A Relatively General Introduction to General Relativity

Mathematical Relativity for Pedestrians

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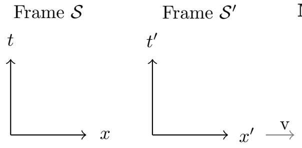
Spacetime in Classical Physics: Galilei and Newton (17th Century)

Postulate (Principle of Relativity, Galilei 1632)

Laws of physics take the same form in every *inertial frame of reference*[a].

[a] = reference frame with no acceleration ("free" particles are at rest or move with constant velocity)

$\mathcal{S} = \mathcal{S}'$
at $t = t' = 0$



Most general linear transformation:

$$\boxed{\begin{aligned} t' &= At + Bx \\ x' &= Dt + Ex \end{aligned}}$$

$$x' = 0 \Leftrightarrow x = vt$$
$$x = 0 \Leftrightarrow x' = -vt'$$

$$\Rightarrow \boxed{\begin{aligned} E &= A \\ D &= -Av \end{aligned}}$$

A,B=?

Postulate (Principle of Absolute Time, Newton 1687)

Time is *absolute*, i.e. $t = t'$ for all inertial frames $\mathcal{S}, \mathcal{S}'$. ($\Rightarrow A = 1$ and $B = 0$)

Proposition (Galilei Transformations)

If an observer $\mathcal{S}'$ moves with constant speed v in x direction with respect to observer $\mathcal{S}$, then

$$t' = t$$

$$x' = x - vt$$

Note: Two invariants: $\Delta t := t_2 - t_1$ and $\Delta r^2 := \Delta x^2 + \Delta y^2 + \Delta z^2$ for two events (t_i, x_i, y_i, z_i).
Summary: Space and time in GALILEI-NEWTON THEORY: $\mathbb{R} \times \mathbb{R}^3$ with Euclidean distance.

Einstein's Special Theory of Relativity I: Basics

↪ Speed of light is *finite*. In vacuum: $c = 299\,792\,458\,\mathrm{m/s}$ (Rømer 1676, Huygens 1678)

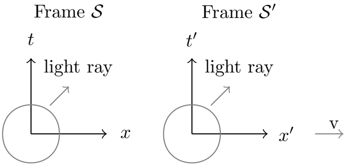
↪ Around 1900, both theoretical considerations (Maxwell, Lorentz, Poincaré...) and experiments (Michelson-Morley, Fizeau...) lead to:

Postulate (Principle of Constant Speed of Light, Einstein 1905)

The speed of light (in vacuum) is independent of the inertial frame of reference.

Together with Principle of Relativity ⇒ Einstein's Special Theory of Relativity.

$\mathcal{S} = \mathcal{S}'$ at $t = t' = 0$



By principle of constant c:

$$c^2t^2 = x^2 + y^2 + z^2$$
$$c^2(t')^2 = (x')^2 + (y')^2 + (z')^2$$

⇒ Determine A, B!

Proposition (Lorentz Transformations; Larmor, Lorentz and Poincaré around 1900)

If an observer $\mathcal{S}'$ moves with speed $0 < v < c$ in x direction with respect to observer $\mathcal{S}$, then

$$t' = \gamma(v)(t - vx/c^2)$$
$$x' = \gamma(v)(x - vt) \quad \text{with} \quad \gamma(v) := (1 - v^2/c^2)^{-1/2}$$

Note: Only invariant: $\Delta s^2 := -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$! ⇒ **Spacetime!**

Einstein's Special Theory of Relativity II: Geometric Formulation

In special relativity, time and space is combined in a *spacetime*, mathematically described by a Minkowski Space:

Definition (Minkowski Space, Minkowski 1907)

Minkowski space is $\mathbb{M}^4 \cong \mathbb{R}^4$ equipped with the non-degenerate bilinear form

$$\eta(x,y) := \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} x^\mu y^\mu = -x^0 y^0 + \sum_{i=1}^{3} x^i y^i \qquad \text{with} \qquad \eta := \mathrm{diag}(-1,1,1,1).$$

