The algebraic approach to QFT

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Algebraic approach

Definition

An algebra $\mathcal{A}$ over $\mathbb{C}$ is called a $*$-algebra if:

- there exists an antilinear involution $*\colon \mathcal{A} \to \mathcal{A}$, called $*$-involution or adjoint, such that $(AB)^* = B^*A^*$ for all $A, B \in \mathcal{A}$;

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A state over $\mathcal{A}$ is a linear functional $\omega\colon \mathcal{A} \to \mathbb{C}$ which is:

- positive $\quad \omega(A^*A) \geq 0 \quad \forall A \in \mathcal{A}$;
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Example

1. $\mathcal{A} := C_c(X, \mathbb{C}) \qquad \omega(f) := \int_X f \, d\mu \qquad \mu$ probability measure
2. $\mathcal{A} = \mathcal{B}(H) \qquad \omega(A) := \mathrm{Tr}(\rho A) \qquad \rho$ density e.g. $\rho = |\psi\rangle\langle\psi|$;

Gelfand, Naimark, Segal Theorem

Theorem (GNS)

$(\mathcal{A}, \omega)$ as above. There exists a GNS triple $(\mathsf{H}_\omega, \mathsf{D}_\omega, \pi_\omega, \Omega_\omega)$ such that:

- $\pi_\omega : \mathcal{A} \to \mathcal{L}(\mathsf{D}_\omega, \mathsf{H}_\omega)$;
- $\mathsf{D}_\omega = \pi_\omega(\mathcal{A})\Omega_\omega$ is dense in H_ω;
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Proof

Define $\langle A|B\rangle_\omega := \omega(A^* B)$.
Then $\mathsf{D}_\omega = \mathcal{A}/\mathcal{I}_\omega, \quad \mathcal{I}_\omega := \{A \in \mathcal{A} | \quad \omega(A^* A) = 0\}$.
Moreover, $\pi_\omega(A)[B] := [AB]$.
Finally $\Omega_\omega := [1_\mathcal{A}]$.

Fock space in a nutshell

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For any $x \in \mathfrak{h}$ the creation operator $a^*(x)$ is defined as

$$a^*(x)\colon \mathfrak{h}^n \to \mathfrak{h}^{n+1}, \qquad a^*(x)\psi_n := \sqrt{n+1}\, S[x \otimes \psi_n],$$
$$\text{CCR/CAR relations:} \qquad [a(x), a^*(y)] = i\langle x|y\rangle 1_{\mathcal{F}(\mathfrak{h})}.$$

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For ω, ω' on $\mathcal{A}$ one can have:

$\surd$ Unitary equivalent GNS triples

$$U\colon \mathsf{H}_\omega \to \mathsf{H}_{\omega'} \qquad U \circ \pi_\omega(A) = \pi_{\omega'}(A) \circ U \qquad U\Omega_\omega = \Omega_{\omega'}.$$

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√ π- Unitary equivalent GNS triples (particle production)

$$U\colon \mathsf{H}_\omega \to \mathsf{H}_{\omega'} \qquad U \circ \pi_\omega(A) = \pi_{\omega'}(A) \circ U \qquad U\Omega_\omega \neq \Omega_{\omega'}.$$

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From the choice of a state one recovers the notion of particle.
The notion of particle is state dependent.

Haag-Kastler axioms

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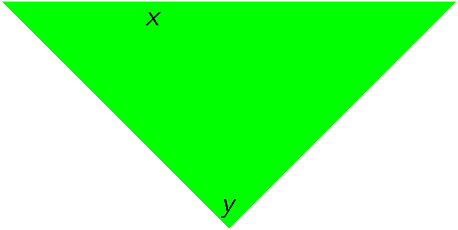
An Haag-Kastler net is a net of $*$-algebras $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ such that:



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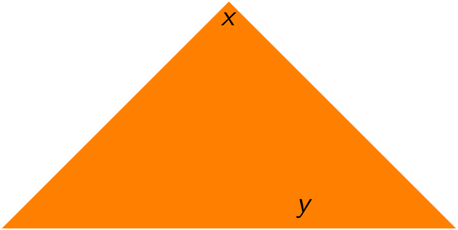

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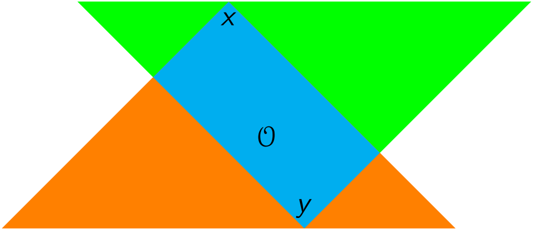

