LEAN: The future of Mathematics

Ignacio Muñoz Jiménez

Università di Genova

2024

Quick Introduction

Topic: computers in mathematical research.

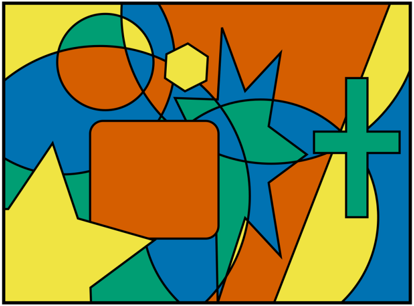
LEAN: proof verifier / proof assistant.

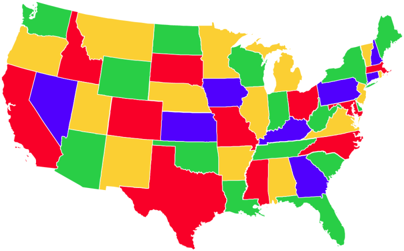
Contents

1. What is a proof verifier?
2. LEAN. How does it work?
3. Try it yourself!
4. Examples and projects.
5. The future of mathematics.

Motivation and Context

Four Color Theorem

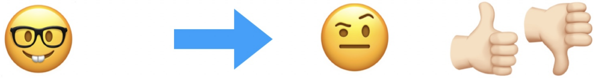




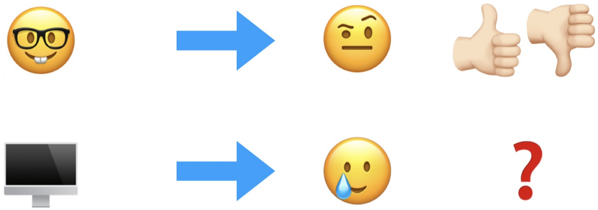
Proved by Keneth Appel and Wolfgang Hanken (1997).

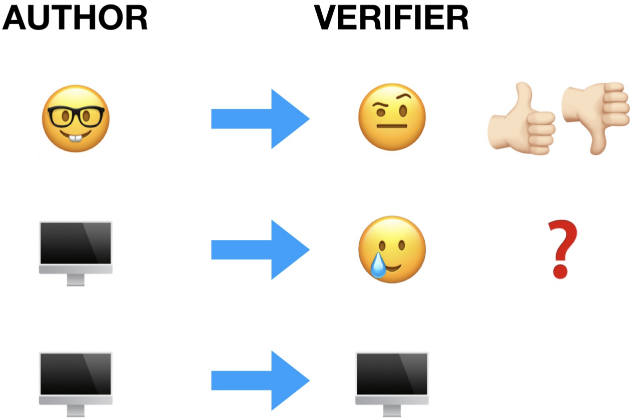
Checking 633 configurations using a computer.

AUTHOR **VERIFIER**

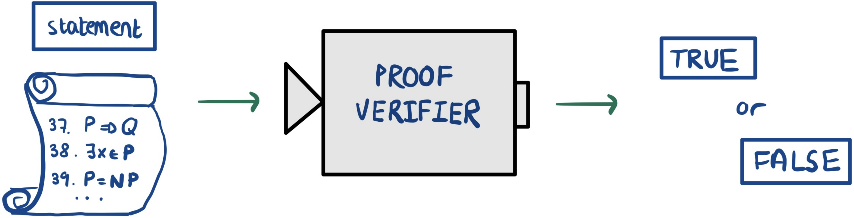


AUTHOR **VERIFIER**




AUTHOR
VERIFIER

What is a proof verifier?



Remarks

1. The proof must be written in a **LANGUAGE** computers can understand.
2. The computers must be able to check logic implications. This include the choice of axioms.

What is NOT a proof verifier?

It does NOT prove the theorems for you!!

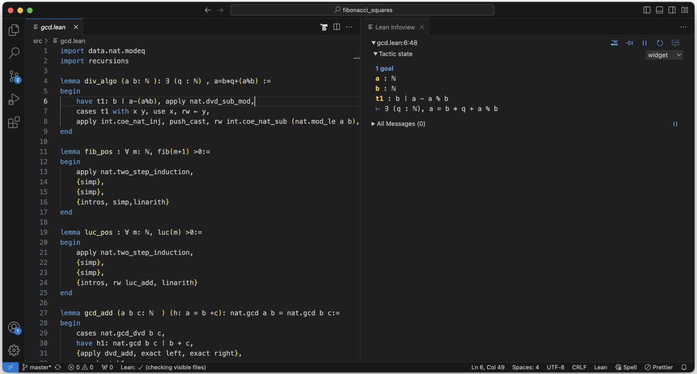
What is NOT a proof verifier?

It does NOT prove the theorems for you!!

for now

What is LEAN?

