The Logic of Quantum Mechanics

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Introduction

Tipical reactions attending the first class in Quantum Mechanics:

Ma che c.... sono tutti 'sti simboli a caso?!?
S.Murro.

Che figata, non capisco una m...!
G.Nosari.

Introduction

Aim of the talk:

- Introduce the first 4 Axioms of Quantum Mechanics.
- Explain the mantra

  "Hilbert spaces provide a natural framework for QM".
- Destroy the mantra.
- Have a tasty break.

No cats were harmed during the production of these slides.

Phase space

Let $\mathscr{S}$ be your favourite physical system (particle, gas, mewing cat...).
Fixing a frame $\mathscr{I}$, $\mathscr{S}$ is described via the phase space $\mathcal{P}$.
$\wp \overset{\text{loc.}}{=} (q^1, \ldots, q^n; p_1, \ldots p_n) \in \mathcal{P}$ describes a configuration of $\mathscr{S}$.
Dynamics is ruled by Hamilton's equations:

$$\mathbb{R} \ni t \mapsto \wp(t) \overset{\text{loc.}}{=} (q^1(t), \ldots, q^n(t); p_1(t), \ldots p_n(t)) \in \mathcal{P}$$
$$\frac{dq^k}{dt} = \frac{\partial H}{\partial p_k}, \qquad \frac{dp_k}{dt} = -\frac{\partial H}{\partial q^k}, \qquad k = 1, \ldots, n,$$

being $H \colon \mathcal{P} \to \mathbb{R}$ the Hamiltonian of the system $\mathscr{S}$.

Proposition and states on $\mathcal{P}$

A proposition P is realized as a Borel set $P \in \mathscr{B}(\mathcal{P})$.

$$\begin{aligned} P \vee Q &\longleftrightarrow P \cup Q \\ P \wedge Q &\longleftrightarrow P \cap Q \\ P \Rightarrow Q &\longleftrightarrow P \subseteq Q \\ \dashv P &\longleftrightarrow \mathcal{P} \setminus P \\ \mathcal{P} &\longleftrightarrow \text{tautology} \\ \emptyset &\longleftrightarrow \text{contradiction.} \end{aligned}$$

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A state ω is a probability measure on $\mathscr{B}(\mathcal{P})$.

$$\omega(P) = \text{probability that } P \text{ is true if the state of } \mathscr{S} \text{ is } \omega.$$

Example: $\omega = \delta_{\wp}, \ \wp \in \mathcal{P}$, is sharp state.

Observables on $\mathcal{P}$

An observable O is a measurable function $O : \mathcal{P} \to \mathbb{R}$.
Observables are completely characterized by a list of propositions parametrized by $E \in \mathscr{B}(\mathbb{R})$:

$$
\begin{aligned}
P_E^{(O)} &\doteq O^{-1}(E) \\
&= \text{The assumed value of } O \text{ on the system belongs to } E.
\end{aligned}
$$

Observables generate a commutative $*$-algebra $\mathscr{A}$ over $\mathbb{C}$.

First step towards QM

A system $\mathscr{S}$ has quantum behaviour if

$$\text{Energy} \times \text{Time} \lesssim \hbar = 6.6262 \cdot 10^{-34} \text{Js}$$

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1800's : Two Slit Experiment: light has wave behaviour.

1905 : Photoelectric effect: light is made by particles.

light : What about make your own business?!?

1924 De Broglie: "Particles and waves are always related".

1926 Schrödinger: "I believe De Broglie.
I have ~~a lovely cat~~ an equation."

1927 Heisenberg: "Guys, we cannot do better than $\Delta x \Delta p = \hbar$".

Pathologies: the crash of logic and probability

There exist incompatible observables A, B i.e.:

- measure A on ω at time $t \Longrightarrow$ outcome α:
- measure B on ω at time $t+\epsilon \Longrightarrow$ outcome β:
- measure A on ω at time $t+2\epsilon \Longrightarrow$ outcome α':

Expectancy: $\alpha' \to \alpha$ as $\epsilon \to 0^+$.

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Expectancy: $\alpha' \to \alpha$ as $\epsilon \to 0^+$.
Fact: α, α' are completely uncorrelated.
Physical interpretation: measurement of B disturbs the measurement of A by changing the state of the system.
Consequence: $P_E^{(A)} \wedge P_{E'}^{(B)}$ has not physical sense.

Proposition on $\mathscr{S}$ are not described by Borel sets.

Hilbert space and propositions

Let $(\mathsf{H}, \langle \cdot \mid \cdot \rangle)$ be an separable Hilbert space over $\mathbb{C}$.
An orthogonal projector $P \in \mathfrak{P}(\mathsf{H})$ is

$$P : \mathsf{H} \to \mathsf{H} \text{ linear}, \qquad P^2 = P^* = P.$$

$$P \text{ proposition} \longleftrightarrow P \in \mathfrak{P}(\mathsf{H})?$$

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Observation: $P, Q \in \mathfrak{P}(\mathsf{H}) \not\Rightarrow PQ \in \mathfrak{P}(\mathsf{H})$

$$\begin{aligned} (PQ)^* &= Q^* P^* \\ &= QP \\ &= PQ \Leftrightarrow [Q, P] = 0! \end{aligned}$$

$[\cdot, \cdot]$ provides a criterion for compatibility.