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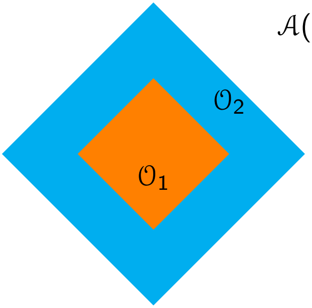

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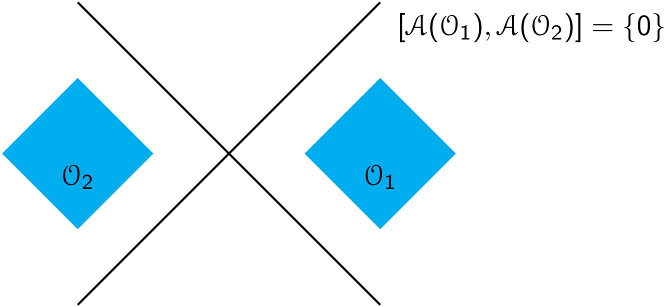

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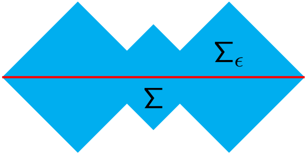

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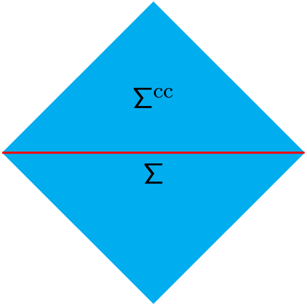

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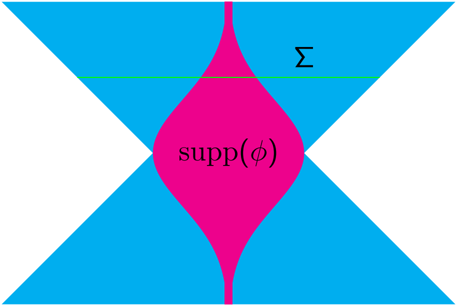
Klein-Gordon model

$$P\phi := (\partial_t^2 - \Delta + m^2)\phi = 0\,, \qquad \phi \in C^\infty_{\mathrm{sc}}(\mathbb{R}^4, \mathbb{R})\,.$$



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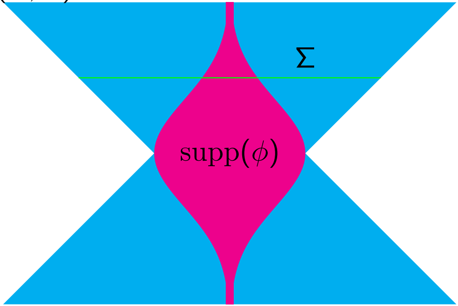
$$\rho\phi := \underline{f} = (\phi|_\Sigma, -i\partial_t\phi|_\Sigma) \in C^\infty_{\mathrm{c}}(\Sigma, \mathbb{C})^{\oplus 2}$$

$$\sigma_\Sigma(\underline{f}, \underline{h}) := \int_\Sigma (f_1 h_2 - h_1 f_2)\, \mathrm{d}\Sigma$$



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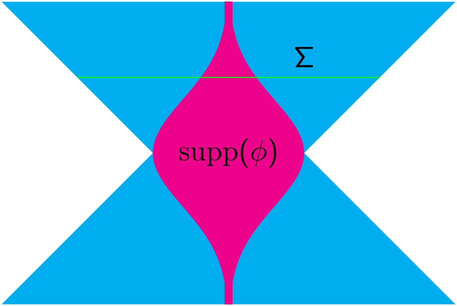

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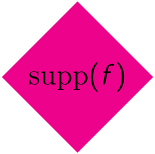

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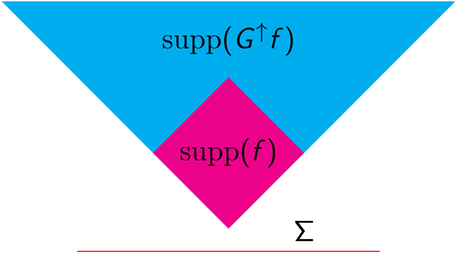

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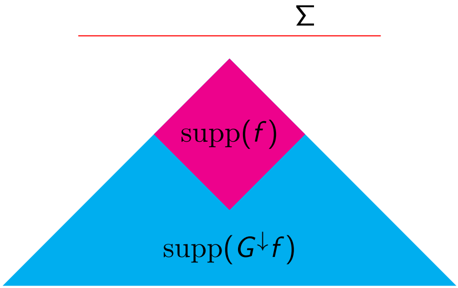