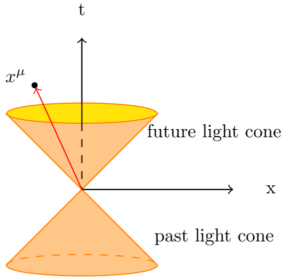
Different types of vectors $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$:

- ↪ $\eta(x,x) > 0 \ldots$ *spacelike*
- ↪ $\eta(x,x) < 0 \ldots$ *timelike*
- ↪ $\eta(x,x) = 0 \ldots$ *lightlike*

Events whose distance vector is ...

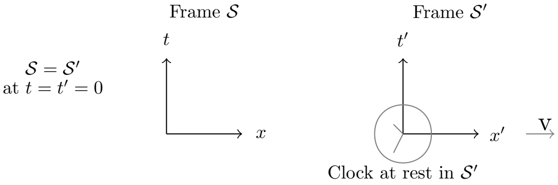
... spacelike are *causally disconnected*.

... timelike/lightlike are *causally connected*.



Einstein's Special Theory of Relativity III: Time Dilation

Example: Time Dilation: Time (and length) are observer dependent:



Cock at rest in $\mathcal{S}'$ moving with speed $0 < v < c$ in x-direction in $\mathcal{S}$.

↪ Time between two "ticks" at t'_1 and t'_2 in $\mathcal{S}'$: $\Delta t' := t'_2 - t'_1$.

↪ Time recorded in $\mathcal{S}$:

$$t_i = \gamma(v)\left(t'_i + \frac{v x'_i}{c^2}\right) \qquad \text{with} \qquad \gamma(v) := \left(1 - \frac{v^2}{c^2}\right)^{-1/2} > 1$$

Since the clock is at rest in $\mathcal{S}'$ ($x'_1 = x'_2$) the time measured in $\mathcal{S}$ is

$$\Delta t := t_2 - t_1 = \gamma(v)\Delta t' \qquad \Rightarrow \qquad \frac{\Delta t}{\Delta t'} = \gamma(v) > 1$$

$\Rightarrow$ **"Moving clocks tick slower for stationary observer"!**

Definition (Proper Time)

The *proper time* of some timelike curve is the time measured by a clock moving on this curve.

From Special to General: The Equivalence Principle

Postulate (Einstein Principle of Equivalence, Einstein 1907)

In small enough regions of spacetime, the laws of physics reduce to those of special relativity

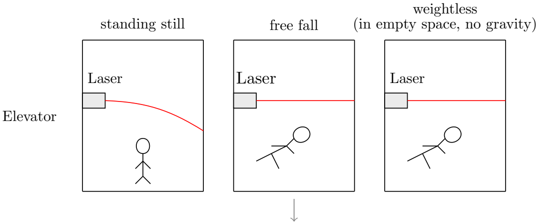
$\Rightarrow$ Natural description of spacetime such that equivalence principle holds:

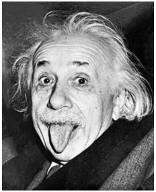
- *Smooth manifold* $\mathcal{M}$ (*locally Euclidean*)
- *Lorentzian metric* g on $\mathcal{M}$ (*metric of signature* $(-,+,+,+)$ *on tangent space*)

$\Rightarrow$ **locally like Minkowski!**

Key Idea of Einstein:

Gravity should no longer be regarded as a force in the conventional sense but rather as a manifestation of the curvature of spacetime, being induced by the presence of matter.





Taken by A. Sesse (1951).
Source: en.wikipedia.org

Lorentzian Geometry I: Basics

Let $\mathcal{M}$ be a smooth manifold of dimension $d \in \mathbb{N}$.

Definition (Pseudo-Riemannian Metric)

A pseudo-Riemannian metric on $\mathcal{M}$ is a smooth symmetric rank 2-tensor field $g \in \Gamma^\infty(T\mathcal{M}^{\otimes_s 2})$, such that $g_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ is a non-degenerate symmetric bilinear form for each $p \in T_p\mathcal{M}$.

