such that $\dim(V_X(I)) = 0$. To solve $F_1 = \cdots = F_s = 0$ we need to compute the Khovanskii-Macaulay matrix in degree $\sum_{i=1}^{n} d_i + \mathrm{HReg}(K[X]) + 1$.

Using Khovanskii bases

Question 2

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Thanks for your attention!