Converse theorems for modular L-functions

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Introduction

- *modular form*: $f : H \to \mathbb{C}$ holomorphic with suitable invariance properties;
- *Dirichlet series*: $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$

Example (Riemann ζ function)

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad \text{verifies} \quad \Phi(s) = \Phi(1-s)$$

$$\text{where} \quad \Phi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

- modular form $\longrightarrow$ Dirichlet series $\longrightarrow$ functional equation.

QUESTION: is it true that a Dirichlet series satisfying a proper functional equation comes from a modular form?

Modular group and modular forms

The *(full) modular group* is defined as

$$SL_2(\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \gamma : z \longmapsto \frac{az+b}{cz+d}$$

$SL_2(\mathbb{Z})$ is generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

which correspond to the transformations on H

$$S : z \longmapsto -\frac{1}{z} \quad \text{e} \quad T : z \longmapsto z + 1$$

Modular group and modular forms

Definition

A *modular form* of weight $k \in \mathbb{Z}_+$ for $SL_2(\mathbb{Z})$ is a function f holomorphic on $H \cup \{\infty\}$ satisfying for any $\gamma \in SL_2(\mathbb{Z})$

$$f(z) = f_{|\gamma}(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{-k} f(\gamma z).$$

$$f \text{ holomorphic at } \infty \iff f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z} = \sum_{n=0}^{\infty} a_n e(nz).$$

Definition

A *cusp form* is a modular form such that $f(\infty) = a_0 = 0$.

Modular group and modular forms

Notation:

- $M_k(SL_2(\mathbb{Z}))$ is the space of modular forms of weight k for the full modular group.
- $S_k(SL_2(\mathbb{Z}))$ is the space of cusp forms of weight k for the full modular group.

Remark

1. Since $SL_2(\mathbb{Z}) = \langle T, S \rangle$ a function f holomorphic on H is a modular form of weight k for $SL_2(\mathbb{Z})$ if and only if it satisfies
   - $f(z) = f(z+1)$.
   - $f(z) = z^{-k} f(-1/z)$.
2. Since $-I \in SL_2(\mathbb{Z})$, if k is odd then $M_k(SL_2(\mathbb{Z})) = 0$. In fact, $f = f_{|-I} \iff f(z) = (-1)^k f(z)$.
3. The definition of the *slash* operator can be extended to $\gamma \in GL_2^+(\mathbb{R})$ $f_{|\gamma}(z) = (\det(\gamma))^{k/2}(cz+d)^{-k} f(\gamma z)$.

The functional equation

Given $f \in S_k(SL_2(\mathbb{Z}))$, with sequence of Fourier coefficients (a_n), we associate the Dirichlet series

$$L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$$

and the *complete* L-function

$$\Lambda(s, f) = (2\pi)^{-s}\Gamma(s)L(s, f).$$

The invariance property $f = f_{|S}$ gives the modular relation

$$f(z) = z^{-k} f(-1/z),$$

which with standard analytic methods leads to the functional equation

$$\Lambda(s, f) = i^k \Lambda(k - s, f).$$

Congruence subgroups

Let $N \geq 1$. The *principal congruence subgroup* of level N is

$$\Gamma(N) = \left\{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}.$$

Definition

A subgroup $\Gamma \subset SL_2(\mathbb{Z})$ is a *congruence subgroup* if there exists an integer $N \geq 1$ such that $\Gamma(N) \subset \Gamma$.

$$\Gamma_1(N) = \left\{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$

$$\Gamma_0(N) = \left\{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}$$

$$\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N)$$

Modular forms for congruence subgroups

Let Γ be a congruence subgroup.We define an equivalence relation over $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{ \infty \}$

$$z \sim w \Longleftrightarrow \text{there exists} \quad \gamma \in \Gamma \quad \text{such that} \quad w = \gamma z.$$

An equivalence class modulo Γ is a *cusp*.

Definition

A *modular form* of weight k for a congruence subgroup Γ is a function f holomorphic on $H^* = H \cup \mathbb{P}^1(\mathbb{Q})$, such that

$$\text{for all} \quad \gamma \in \Gamma \quad f = f_{|\gamma}.$$

A *cusp form* is a modular form vanishing at all cusps.

The functional equation

Observe that, if $N > 1$, the matrix $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \notin \Gamma_0(N)$.