By Sylvester's Law of Inertia (1852), for every symmetric non-degenerate bilinear form $B : V \times V \to \mathbb{R}$ on a finite-dimensional $\mathbb{R}$-vector space V, there exists a basis such that

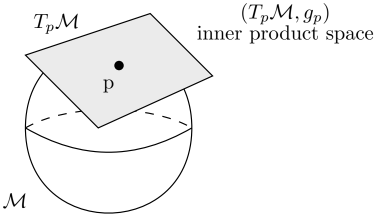
$$B = \mathrm{diag}(-1, \dots, -1, 1, \dots, 1) \qquad \dots \textit{signature of } B$$

- $\hookrightarrow$ A metric g of signature $(+, \dots, +)$ is called *Riemannian*.
- $\hookrightarrow$ A metric g of signature $(-, +, \dots, +)$ is called *Lorentzian*.

If (U, φ) is a local chart of $\mathcal{M}$, locally:

$$g|_U = \sum_{\mu,\nu} g_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$$

for components $g_{\mu\nu} \in C^\infty(U)$, where $x^\mu := \mathrm{pr}^\mu \circ \varphi$ denote the coordinates.



Lorentzian Geometry II: Examples

Examples of Lorentzian Manifolds:

(1) Minkowski space: $\mathcal{M} \cong \mathbb{R}^4$ with Minkowski metric $\eta_p := \mathrm{diag}(-1, 1, 1, 1)$ on each tangent space $T_p\mathcal{M} \cong \mathbb{R}^4$, i.e.

$$\eta = -\mathrm{d}t \otimes \mathrm{d}t + \sum_{i=1}^{3} \mathrm{d}x^i \otimes \mathrm{d}x^i.$$

(2) Let $\mathcal{M} \cong I \times \Sigma$ with $I \subset \mathbb{R}$, where (Σ, h) is a 3D Riemannian manifold. Then

$$g := -N^2 \mathrm{d}t \otimes \mathrm{d}t + h$$

with $N \in C^\infty(\mathcal{M})$ and $N > 0$ (called *lapse*) is a Lorentzian manifold.

$\Rightarrow$ Manifolds of this type are called *globally hyperbolic*.
$\Rightarrow$ Play an important role in the Cauchy problem of general relativity and in Quantum Field Theory on Curved Spacetime.

(3) In general, every non-compact and connected manifold admits a Lorentzian metric. In the compact case, there are topological obstructions:

Example

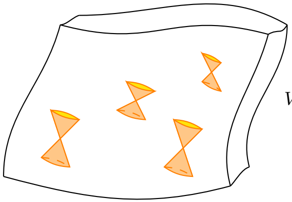
Only spheres $\mathbb{S}^n$ with $n \in \mathbb{N}$ odd admit Lorentzian metrics.

Lorentzian Geometry III: Causality

Let $(\mathcal{M}, g)$ be a Lorentzian manifold. Tangent vectors $v \in T_p\mathcal{M}$ can be divided as follows:

$\hookrightarrow$ $g_p(v,v) > 0 \ldots$ *spacelike*
$\hookrightarrow$ $g_p(v,v) < 0 \ldots$ *timelike*
$\hookrightarrow$ $g_p(v,v) = 0 \ldots$ *lightlike*

Using this, we can define at every point a corresponding *light cone* $V_p \subset T_p\mathcal{M}$:



$$V_p := \{v \in T_p\mathcal{M} \mid g_p(v,v) < 0\}$$

A vector field $X : \mathcal{M} \to T\mathcal{M}$ is called space/time/lightlike if it is so at every point. In general, there is no *global* notion of time. If this is the case, $(\mathcal{M}, g)$ is called *time-orientable*:

Definition (Time-Orientation)

A global timelike vector field X on $\mathcal{M}$ is called "time-orientation". If such a vector field exists, $(\mathcal{M}, g)$ is called "time-orientable".