Hilbert space and propositions

Let $(\mathsf{H}, \langle \cdot \mid \cdot \rangle)$ be an separable Hilbert space over $\mathbb{C}$.

QM,Axiom 1

A proposition P is realized as an orthogonal projector $P \in \mathfrak{P}(\mathsf{H})$.

$$\begin{aligned}
P, Q \text{ compatible} &\longleftrightarrow [P, Q] = 0 \\
\text{if } P, Q \text{ are compatible } P \vee Q &\longleftrightarrow P + Q - PQ \\
\text{if } P, Q \text{ are compatible } P \wedge Q &\longleftrightarrow PQ \\
\text{if } P, Q \text{ are compatible } P \Rightarrow Q &\longleftrightarrow P \leq Q \\
\dashv P &\longleftrightarrow I - P \\
I &\longleftrightarrow \text{tautology} \\
0 &\longleftrightarrow \text{contradiction.}
\end{aligned}$$

States

A state is a map which associates a probabilistic value to each proposition.

QM,Axiom 2

A state ω is a map $\omega : \mathfrak{P}(\mathsf{H}) \to [0,1]$ such that:

- $\omega(I) = 1$;
- if $(P_n)_n \subset \mathfrak{P}(\mathsf{H})$ are such that $P_n P_m = 0$ for $n \neq m$

$$\omega\left(\bigvee_{n\geq 0} P_n\right) = \sum_{n\geq 0} \omega(P_n).$$

Example: $\psi \in \mathsf{H}, \ \|\psi\| = 1$

$$\omega_\psi(P) = \langle \psi \mid P\psi \rangle .$$

A bit more about states

ω_1, ω_2 states $\not\Rightarrow \alpha\omega_1 + \beta\omega_2$

The state space of $\mathscr{S}$, $\mathcal{S}(\mathsf{H})$, is not linear.

A bit more about states

$$\omega_1, \omega_2 \text{ states} \Rightarrow \lambda\omega_1 + (1-\lambda)\omega_2, \quad \lambda \in [0,1]$$

The state space of $\mathscr{S}$, $\mathcal{S}(\mathsf{H})$, is convex.

ω is said to be:

pure: if it cannot be decomposed as $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$, $\lambda \in (0,1)$, $\omega_{1,2} \neq \omega$;

mixture: if it is not pure.

A bit more about states

$$\omega_1, \omega_2 \text{ states} \Rightarrow \lambda\omega_1 + (1-\lambda)\omega_2, \quad \lambda \in [0,1]$$

The state space of $\mathscr{S}$, $\mathcal{S}(\mathsf{H})$, is convex.

ω is said to be:

pure: $\psi \in \mathsf{H}, \quad \|\psi\| = 1, \quad \omega_\psi(P) = \langle \psi \mid P\psi \rangle$.

mixture: $(\psi_n)_n \in \mathsf{H}, \quad \langle \psi_n \mid \psi_m \rangle = \delta_{n,m}, \quad c_n \geq 0, \quad \sum_n c_n = 1$

$$\omega(P) = \sum_n c_n \langle \psi_n \mid P\psi_n \rangle = \mathrm{Tr}\,(TP)\,,$$

being $T\phi = \sum_n c_n \langle \psi_n \mid \phi \rangle \psi_n, \quad \mathrm{Tr}(A) = \sum_n \langle \phi_n \mid A\phi_n \rangle$.

A bit more about states

Hp: H with finite dimension≥ 3 or infinite dimensional and separable.

Theorem (Gleason)

For each state ω there exists an positive trace class operator T such that $\omega(P) = \mathrm{Tr}(TP)$.

Theorem (Kochen-Specker)

There is no state ω taking values in $\{0, 1\}$.

Quantum Mechanics does not admit sharp states.

Post-measurement states

Let ω be the state of $\mathscr{S}$ at a certain time t.

Which state describes $\mathscr{S}$ after a positive measure of P?

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Which state describes $\mathscr{S}$ after a positive measure of P?

Positive outcome for $P \Rightarrow \omega(P) > 0$.
Heuristic argument: assume $\omega = \omega_\psi$.

$$\omega_\psi \quad \to \quad \omega_{\widehat{P\psi}} \qquad \widehat{P\psi} \doteq \frac{P\psi}{\|P\psi\|}$$

$$\omega_\psi(P) = \langle \psi \mid P\psi \rangle = \langle P\psi \mid P\psi \rangle = \|P\psi\|^2$$

$$\omega_{\widehat{P\psi}}(Q) = \frac{\langle P\psi \mid QP\psi \rangle}{\|P\psi\|^2} = \frac{\langle \psi \mid PQP\psi \rangle}{\omega_\psi(P)} = \frac{\omega_\psi(PQP)}{\omega_\psi(P)}.$$

Post-measurement states

Let ω be the state of $\mathscr{S}$ at a certain time t.