So, *how can we get a functional equation in this case?*

Consider the *Fricke involution* $\omega_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. It normalises the group $\Gamma_0(N)$, i.e. if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$,

$$\gamma' = \omega_N \gamma \omega_N^{-1} = \begin{pmatrix} d & -c/N \\ -Nb & a \end{pmatrix} \in \Gamma_0(N).$$

Then, if $f \in S_k(\Gamma_0(N))$, for all $\gamma \in \Gamma_0(N)$,

$$f_{|\omega_N \gamma} = f_{|\gamma' \omega_N} = f_{|\omega_N} \Longrightarrow f_{|\omega_N} \in S_k(\Gamma_0(N)).$$

Now, the operator $|\omega_N$ is an involution ($f_{|\omega_N^2} = f$) and an endomorphism of the space of cusp forms, so it only has eigenvalues ± 1, hence

$$S_k(\Gamma_0(N)) = S_k^+(\Gamma_0(N)) \oplus S_k^-(\Gamma_0(N)),$$

where $S_k^\pm(\Gamma_0(N))$ are the eigenspaces corresponding to ± 1 respectively.

The functional equation

Then, if $f \in S_k(\Gamma_0(N))$, we have

$$f = \pm f_{|\omega_N} \Longleftrightarrow f(z) = \pm N^{-k/2} z^{-k} f\left(- \frac{1}{Nz} \right).$$

This modular relation implies the functional equation

$$\Lambda(s, f) = \pm i^k \Lambda(k - s, f),$$

where, as usual,

$$\Lambda(s, f) = \left(\frac{\sqrt{N}}{2\pi} \right)^s \Gamma(s) L(s, f)$$

and $L(s, f)$ is the Dirichlet series associated to f.

Hecke's converse theorem

Let now $N = 1, 2, 3, 4$. Consider the Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

convergent in a right half plane, and assume the functional equation

$$\Lambda(s) = \pm i^k \Lambda(k - s), \quad \text{where} \quad \Lambda(s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(s).$$

Then, $f \in S_k(\Gamma_0(N))$, where

$$f(z) = \sum_{n=1}^{\infty} a_n e(nz).$$

The result is due to the fact that, for these values of the level N,

$$\Gamma_0(N) = \langle T, W_N \rangle, \quad \text{with} \quad W_N = \omega_N T^{-1} \omega_N.$$

In fact, by Mellin inversion formula, the functional equation gives

$$f(iy) = \pm i^k N^{-k/2} y^{-k} f\left(\frac{i}{Ny}\right) \iff f = \pm f_{|\omega_N}.$$

Then,

- Fourier expansion $\Longrightarrow f_{|T} = f$.
- $f_{|\omega_N} = \pm f \Longrightarrow f_{|W_N} = f$.

What about $N \geq 5$?

- the number of generators of $\Gamma_0(N)$ grows.
- the space of L-function satisfying the functional equation has infinite dimension.
- the space of modular/cusp forms for $\Gamma_0(N)$ is always finite dimensional.
- the single functional equation is not enough to conclude.
- we need additional conditions on the Dirichlet series.

Two main approaches to the matter:

- the *twist-theory*: assume that proper twists by Dirichlet characters of the original L-function satisfy a functional equation. (Weil)
- Conrey-Farmer's converse theorem: besides the usual functional equation assume the existence of a proper *Euler product*.

Modular forms and Dirichlet characters

Definition

A *Dirichlet character* modulo $N \geq 1$ is a group homomorphism $\psi : \mathbb{Z} \to \mathbb{C}^*$ periodic of period N, such that $\psi(a) \neq 0 \quad \text{iff} \quad (N, a) = 1$.

So, let ψ be a Dirichlet character modulo N, $M_k(\Gamma_0(N), \psi)$ is the space of the modular forms $f \in M_k(\Gamma_1(N))$ such that

$$\text{for all} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \quad f_{|\gamma} = \psi(d) f.$$

The following decompositions hold

$$M_k(\Gamma_1(N)) = \bigoplus_{\psi \pmod N} M_k(\Gamma_0(N), \psi)$$

$$S_k(\Gamma_1(N)) = \bigoplus_{\psi \pmod N} S_k(\Gamma_0(N), \psi)$$

The functional equation

Let $f \in S_k(\Gamma_0(N), \psi)$ and let $g = f_{|\omega_N}$. Then

$$g_{|\gamma} = f_{|\omega_N \gamma} = f_{|\gamma' \omega_N} = \psi(a) f_{|\omega_N} = \bar{\psi}(d) g \Longrightarrow g \in S_k(\Gamma_0(N), \bar{\psi}).$$