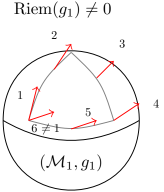
Note: Both a topological and geometrical concept!

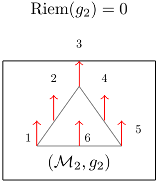
$(\mathcal{M}, g)$ has a unique connection $\nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ called *Levi-Civita connection*.

$$\nabla_X Y \ldots \textit{directional derivative of } Y \textit{ w.r.t. } X$$

Curvature Tensor : $\mathrm{Riem}_g(X, Y, Z) := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

Parallel Transport: shift vector parallel along a curve





Proposition

A Lorentzian manifold is flat $(\mathrm{Riem}_g = 0)$ if and only if it is locally isometric to Minkowski space.

- ↪ Curvature tensor can be identified with a smooth rank 4 tensor field ⇒ Locally: $R_{\alpha\beta\gamma\delta}$.
- ↪ Other important curvature tensors ($g^{\mu\nu}$ denotes the components of the inverse metric):

$$\mathrm{Ric}(g)_{\mu\nu} := g^{\mu\nu} R_{\alpha\mu\beta\nu} \ldots \textit{Ricci tensor}$$

$$\mathrm{Scal}(g) := g^{\mu\nu} \mathrm{Ric}(g)_{\mu\nu} \ldots \textit{scalar curvature}$$

Einstein Equations

In the following, let $(\mathcal{M}, g)$ be a 4D Lorentzian manifold.

Definition (Einstein's Field Equations, Einstein 1915)

Let T be a smooth divergence-free 2-tensor field and $\Lambda \in \mathbb{R}$. Einstein's field equations are

$$\underbrace{\mathrm{Ric}(g) - \frac{1}{2}\mathrm{Scal}(g)g + \Lambda g}_{\text{curvature of spacetime}} = \underbrace{\frac{8\pi G}{c^4}T}_{\text{matter content}}$$

- ↪ T describes the matter content. Energy-momentum conservation: $\nabla^\lambda T_{\lambda\nu} = 0$.
- ↪ LOVELOCK'S THEOREM (1971): LHS is only symmetric divergent-free 2-tensor field constructed out of g involving only up to second derivatives
- ↪ Equations reduce to POISSON EQUATIONS of Newtonian gravity in appropriate limit.
- ⇒ Special case: Vacuum $T = 0$. In this case, Einstein equations equivalent to

$$\boxed{\mathrm{Ric}(g) = \Lambda g}$$

Side Fact: $(\mathcal{M}, g)$ with g such that $\mathrm{Ric}(g) \propto g$ are called *Einstein manifolds*.

- ↪ Many applications in differential geometry, but examples hard to find!

> *"[...] try to find one yourself [an example] which is not in our book. And if you succeed, please write to us immediately! [...] The author will be happy to stand you a meal in a starred restaurant in exchange for one of these!"*
>
> – Besse in *Einstein manifolds*, Springer, 1987.

Cauchy Problem for Einstein's Equations I

Einstein's Field Equations on $\mathcal{M}$ $\widehat{=}$ *highly non-trivial 2nd order system of quasilinear PDE's for components* $g_{\mu\nu}$

Leading order contribution:

$$\begin{aligned}
&\mathrm{Ric}(g)_{\mu\nu} - \frac{1}{2}\mathrm{Scal}(g) g_{\mu\nu} + \Lambda g_{\mu\nu} = \\
&= \frac{1}{2}\sum_{\alpha,\beta} g^{\alpha\beta}(\partial_\alpha\partial_\nu g_{\mu\beta} + \partial_\mu\partial_\beta g_{\alpha\nu} - \partial_\alpha\partial_\beta g_{\mu\nu} - \partial_\mu\partial_\nu g_{\alpha\beta}) \\
&\quad - \frac{1}{2} g_{\mu\nu} \sum_{\alpha\beta,\rho,\sigma} g^{\alpha\beta} g^{\rho\sigma}(\partial_\alpha\partial_\rho g_{\sigma\beta} - \partial_\alpha\partial_\beta g_{\rho\sigma}) \\
&\quad + \textit{many lower order terms involving } g \textit{ and } \partial g.
\end{aligned}$$

$\hookrightarrow$ neither hyperbolic, elliptic nor parabolic!