Which state describes $\mathscr{S}$ after a positive measure of P?

Positive outcome for $P \Rightarrow \omega(P) > 0$.

QM,Axiom 3

If $\mathscr{S}$ is in a state ω at time t and a proposition $P \in \mathfrak{P}(\mathsf{H})$ is true after a measurement at time t, then immediately afterwards the system's state collapses into

$$\omega_P(\cdot) = \frac{\omega(P \cdot P)}{\omega(P)}.$$

Observables

"Observables are completely characterized by a list of propositions parametrized by $E \in \mathscr{B}(\mathbb{R})$."

QM,Axiom 4

An observable O is a map $\mathscr{B}(\mathbb{R}) \ni E \mapsto P_E^{(O)} \in \mathfrak{P}(\mathsf{H})$ such that:

- $[P_E^{(O)}, P_{E'}^{(O)}] = 0$;
- $P_{E \cap E'}^{(O)} = P_E^{(O)} \wedge P_{E'}^{(O)}$;
- $P_{\mathbb{R}}^{(O)} = I$;
- $P_{\bigcup_n E_n}^{(O)} = \bigvee_n P_{E_n}^{(O)}$.

$$P_E^{(O)} \stackrel{?}{=} O^{-1}(E)$$

Observables

Consider an observable O with discrete spectrum $\sigma(O) = \{a_n\}_n \subset \mathbb{R}$.
$P_n^{(O)} =$ the measured value of O is precisely a_n.
Average of O on ω:

$$\sum_n a_n \, \omega\left(P_n^{(O)}\right) =: \Omega\left(\sum_n a_n \; P_n^{(O)}\right) =: \Omega(\mathcal{O})$$

$$\mathcal{O} : \mathsf{H} \to \mathsf{H} \text{ is self-adjoint.}$$

Observables can be regarded as self-adjoint operators over H.
States can be regarded as maps on observables such that

$$\Omega(\mathcal{O}) = \text{ Average of } \mathcal{O} \text{ on the state } \Omega.$$

Observables

Observables can be regarded as self-adjoint operators.
States can be regarded as maps on observables.

Theorem ($\sim$ Spectral decomposition)

For each observable O there exists a (possibly unbounded) self-adjoint operator $\mathcal{O}$ on H defined as

$$\mathcal{O} \doteq \int_{\sigma(O)} \lambda \, P_\lambda^{(O)} .$$

Observables generate a non commutative $*$-algebra $\mathscr{A}$ over $\mathbb{C}$.

Algebraic approach and GNS Theorem

Definition (∗-algebra)

A ∗*-algebra* $\mathscr{A}$ *over* $\mathbb{C}$ *is an algebra* $\mathscr{A} = \mathrm{Alg}(+, \cdot)$ *over* $\mathbb{C}$ *with an involution* $* : \mathscr{A} \to \mathscr{A}$ *such that*

$$(a \cdot b)^* = b^* \cdot a^*, \quad (\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^*.$$

A state ω *on a* ∗*-algebra* $\mathscr{A}$ *is a linear functional* $\omega \colon \mathscr{A} \to \mathbb{C}$ *which is:*

- *positive:* $\omega(a^* \cdot a) \geq 0$;
- *normalized:* $\omega(1_{\mathscr{A}}) = 1$.

Physical observables are self-adjoint elements $a = a^* \in \mathscr{A}$.

Algebraic approach and GNS Theorem

Theorem (Gelfand-Naimark-Segal)

Let ω be a state on a $*$-algebra $\mathscr{A}$. There exists a quadruple $(\mathsf{D}_\omega, \mathsf{H}_\omega, \pi_\omega, \Omega)$ such that:

- D_ω *is a dense subspace in* H_ω;
- $\pi_\omega : \mathscr{A} \to \mathcal{L}(\mathsf{H}_\omega)$ *is a $*$-representation of $\mathscr{A}$ onto D_ω-defined operator on* H_ω.
- $\pi_\omega(\mathscr{A})\Omega = \mathsf{D}_\omega$.
- $\omega(a) = \langle \Omega \mid \pi_\omega(a)\Omega \rangle$.

$(\mathsf{D}_\omega, \mathsf{H}_\omega, \pi_\omega, \Omega)$ *is unique up to unitary isomorphism.*

ω, ω' can induce different GNS representations of $\mathscr{A}$!

I would like to make a confession that may seem immoral:
I do not believe absolutely in Hilbert spaces anymore.
von Neumann (1935)