So, f, g are related by equation

$$f(i/y) = N^{-\frac{k}{2}} i^k g(iy/N),$$

which, with the usual analytic technique, leads to the functional equation

$$\Lambda(s, f) = i^k \Lambda(k - s, g),$$

where

$$\Lambda(s, f) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(s, f) \qquad L(s, f) = \sum_{n=1}^{\infty} a_n e(nz)$$

$$\Lambda(s, g) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(s, g) \qquad L(s, g) = \sum_{n=1}^{\infty} b_n e(nz).$$

The twist theory

Let N, D be coprime integers, ψ a character modulo N, χ a primitive character modulo D and $f \in S_k(\Gamma_0(N), \psi)$.

$$f(z) = \sum_{n=1}^{\infty} a_n e(nz) \longrightarrow f_\chi(z) = \sum_{n=1}^{\infty} a_n \chi(n) e(nz).$$

It can be proved that $f_\chi \in S_k(\Gamma_0(ND^2), \psi\chi^2)$ and if $g = f_{|\omega_N}$

$$\Lambda_1(s,\chi) = i^k \omega(\chi) \Lambda_2(k-s, \bar{\chi}),$$

where $\omega(\chi) = \psi(D)\chi(N)\tau_\chi^2 D^{-1}$, τ_χ is the Gauss sum,

$$L_1(s,\chi) = \sum_{n=1}^{\infty} a_n \chi(n) n^{-s} \qquad L_2(s,\bar{\chi}) = \sum_{n=1}^{\infty} b_n \bar{\chi}(n) n^{-s}$$

$$\Lambda_1(s,\chi) = \left(\frac{D\sqrt{N}}{2\pi}\right)^s \Gamma(s) L_1(s,\chi) \qquad \Lambda_2(s,\bar{\chi}) = \left(\frac{D\sqrt{N}}{2\pi}\right)^s \Gamma(s) L_2(s,\bar{\chi})$$

and (b_n) is the sequence of Fourier coefficients of g at ∞.

Weil's converse theorem

Assumptions:

- $(a_n), (b_n)$, $n \geq 1$ with polynomial growth.
- S finite set of primes, including those dividing N.
- For $D = 1$ or $D \notin S$, for any primitive character $\chi \pmod{D}$, $\Lambda_1(s, \chi)$, $\Lambda_2(s, \bar{\chi})$ are EBV and satisfy

$$\Lambda_1(s, \chi) = i^k \omega(\chi) \Lambda_2(k - s, \bar{\chi}).$$

Then

$$f(z) = \sum_{n=1}^{\infty} a_n e(nz) \in S_k(\Gamma_0(N), \psi)$$

$$g(z) = \sum_{n=1}^{\infty} b_n e(nz) \in S_k(\Gamma_0(N), \bar{\psi}).$$

Moreover,

$$g = f_{|\omega_N}.$$

Weil's converse theorem: sketch of the proof

- By Mellin inversion formula, using the functional equation for twists,

$$f_\chi = \omega(\chi) g_{\bar{\chi}}{}_{\left|\begin{pmatrix} 0 & -1/(D^2N) \\ 1 & 0 \end{pmatrix}\right.} \quad \text{so} \quad f_\chi{}_{\left|\begin{pmatrix} 0 & -1 \\ D^2N & 0 \end{pmatrix}\right.} = \omega(\chi) g_{\bar{\chi}}.$$

  In particular, choosing $D = 1$, we get $g = f_{|\omega_N}$.
- For $D, s \notin S$ distinct odd primes

$$g_{\left|\begin{pmatrix} D & -r \\ -Nm & s \end{pmatrix}\right.} = \bar{\psi}(s) g \quad \text{where} \quad Ds - rNm = 1.$$

Weil's converse theorem: sketch of the proof

- We want to prove that for $a, b, c, d \in \mathbb{Z}$ such that $ad - bNc = 1$

$$g_{|\begin{pmatrix} a & b \\ Nc & d \end{pmatrix}} = \bar{\psi}(d) g.$$

By Dirichlet's theorem

Theorem

Given m, n coprime integers, the arithmetic progression

$$m, m+n, m+2n, ..., m+kn, ...$$

contains infinitely many primes.