$\Rightarrow$ **Key observation:** Heavily overdetermined system (16 equations for 10 unknowns) (*"general covariance"*)

$\Rightarrow$ In a clever coordinate system, Einstein equations become hyperbolic:

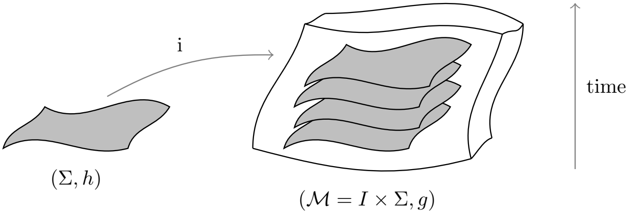
$$\mathrm{Ric}(g)_{\mu\nu} = -\frac{1}{2}\Box_g^S g_{\mu\nu} + \textit{terms involving } g \textit{ and } \partial g.$$

where $\Box_g^S := g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ denotes wave operator acting on scalars.

Cauchy Problem for Einstein's Equations II

General relativity as an initial value problem:

- ↪ **Initial Data**: 3D Riemannian manifold (Σ, h) and 2-tensor field k on Σ, satisfying constraints.
- ↪ A **Cauchy development** of (Σ, h, k) is a 4D Lorentzian manifold $(\mathcal{M}, g)$ solving Einstein's equations, such that
  1. $\mathcal{M} \cong I \times \Sigma$ with $I \subset \mathbb{R}$ ("time") and such that Σ is a spacelike embedded hypersurface.
  2. $i^* g = h$ with embedding $i : \Sigma \to \mathcal{M}$ and k is the *extrinsic curvature* of hypersurface Σ in $\mathcal{M}$.



Theorem (Choquet-Bruhat (1952); Choquet-Bruhat, Geroch (1969))

Let (Σ, h, k) be an admissible set of initial data. Then there exists a Cauchy development $(\mathcal{M}, g)$, which is unique up to isometry.

Note: Unlike other PDEs formulated on fixed background we are solving for *spacetime itself*!
Example: Initial data $(\mathbb{R}^3, \delta, 0)$ leads to Minkowski space $(\mathbb{R}^4, \eta)$.

The Schwarzschild Solution and Black Holes

First (vacuum) solution: Schwarzschild 1916 (during world war I): Manifold $\mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{S}^2$ with

$$g = -c^2\left(1 - \frac{2GM}{c^2 r}\right)\mathrm{d}t \otimes \mathrm{d}t + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\Omega^2_{\mathbb{S}^2},$$

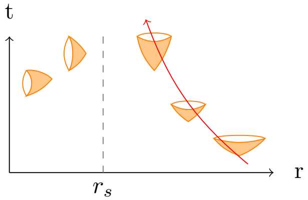
where $\mathrm{d}\Omega^2_{\mathbb{S}^2}$ denotes the standard Riemannian metric of the 2-sphere $\mathbb{S}^2$ and $M \in \mathbb{R}_{>0}$ a mass.

$\Rightarrow$ For $r > R$ describes gravitational field outside of a spherically symmetric object of mass M and radius R (star, planets, ...)

Theorem (Birkhoff 1923)

The Schwarzschild metric is the only spherically symmetric vacuum solution with $\Lambda = 0$.

