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Introduction to Noncommutative Geometry in Physics

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Riemannian Geometry

- Points $\boldsymbol{x}$ elements of a smooth (topological) manifold $\boldsymbol{M}$.
- Tangent spaces at each points $\boldsymbol{T_x M}$.
- Positive definite inner product $\boldsymbol{g_x}$ on $T_x M$ called metric tensor.

  $\longrightarrow g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$ where $\partial_\mu, \partial_\nu \in TM$.
- Riemannian manifold $(M, g) \longrightarrow$ infinitesimal length $\boldsymbol{ds^2 = g_{\mu\nu} dx^\mu dx^\nu}$ with $dx^\mu \in T^* M$.
- The distance between two points x_1 and x_2 of M is $d(x_1, x_2) = \inf \int_{x_1}^{x_2} ds$.
- Observable (any physical quantity that can be measured) $\equiv \boldsymbol{f(x) \in C^\infty(M)}$.

  $\longrightarrow$ Partial differential equations for mechanics, locality principle...

$\longrightarrow$ **Classical Mechanics.**

Geometry and Physics

- Geometry in physics:
  - Geometry is a representation where the processes of reality are made intelligible.
  - It encodes mathematically what we know about spacetime.
  - Spacetime is a concept that comes from our **observations** of the world.
- Two ways to see geometry with respect to **observables**:
  - Experimental way: observables collected in $\mathcal{C}^\infty(M) \xrightarrow{\textit{deduction}}$ geometry of spacetime (M, g).
  - Intuitive way: geometry of spacetime $(M, g) \xrightarrow{\textit{then}}$ observables in $\mathcal{C}^\infty(M)$ defined on M.

    $\longrightarrow$ Problem: Spacetime is not an **observable**.
- Useful generalisations of geometry in physics:
  - **Euclidean geometry to Riemannian geometry: General Relativity.**
  - Many arguments to generalise again our geometrical framework...

$\rightsquigarrow$ **Good candidate: Noncommutative Geometry.**

Quantum Mechanics (QM)

- Observables became operators $\sim \mathcal{C}^{\infty}(M)$ is **replaced by a noncommutative algebra $\boldsymbol{\mathcal{A}}$.**
  $\longrightarrow$ *Uncertainty principle.*

- $\mathcal{A}$ can be represented as acting on a Hilbert space $\boldsymbol{\mathcal{H}}$.

- $a \in \mathcal{A}$ represent a physical observable (position, momentum, spin..).

- $\psi \in \mathcal{H}$ represent the physical state whose evolution is given by **Schrödinger equation**.
  $\longrightarrow$***New mechanics****, superposition principle and entanglement.*

  Processes in M $\quad\longrightarrow\quad$ Processes in $\mathcal{H}$

- The theory is non deterministic (collapse of ψ during the measurement process).

  $\longrightarrow$ *Expectation value of an observable a for a quantum state ψ is $\langle\psi, a\psi\rangle$.*

The novelty in Quantum Mechanics (QM)

- **Observables:** functions f of Classical Mechanics (CM) $\longrightarrow$ operators a in an algebra $\mathcal{A}$.
- $\rightsquigarrow$ **Non-commuting observables ($[a, b] \neq 0$).**
- Physical states: Points in phase space $\longrightarrow$ elements in a Hilbert space $\mathcal{H}$.

$\rightsquigarrow$ **Feeling of a new notion of space/geometry (to be discovered).**
(at least for phase space.)

$\rightsquigarrow$ **How to extract information about an underlying space from the observables ($\mathcal{A}$)?**

Ground of the framework: a C^*-Algebra $\mathcal{A}$.

Starting from a C^*-Algebra $\mathcal{A} \longrightarrow$ the link with Hilbert spaces:

Theorem (Gelfand-Naimark)

Every abstract C^*-algebra $\mathcal{A}$ is isometrically $*$-isomorphic to a concrete C^*-algebras of bounded operators on a Hilbert space $\mathcal{H}$.

$$a \in \mathcal{A} \text{ a } C^*\text{-algebra} \leftrightarrow \text{bounded operator in } \mathcal{B}(\mathcal{H}) \equiv \text{Observable in QM}$$

$\rightsquigarrow$ **Does $\mathcal{A}$ and $\mathcal{H}$ contain information on an underlying "space"?**

The connection with Topology

Theorem (Gelfand-Naimark theorem (1943))

If a C^*-algebra is ***commutative*** *then it is an algebra of continuous functions on some (locally compact, Hausdorff) topological space.*