- LEAN is a proof verifier (proof assistant).
- Developed in 2013 by Leonardo de Moura at Microsoft Research (now Amazon Web Services).
- It is open sourced.



How does it work?

Logic exercise

Logic exercise

Let P, Q be propositions.

Lemma 1. *(Logic Exercise)* $P \to (Q \to P)$.

Proof. Assume P is True (Hypothesis P). Then we have to prove $Q \to P$.

Assume P is True (Hypothesis Q). Then we have to prove P.

We know by Hypothesis P that P is True. This finishes the proof. □

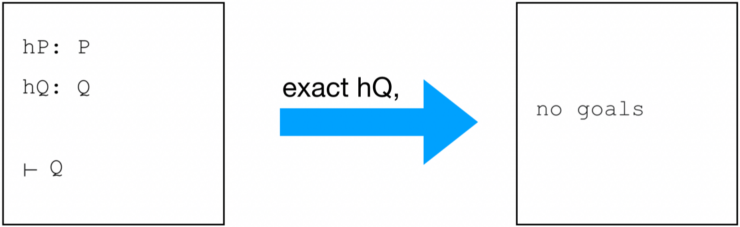
```
variables (P Q : Prop)

lemma logic_exercise: P->Q->P

begin

    intro hP,

    intro hQ,

    exact hP,

end
```

Example 1

exact (+hypothesis)

Provides the exact proof of the goal.

Example:

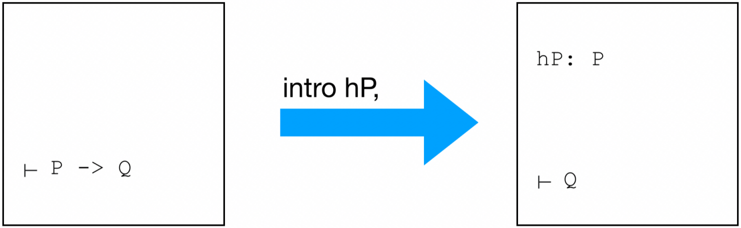


Some basic *tactics*

intro (+hypothesis' name)

Given a goal $P \to Q$, it introduces P as an hypothesis and turns the goal into Q.

Example:

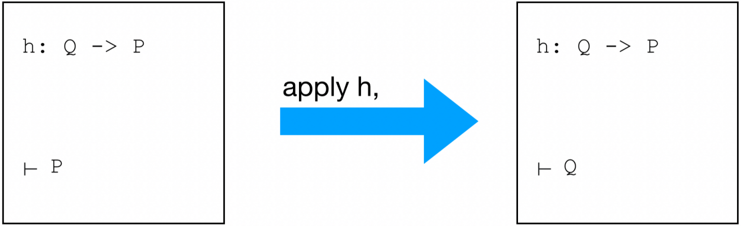


Some basic *tactics*

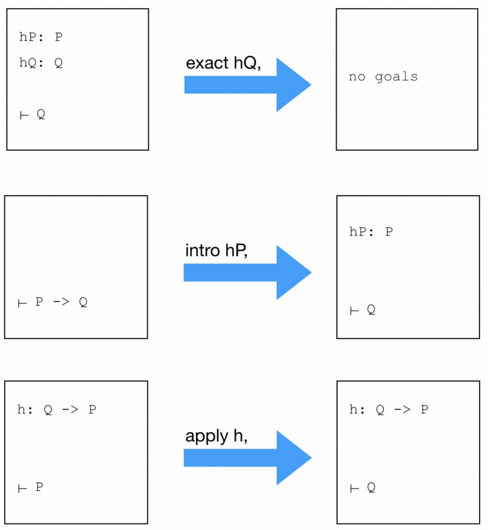
apply (+hypothesis)

Given a goal $P \to Q$ and a hypothesis $Q \to P$,turns the goal into P.

Example:



Try it yourself!

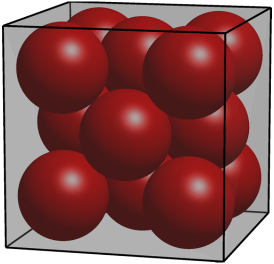


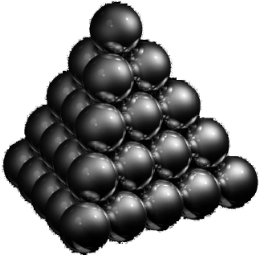
https://shorturl.at/bikrG

Example 2

People and Projects

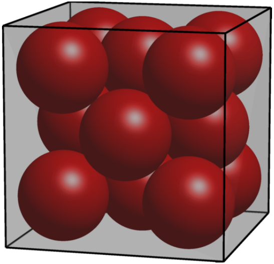
Kepler Conjecture. Sphere Packaging in 3 dimensions. Minimizing a function of 150 variables.