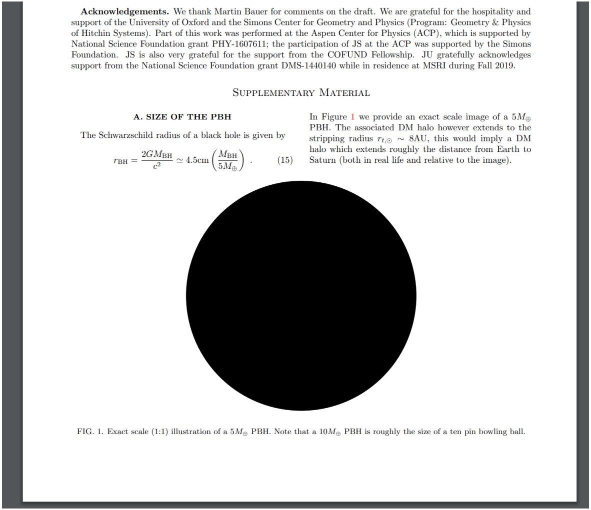
Something strange happens at $r_s := 2GM/c^2$: (sun: $r_s \cong 3\,\mathrm{km} \ll R \cong 670\,000\,\mathrm{km}$)



Questions: What happens if an object is so massive that its radius R is $R < r_s$? $\Rightarrow$ **Black Hole**

Accurate Sketch of a Black Hole

Accurate and detailed sketch of a black hole:



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SUPPLEMENTARY MATERIAL

A. SIZE OF THE PBH

The Schwarzschild radius of a black hole is given by

$$r_{\rm BH} = \frac{2GM_{\rm BH}}{c^2} \simeq 4.5{\rm cm}\left(\frac{M_{\rm BH}}{5M_\oplus}\right) . \qquad (15)$$

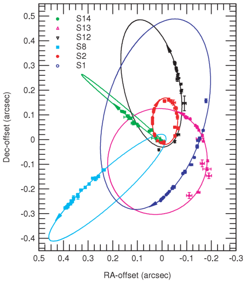
In Figure 1 we provide an exact scale image of a $5M_\oplus$ PBH. The associated DM halo however extends to the stripping radius $r_{t,\odot} \sim 8$AU, this would imply a DM halo which extends roughly the distance from Earth to Saturn (both in real life and relative to the image).

FIG. 1. Exact scale (1:1) illustration of a $5M_\oplus$ PBH. Note that a $10M_\oplus$ PBH is roughly the size of a ten pin bowling ball.

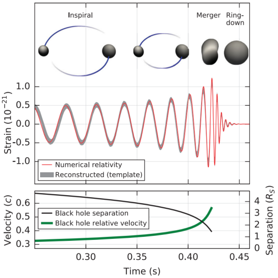
Source: Scholtz, Unwin: *What if Planet 9 is a Primordial Black Hole?* Physical Review Letters 125, 2020. Preprint: arXiv:1909.11090 [hep-th].

Physical Evidence of Black Holes

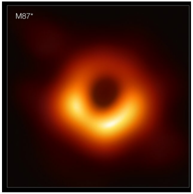
(1) **Indirect** (Genzel, Ghez et al.):

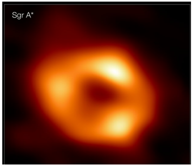


(2) **Indirect** (LIGO & Virgo):



(3) **Direct** (EHT):





Sources:

(1) Eisenhauer et al.: *SINFONI in the Galactic Center: young stars and IR flares in the central light month*. The Astrophysical Journal 628(1), pages 246-259, 2005. Preprint: arXiv:astro-ph/0502129 [astro-ph].

(2) LIGO Scientific and Virgo collaboration: *Observation of Gravitational Waves from a Binary Black Hole Merger*. Physical Review Letters 116, 2016. Preprint: arXiv:1602.03837 [gr-qc].

(3) The EHT collaboration: *First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole*. The Astrophysical Journal Letters 875(1), 2019. Preprint: arXiv:1906.11238 [astro-ph.GA].

The EHT collaboration: *First Sagittarius A* Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole in the Center of the Milky Way*. The Astrophysical Journal Letters 930(2), 2022.

Mathematical Results: Singularity Theorems

In a Schwarzschild black hole ...

... no observer or light-ray entering the region $r < r_s$ can leave.

... any causal curve starting in region $r < r_s$ terminates at $r = 0$ in finite proper time.

