An introduction to the category of equilogical spaces

Giulia Frosoni

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Categories

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There is an operation of **composition** of arrows:

$$A \xrightarrow{f} B \xrightarrow{g} C$$

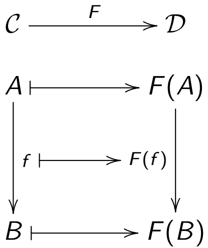
$$A \xrightarrow{g \circ f} C$$

such that

- composition is **associative**: $(g \circ f) \circ h = g \circ (f \circ h)$;
- for each object X, there is the so-called **identity arrow** $1_X : X \to X$ such that for every arrow $f : A \to B$, $1_B \circ f = f \circ 1_A = f$.

Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A **functor** F between $\mathcal{C}$ and $\mathcal{D}$ associates every object of $\mathcal{C}$ with an object of $\mathcal{D}$ and every arrow of $\mathcal{C}$ with an arrow of $\mathcal{D}$ in the following way:



Moreover,

- $F(1_A) = 1_{F(A)}$ for every object $A \in \mathcal{C}$;
- $F(g \circ f) = F(g) \circ F(f)$.

Properties of functors

A functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is

- **Faithful**: given arrows $f, g : A \to B$ of $\mathcal{C}$, if $F(f) = F(g)$ then $f = g$.
- **Full**: for every arrow $g : F(A) \to F(B)$ of $\mathcal{D}$, there exists an arrow $f : A \to B$ of $\mathcal{C}$ such that $F(f) = g$.
- **Essentially surjective on objects**: for every object D in $\mathcal{D}$, there exists an object C in $\mathcal{C}$ such that $F(C)$ is isomorphic to D.

Let A and B be two objects in a category $\mathcal{C}$. A and B are **isomorphic** if there exist two arrows $f : A \to B$ and $g : B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$.

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$\mathcal{C}$ and $\mathcal{D}$ are **equivalent** if there exists a functor F between $\mathcal{C}$ and $\mathcal{D}$ which is faithful, full and essentially surjective on objects.

Two equivalent categories have the same "categorical properties".

Equ: equilogical spaces (Scott, 1976)

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Arrows: $[f]_{\mathbf{E}\to\mathbf{F}} : (E, \tau_E) \to (F, \tau_F)$ is an equivalence class of continuous functions between the topological spaces which, in addition, preserve the equivalence relations, that is

$$\text{for all } e, e' \in E, \text{ if } e \equiv_E e' \text{ then } f(e) \equiv_F f(e')$$

and moreover,

$$f \equiv_{\mathbf{E}\to\mathbf{F}} f' \text{ when for all } e, e' \in E, \text{ if } e \equiv_E e' \text{ then } f(e) \equiv_F f'(e')$$

We recall the T_0 **separation axiom** for a topological space (X, τ_X):

for any two different points x and y, there is an open set which contains one of these points and not the other

or equivalently

$$\text{for all } x, y \in X, \text{ if } \{U \in \tau_X | x \in U\} = \{U \in \tau_X | y \in U\}, \text{ then } x = y.$$

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Equ embeds the category Top_0 of T_0 topological spaces and continuous functions:

$$\mathrm{Top}_0 \hookrightarrow \mathrm{Equ}$$
$$(X, \tau_X) \mapsto (X, \tau_X, =)$$

When Universal Algebra meets Category Theory

A **lattice** is a poset $(P, \leq)$ where every pair x, y of elements have a least upper bound $(x \vee y)$ and a greatest lower bound $(x \wedge y)$. It is standard notation to indicate the lattice with $(L, \wedge, \vee)$.

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A lattice P is called **complete** if for any $S \subseteq P$, the least upper bound $\bigvee S$ of S and the greatest lower bound $\bigwedge S$ of S exist.

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Examples

- Let X be a set. $(\mathscr{P}(X), \cap, \cup)$ is a complete lattice.
- Any non-empty finite lattice is trivially complete.
- If (X, τ_X) is a topological space, then $(\tau_X, \wedge, \cup)$ is a complete lattice where, given $U, V \in \tau_X$, $U \wedge V$ is the greatest open set contained in $U \cap V$.

We can see a complete lattice as a topological space, endowing the lattice with the following T_0 topology:

Scott topology

Let $(L, \leq)$ be a complete lattice. The *Scott topology* τ_{Sc} on L consists of those subsets $U \subseteq L$ such that:

- U is upward closed: if $x \in U$ and $y \in L$ and $x \leq y$, then $y \in U$.
- For every subset $S \subseteq L$, if $\bigvee S \in U$ then there exists a finite subset $S_0 \subseteq S$ such that $\bigvee S_0 \in U$.

If $(L, \leq)$ is a complete lattice, (L, τ_{Sc}) is a T_0 topological space.

CLat: category of complete lattices.

Objects: complete lattices

Arrows: functions between lattices, continuous under the Scott topology (Scott-continuous functions).