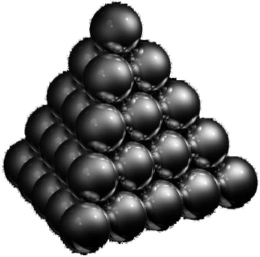




People and Projects

Kepler Conjecture. Sphere Packaging in 3 dimensions. Minimizing a function of 150 variables.





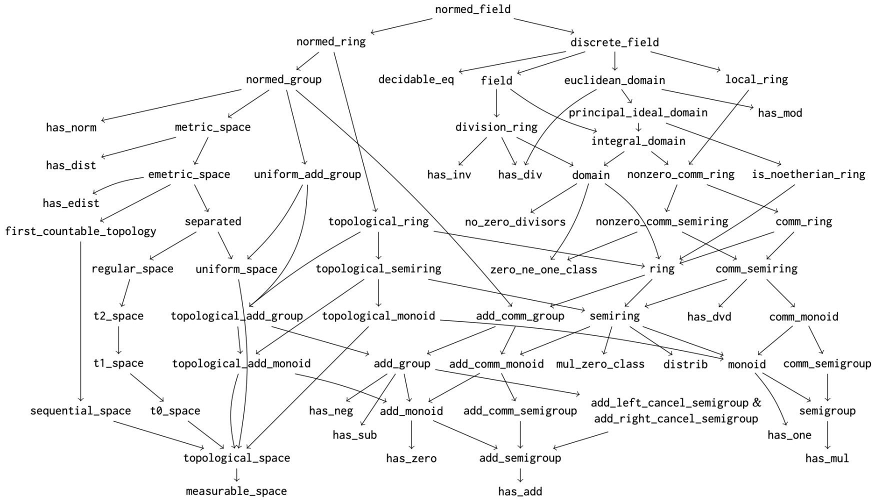
Kevin Buzzard (ICL)→ **Xena's Project**.
Formalize the Imperial's Undergraduate Curriculum.

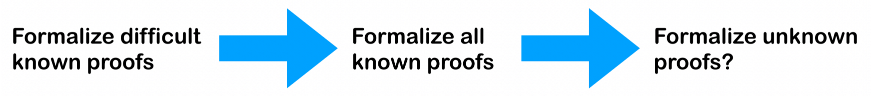
MATHLIB

- Unified library of mathematics.
- Nov. 2023: +127.000 theorems, +70.000 definitions.

Compare to Wikipedia or The Stacks project.

Dependency graph



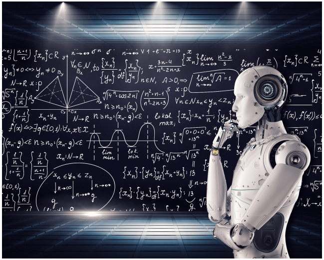


Use LEAN to generate new knowledge. Proof assistant.

1. LEAN + AI.
2. LEAN on publishing articles.
3. LEAN on education.

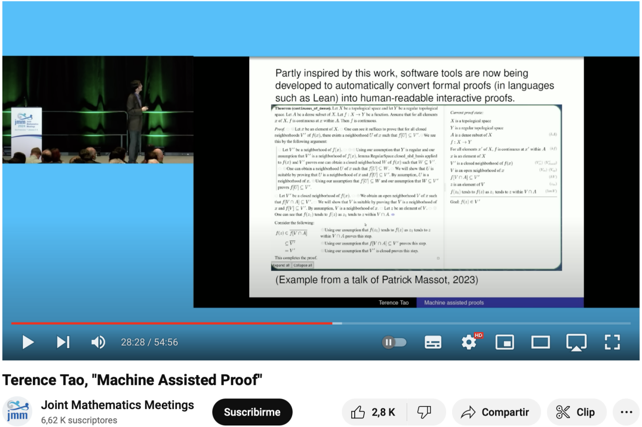
The future

Artificial Intelligence



- Solves very easy proofs.
- Uncover hidden connections.
- Propose proof strategies.

Research and scientific articles



Education: Xena's Project: LEAN as a game.

To learn more

More info about LEAN:

- Lean Webpage: `https://lean-lang.org`
- Lean Comunity / Mathlib: `https://leanprover-community.github.io`
- The Xena Project: `https://www.ma.imperial.ac.uk/~buzzard/xena/`

Where to learn LEAN:

- Natural Number Game: `https://adam.math.hhu.de`
- More references: `https://leanprover-community.github.io/learn.html`

Videos about LEAN:

- Terence Tao, "Machine Assisted Proof": `https://www.youtube.com/watch?v=AayZuuDDKP0`
- Microsoft research: `https://www.microsoft.com/en-us/research/project/lean/`