- $(a, Nc) = (d, Nc) = 1$, so there exist u, v such that

$$D = a - uNc \qquad s = d - vNc$$

and $D, s \notin S$ are distinct primes, since S is a finite set.

Weil's converse theorem: sketch of the proof

- Taking $r = -av + uvNc + b - ud$, we have

$$\begin{pmatrix} a & b \\ Nc & d \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D & r \\ Nc & s \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$$

hence

$$g_{|\begin{pmatrix} a & b \\ Nc & d \end{pmatrix}} = g_{|\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D & r \\ Nc & s \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}} = g_{|\begin{pmatrix} D & r \\ Nc & s \end{pmatrix}}.$$

Then we conclude

$$g_{|\begin{pmatrix} a & b \\ Nc & d \end{pmatrix}} = \bar{\psi}(s) g = \bar{\psi}(d) g.$$

Conrey-Farmer's converse theorem

$$L(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad \text{and} \quad f(z) = \sum_{n=1}^{\infty} a_n e(nz)$$

- L satisfies the functional equation

$$\Lambda(s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(s) = \pm i^k \Lambda(k-s)$$

- L has an *Euler product* of the form

$$L(s) = \prod_p L_p(s)$$

$$\begin{aligned}
L_p(s) &= (1 - a_p p^{-s} + p^{k-1-2s})^{-1} && \text{if} \quad p \nmid N \\
L_p(s) &= (1 - p^{k/2-1-s})^{-1} && \text{if} \quad p \mid\mid N \\
L_p(s) &= 1 && \text{if} \quad p^2 \mid N
\end{aligned}$$

Conrey-Farmer's converse theorem

Then, for $5 \leq N \leq 17$ and $N = 23$, we have

$$f \in S_k(\Gamma_0(N)).$$

- Hecke and Atkin-Lehner operators for p prime

$$T_p, U_p : S_k(\Gamma_0(N)) \to S_k(\Gamma_0(N))$$

$$T_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} + \sum_{a=0}^{p-1} \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix} \quad \text{and} \quad U_p = \sum_{a=0}^{\infty} \begin{pmatrix} p & a \\ 0 & p \end{pmatrix}$$

- the Euler product is equivalent to

$$T_p f = a_p f, \quad U_p f = f_{\Big| \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}, \quad U_p f = 0$$

respectively for $p \nmid N$, $p \,||\, N$ and $p^2 \mid N$.

Conrey-Farmer's converse theorem

- For any value of the level N
  - $f_{|T} = f$ (by the Fourier expansion)
  - $f_{|\omega_N} = \pm f$ (by the functional equation) $\Longrightarrow f_{|W_N} = f$.
- Let $N = 5, 7, 9$. In these cases

$$\Gamma_0(N) = \langle W_N, T, M_2 \rangle \quad \text{where} \quad M_2 = \begin{pmatrix} 2 & -1 \\ -N & (N+1)/2 \end{pmatrix}.$$

  - $2 \nmid N \Longrightarrow T_2 f = a_2 f$.
  - combining with $f_{|T} = f$ and $f_{|\omega_N} = \pm f$, we get $f_{|M_2} = f$.
  - then for all $\gamma \in \Gamma_0(N)$, $f_{|\gamma} = f$.
- for $N = 6, 8, 10, 12, 16$ we again only use the local factor at $p = 2$, distinguishing $2 \mid\mid N$ and $4 \mid N$.

Conrey-Farmer's converse theorem

- let $N = 11, 14, 15, 17, 23$. In this cases, the key point is

Lemma

Let $\gamma \in SL_2(\mathbb{Z})$ *satisfy*

$$f_{|(1-\gamma)\varepsilon} = f_{|(1-\gamma)},$$

where $\varepsilon \in SL_2(\mathbb{R})$ *is an elliptic matrix of infinite order. Then,* $f_{|\gamma} = f$.

We recall that a matrix $\varepsilon \in SL_2(\mathbb{R})$ is *elliptic* iff $|tr(\varepsilon)| < 2$. Equivalently, its eigenvalues have absolute value 1, but they are not roots of unity.

- for $N = 13$, the converse theorem has been proved in a more recent paper (2006), assuming the usual functional equation and the Euler product for the local factors at the primes $p = 2, 3$.

Thank you for your attention!