... Curvature invariants diverge at $r = 0$, e.g.

$$\|\mathrm{Riem}_g\|_g^2 = \sum_{\alpha,\beta\gamma\delta} R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = \frac{12 r_s^2}{r^6} \stackrel{r\to 0}{\to} \infty$$

$\Rightarrow$ Prototypical example of *gravitational singularity*!

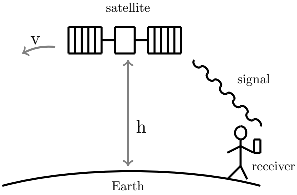
Question: Are gravitational singularities *mathematical artifacts* or *physical predictions*?

Theorem (Penrose 1965)

Let $(\mathcal{M}, g)$ be a solution of Einstein's equations with energy-momentum tensor T. Under certain (reasonable) assumptions on T ("*energy conditions*") and certain (reasonable) causality assumptions, $(\mathcal{M}, g)$ is lightlike geodesically incomplete whenever it admits a closed trapped surface[a].

[a] Roughly speaking: surface where the gravitational field is so strong that outgoing photons are dragged inwards.

Remark: Similar theorem for timelike geodesically incompleteness (Hawking 1966). (*Cosmology*)



Metric of earth can be approximated as $(r > R_\oplus)$

$$g = -c^2\left(1 + \frac{2\Phi(r)}{c^2}\right)\mathrm{d}t \otimes \mathrm{d}t + \left(1 - \frac{2\Phi(r)}{c^2}\right)(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z)$$

with gravitational potential $\Phi(r) = -GM_\oplus/r$ and $r = \sqrt{x^2+y^2+z^2}$.

$\Rightarrow$ Proper Time difference between (stationary) receiver and satellite:

$$\frac{\Delta\tau_r}{\Delta\tau_s} \cong \left(1 + \underbrace{\frac{\Phi(r_\oplus) - \Phi(r_\oplus + h)}{c^2}}_{\text{gravitational time dilation}} + \underbrace{\frac{1}{2}\frac{v^2}{c^2}}_{\text{special relativistic time dilation}}\right)$$

Example: Satellite with v=14 000 km/h and h=20 200 km. After one day on earth $(\Delta\tau_r = 24\,\mathrm{h})$:

$$\delta\tau = \Delta\tau_r - \Delta\tau_s \cong -38.5\,\mu\mathrm{s} = \begin{cases} -45.7\,\mu\mathrm{s} & \textit{gravitational time dilation} \\ +7.3\,\mu\mathrm{s} & \textit{special relativistic time dilation} \end{cases}.$$

$\Rightarrow$ Without taking $\delta\tau$ into account, we have a distance deviation of $|c \cdot \delta\tau| \cong 11.5\,\mathrm{km}$ per day!

Conclusion and Outlook

↪ General relativity is a well-established theory with many applications in *physics*, *pure mathematics* and in our *daily life*.

↪ Many more topics and applications (*Numerical Relativity*, *Cosmology*, . . .)

↪ General relativity is not the end of the story!

⇒ not compatible with the "second half of modern physics": QUANTUM FIELD THEORY

⇒ The ultimate quest for a QUANTUM THEORY OF GRAVITY.

⇒ **Many candidates:** (partially related)

- String Theory
- Loop Quantum Gravity
- Spin Foam Models
- Group Field Theory
- Matrix and Tensor Models
- Simplicial Quantum Gravity
- Causal Dynamical Triangulation
- Causal Set Theory
- Causal Fermion Systems
- Asymptotic Safety for Gravity
- Noncommutative Geometry
- Twistor Theory
- . . .

⇒ Many open questions. One Step back: *Perturbative Quantum Gravity* (=linearized gravity as perturbative quantum field theory), *Semiclassical Gravity*, *Quantum Field Theory on Curved Spacetime*, . . .

"In so far as theories of mathematics speak about reality, they are not certain, and in so far as they are certain, they do not speak about reality."

– Einstein in *Geometrie und Erfahrung*, Springer, 1921.