Algebraic lattices

Let $(L, \leq)$ be a complete lattice. An element $c \in L$ is called **compact** if whenever $c \leq \bigvee S$ for a subset $S \subseteq L$, then there exists a finite subset $S_0 \subseteq S$ such that $c \leq \bigvee S_0$.

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Example

If X is a set, then $(\mathscr{P}(X), \cup, \cap)$ is an algebraic lattice and the compact elements in $\mathscr{P}(X)$ are the finite subsets of X.

ALat: category of algebraic lattices.

Objects: algebraic lattices

Arrows: Scott-continuous functions.

PEqu: partial equilogical spaces

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Objects: $\mathbf{A} = (A, \tau_{Sc}, \approx_A)$, where A is an algebraic lattice, τ_{Sc} is the Scott topology on A and $\approx_A$ is a partial equivalence relation on A, namely it is symmetric and transitive, but not necessary reflexive.

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Arrows: $[f]_{\mathbf{A}\to\mathbf{B}} : (A, \tau_{Sc}) \to (B, \tau_{Sc})$ is an equivalence class of Scott-continuous functions which preserve the partial equivalence relations, namely

$$\text{for all } a, a' \in A, \text{ if } a \approx_A a' \text{ then } f(a) \approx_B f(a')$$

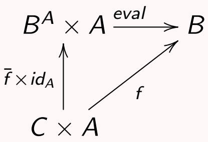
and moreover,

$$f \approx_{\mathbf{A}\to\mathbf{B}} f' \text{ when for all } a, a' \in A, \text{ if } a \approx_A a' \text{ then } f(a) \approx_B f'(a').$$

PEqu is cartesian closed.

Cartesian closure

A category **C** is *cartesian closed* if it has finite products and, given two objects A and B in **C**, there exists an object B^A and an arrow $eval : B^A \times A \to B$ such that for every object C in **C** and for every arrow $f : C \times A \to B$, there exists a unique arrow $\bar{f} : C \to B^A$ such that the following diagram is commutative:



If $\mathcal{A} = (A, \tau_{Sc}, \approx_A)$ and $\mathcal{B} = (B, \tau_{Sc}, \approx_B) \in$ PEqu, the exponential object in PEqu is $\mathcal{B}^{\mathcal{A}} = (B^A, \tau_{Sc}, \approx_{B^A})$ where $B^A = \{f : A \to B | f \text{ is Scott-continuous}\}$ (B^A is an algebraic lattice!) and $\approx_{B^A}$ is the same as $\approx_{A \to B}$.

Equivalence between PEqu and Equ

PEqu and Equ are equivalent. Indeed, the following functor is full, faithful and essentially surjective on objects:

$$\begin{array}{ccc}
\mathsf{PEqu} & \xrightarrow{\quad R \quad} & \mathsf{Equ} \\
 & & \\
(A, \tau_{Sc}, \approx_A) & \longmapsto & (dom(A), \tau_{dom(A)}, \equiv_{dom(A)}) \\
\Big\downarrow & [f] \longmapsto [f|_{dom(A)}] & \Big\downarrow \\
(B, \tau_{Sc}, \approx_B) & \longmapsto & (dom(B), \tau_{dom(B)}, \equiv_{dom(B)})
\end{array}$$

where $dom(A) = \{a \in A | a \approx_A a\}$, $\tau_{dom(A)}$ is the inherited topology and $\equiv_{dom(A)}$ is the restriction of $\approx_A$ to $dom(A)$.

Therefore, Equ is a cartesian closed category which embeds Top_0 (Top_0 is NOT cartesian closed).

Natural Transformations

Let $\mathcal{C}$ and $\mathcal{D}$ two categories and $F, G : \mathcal{C} \to \mathcal{D}$ two functors between these categories. A **natural transformation** $\eta : F \to G$ is a class of arrows defined as follows:

- for every object X of $\mathcal{C}$, $\eta_X : F(X) \to G(X)$ is an arrow of $\mathcal{D}$, and
- for every arrow $f : X \to Y$ of $\mathcal{C}$, the following diagram is commutative:

$$
\begin{array}{ccc}
F(X) & \xrightarrow{F(f)} & F(Y) \\
{\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\
G(X) & \xrightarrow[G(f)]{} & G(Y)
\end{array}
$$

Monads

A **monad** on a category $\mathcal{C}$ is a triple (T, η, μ), where T is an endofunctor on $\mathcal{C}$ and η and μ are two natural transformations, $\eta : Id_{\mathcal{C}} \to T$ (where $Id_{\mathcal{C}}$ is the identical functor on $\mathcal{C}$) and $\mu : T^2 \to T$ (T^2 stands for $T \circ T$), such that the following diagrams are commutative

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array} \qquad\qquad \begin{array}{ccc} T & \xrightarrow{\eta T} & T^2 \\ {\scriptstyle T\eta}\downarrow & \searrow\!\!\!\!= & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array}$$

where

- $T\mu : T^3 \to T^2$ is a natural transformation such that for every object x of $\mathcal{C}$, $(T\mu)_x = T(\mu_x)$;
- $\mu T : T^3 \to T^2$ is a natural transformation such that for every object x of $\mathcal{C}$, $(\mu T)_x = \mu_{T(x)}$.