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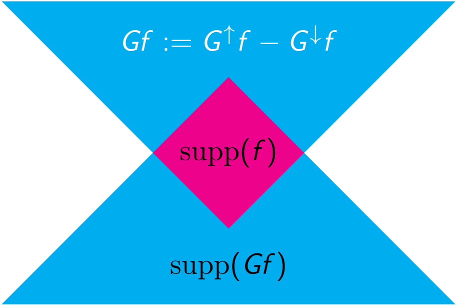

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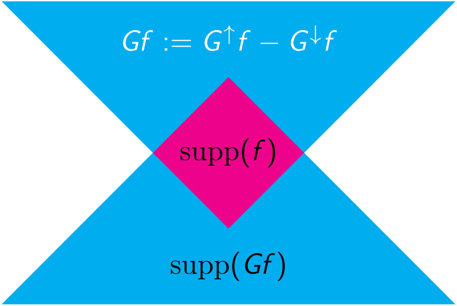

$$\{0\} \to C^\infty_{\mathrm{c}}(\mathbb{R}^4) \xrightarrow{P} C^\infty_{\mathrm{c}}(\mathbb{R}^4) \xrightarrow{G} C^\infty_{\mathrm{sc}}(\mathbb{R}^4) \xrightarrow{P} C^\infty_{\mathrm{sc}}(\mathbb{R}^4) \to \{0\}$$

Quantization

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Let $\ell(\mathcal{O}) := C_c^\infty(\mathcal{O}, \mathbb{C})/PC_c^\infty(\mathcal{O}, \mathbb{C})$.
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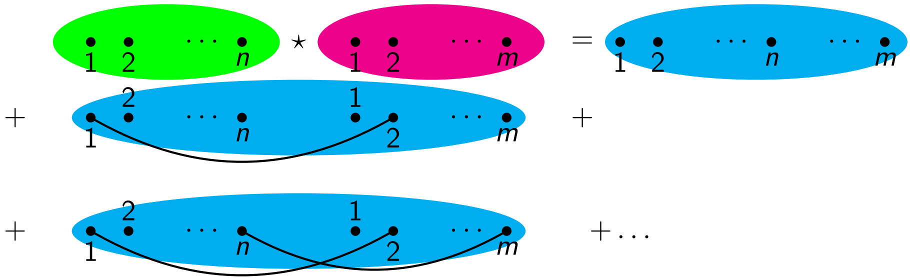
$$\begin{aligned} & (f_1 \odot \cdots \odot f_n) \star (h_1 \odot \cdots \odot h_m) = f_1 \odot \cdots \odot f_n \odot h_1 \odot \cdots \odot h_m \\ + \quad & \frac{i\hbar}{2} \sum_{j,k} \mathsf{G}(f_j, h_k) f_1 \odot \cdots \widehat{f_j} \odot \cdots f_n \odot h_1 \odot \cdots \widehat{h_k} \cdots \odot h_m \\ + \quad & \text{higher order contractions} \end{aligned}$$

$$\mathcal{A}(\mathcal{O}) := \mathbb{C} \oplus \ell(\mathcal{O}) \oplus \ell(\mathcal{O})^{\odot 2} \oplus \cdots$$



Quantization (concretely)

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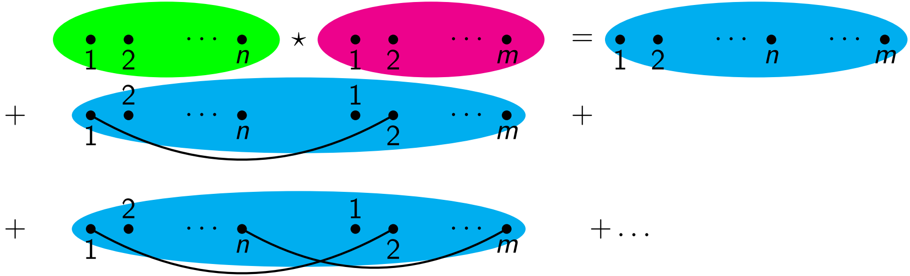

$$f_1 \star f_2 = f_1 \odot f_2 + \frac{i}{2}\hbar G(f_1, f_2)\,.$$

Haag-Kastler properties

√ isotony $\quad \ell(\mathcal{O}_1) \subseteq \ell(\mathcal{O}_2) \quad \text{if} \quad \mathcal{O}_1 \subseteq \mathcal{O}_2;$

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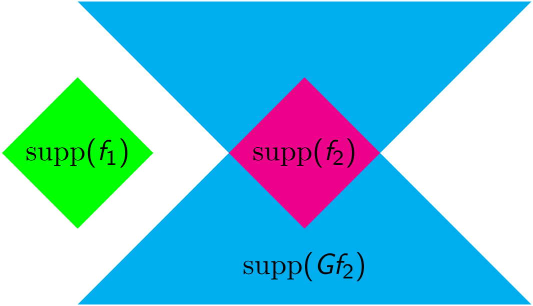
$$G(f_1, f_2) = 0$$



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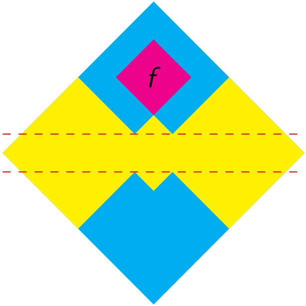

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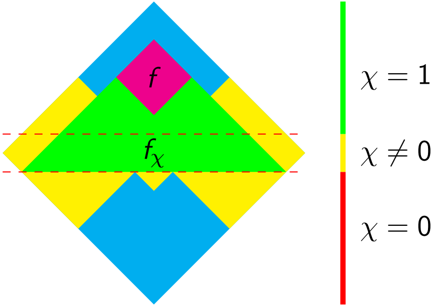
$$f \mapsto f_\chi := f - P\left(\chi G^{\downarrow} f\right)$$



States on $\mathcal{A}(\mathcal{O})$

A state $\omega\colon \mathcal{A}(\mathcal{O}) \to \mathbb{C}$ is determined by its n-points functions

$$\omega_n(f_1, \dots, f_n) := \omega(f_1 \star \dots \star f_n)\,, \qquad \omega_n \in C^\infty_{\mathrm{c}}(\mathcal{O}^n)'\,, \qquad n \in \mathbb{N}\,.$$

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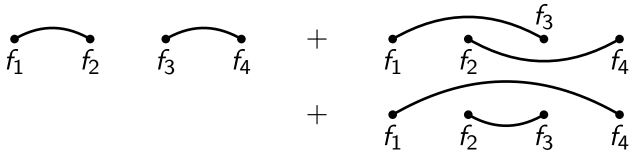
Proposition

A state $\omega\colon \mathcal{A}(\mathcal{O}) \to \mathbb{C}$ is quasi-free iff

$$\omega_{2n+1} \equiv 0\,, \qquad \omega_{2n}(f_1, \ldots, f_{2n}) = \sum_{i_k < i_{k+1}} \prod_{k=1}^{n-1} \omega_2(f_{i_k}, f_{i_{k+1}})\,, \qquad \forall n \in \mathbb{N}\,.$$

$\omega_4(f_1, f_2, f_3, f_4) =$



States on $\mathcal{A}(\mathcal{O})$

Constraints on ω_2:

$$\omega_2(f_1, f_2) - \omega_2(f_2, f_1) = i\mathrm{G}(f_1, f_2), \qquad \omega_2(\overline{f}, f) \geq 0,$$
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Modes decomposition:

$$\omega_2(x, x') = \int_{\mathbb{R}^3} \left(b_+(k) e^{iE_k(t-t')} + b_-(k) e^{-iE_k(t-t')} \right) e^{i\underline{k}\cdot(\underline{x}-\underline{x}')} \frac{\mathrm{d}\underline{k}}{E_k}\,,$$
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Vacuum: $b_+ \equiv 1, \qquad b_- \equiv 0.$

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Modes decomposition:

$$\omega_2^\beta(x, x') = \int_{\mathbb{R}^3} \left(\frac{1}{1 - e^{-\beta E_k}} e^{iE_k(t-t')} + \frac{e^{-\beta E_k}}{1 - e^{-\beta E_k}} e^{-iE_k(t-t')} \right) e^{i\underline{k}\cdot(\underline{x}-\underline{x}')} \frac{\mathrm{d}\underline{k}}{E_k}\,,$$
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The choice of $b_\pm$ determines the physics described by ω.
KMS (thermal) state: $b_\pm(k) = \pm(1 - e^{\mp\beta E_k})^{-1}, \qquad \beta > 0.$

States on $\mathcal{A}(\mathcal{O})$

Constraints on ω_2:

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Physically admissible states: $b_- \in \mathcal{S}(\mathbb{R}^3)$.

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$$(\partial_t^2 - \Delta + m^2)\phi + \lambda\phi^3 = 0 \,.$$

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3. Curved backgrounds: Hadamard states, Gauge theories, Semi-Classical Einstein Equations, Quantum Gravity,...