Sierpinski space

Let $\Sigma = \{0, 1\}$ and let $\tau_\Sigma = \{\emptyset, \{1\}, \{0, 1\}\}$. Then τ_Σ is a T_0 topology on Σ and it corresponds to the Scott topology on Σ, seen as a complete (actually, algebraic) lattice.

Since Σ is a T_0 topological space, we can see it as an equilogical space:

$$(\Sigma, \tau_\Sigma, =)$$

Note that the previous equilogical space is also an object of PEqu.

Towards the double-exponential monad

We can build the following controvariant endofunctor on Equ:

$$\mathrm{Equ} \xrightarrow{\Sigma^{(-)}} \mathrm{Equ}^{op}$$

$$\begin{array}{ccc}
\mathbf{A} & \longmapsto & \Sigma^{\mathbf{A}} \\
\Big\downarrow f & f \longmapsto -\circ f & \Big\uparrow \\
\mathbf{B} & \longmapsto & \Sigma^{\mathbf{B}}
\end{array}$$

Therefore, $\Sigma^{\Sigma^{(-)}} = \Sigma^{(-)} \circ \Sigma^{(-)}$ is a covariant endofunctor on Equ.

It is possible to build two natural transformations, η and μ, such that $(\Sigma^{\Sigma^{(-)}}, \eta, \mu)$ determines a monad on Equ, called the **double-exponential monad** on Equ.

Algebras for a monad

Let (T, η, μ) be a monad on $\mathcal{C}$. The category $\mathcal{C}^T$ of T-algebras consists of:

Objects: pairs (x, α), where x is an object of $\mathcal{C}$ and $\alpha : T(x) \to x$ an arrow of $\mathcal{C}$ called *structure map* such that the following diagrams are commutative

$$\begin{array}{ccc} T^2(x) & \xrightarrow{T\alpha} & T(x) \\ {\scriptstyle \mu_x}\downarrow & & \downarrow{\scriptstyle \alpha} \\ T(x) & \xrightarrow[\alpha]{} & x \end{array} \qquad\qquad \begin{array}{ccc} x & \xrightarrow{\eta_x} & T(x) \\ & {\scriptstyle 1_x}\searrow & \downarrow{\scriptstyle \alpha} \\ & & x \end{array}$$

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Arrows: $f : (x, \alpha) \to (x', \alpha')$ is an arrow $f : x \to x'$ of $\mathcal{C}$ such that the following diagram is commutative

$$\begin{array}{ccc} T(x) & \xrightarrow{Tf} & T(x') \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha'} \\ x & \xrightarrow[f]{} & x' \end{array}$$

Algebras for the double-exponential monad on Equ: the category $\mathrm{Equ}^{\Sigma^{\Sigma^{(-)}}}$

$\Sigma^{\Sigma^{(-)}}$-algebras on Equ: pairs $(\mathbf{A}, \alpha)$, where $\mathbf{A} \in$ Equ and $\alpha : \Sigma^{\Sigma^{\mathbf{A}}} \to \mathbf{A}$ is the structure map on $\mathbf{A}$.

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$\Sigma^{\Sigma^{(-)}}$**-algebra homomorphisms**: $f : (\mathbf{A}_1, \alpha_1) \to (\mathbf{A}_2, \alpha_2)$ arrow of Equ between $\mathbf{A}_1$ and $\mathbf{A}_2$ which, in addition, makes the following diagram commutative

$$\begin{array}{ccc} \Sigma^{\Sigma^{\mathbf{A}_1}} & \xrightarrow{\Sigma^{\Sigma^{f}}} & \Sigma^{\Sigma^{\mathbf{A}_2}} \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} \\ \mathbf{A}_1 & \xrightarrow[f]{} & \mathbf{A}_2 \end{array}$$

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Aim: investigating the category $\mathsf{Equ}^{\Sigma^{\Sigma^{(-)}}}$.

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Frames

A *frame* $(F, \wedge, \vee)$ is a complete lattice such that finite meets distribute over arbitrary joins, namely

$$a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge b_i)$$

where $a \in F$ and $b_i \in F$ for all $i \in I$.

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where $a \in F$ and $b_i \in F$ for all $i \in I$.

We prove that there is an adjunction between the category $\mathrm{Equ}^{\Sigma^{\Sigma^{(-)}}}$ and the category Frm of frames, but it does not seem that the two categories are equivalent.

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- Alternative description of Σ^{Σ^X} where X is a $\Sigma^{\Sigma^{(-)}}$-algebra on Top_0.
- **Next goal**: $\mathsf{Top}_0^{\Sigma^{\Sigma^{(-)}}}$: $\Sigma^{\Sigma^{(-)}}$-algebras on Top_0.

Thank you for your attention!