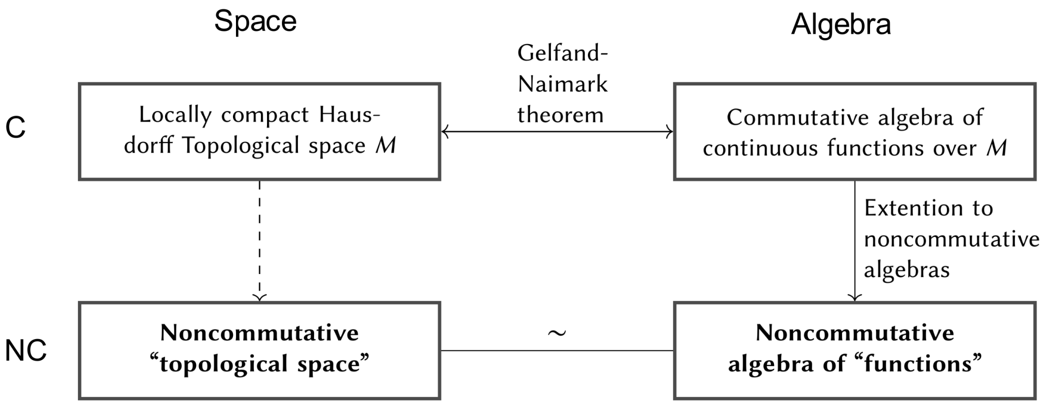
Gelfand transform: $\mathcal{A} \to C_0(M(\mathcal{A}))$ $M(\mathcal{A}) \equiv$ topological space

Algebra	Topology
Commutative C^*-algebra $\mathcal{A}$	Topological space M
Projectionless	Connectedness
Projection/Pure states	Point
Isomorphism	Homeomorphism
Unital	Compact
tensor product	Cartesian product

Figure: Equivalences between algebraic and topological properties.

$\rightsquigarrow$ **What happens with noncommutative algebras?**

Noncommutative Topology



C and NC: Commutative and NonCommutative.

$\rightsquigarrow$ **How to obtain something like a (differential) geometry?**

How to get a "Geometry" from $(\mathcal{A}, \mathcal{H})$?

Answer: Introduce a Differential Structure!

Differential Structure $\longrightarrow$ *study of observable/function/operator's variations.*

Two main kinds of differential structures in NCG:

- Those resulting from variations in the algebra $\mathcal{A}$:
  The derivation-based differential structures.
  Dubois-Violette, Kerner, Madore, Masson, Michor, 1988

- Those resulting from variations in the Hilbert space $\mathcal{H}$:
  The Spectral triples-based differential structures.
  Connes, Lott $\sim$ *1990*

The differential structure will play the (equivalent) role of tangent spaces in geometry.

$\longrightarrow$ Define a metric on it.

$\longrightarrow$ Define differential forms on it...

Spectral Triples

Definition (Spectral triple)

A Spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is the data of an involutive unital algebra $\mathcal{A}$ represented by bounded operators on a Hilbert space $\mathcal{H}$, and of a self-adjoint operator D acting on $\mathcal{H}$ such that the resolvent $(i + D^2)^{-1}$ is compact and that for any $a \in \mathcal{A}$, $[D, a]$ is a bounded operator.

- Real (J) and Even (γ) Spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$:

J anti-unitary operator $\langle J\psi_1, J\psi_2\rangle = \langle \psi_2, \psi_1\rangle$: • $J^2 = \epsilon$ • $JD = \epsilon' DJ$ • $J\gamma = \epsilon'' \gamma J$ • $\epsilon, \epsilon', \epsilon'' = \pm 1 \longrightarrow$ **define the KO dimension.**	γ a $\mathbb{Z}_2$-grading on $\mathcal{H}$, $\forall a \in \mathcal{A}$: • $\gamma^2 = 1$ • $\gamma^\dagger = \gamma$ • $\gamma D + D\gamma = 0$ • $\gamma\pi(a) = \pi(a)\gamma$

- Commutant property: $[a, Jb^* J^{-1}] = 0$.
- First-order condition $[[D, a], Jb^* J^{-1}] = 0$.

Connes Reconstruction Theorem

Hint: The Dirac operator $\not\partial \equiv -i\gamma^\mu \partial_\mu$ embodies the metric properties of the manifold through the relation $\{\gamma^\mu, \gamma^\nu\} = 2g_{\mu\nu}\mathbb{1}_{2^m}$.

Theorem (Connes Reconstruction Theorem)

There is a one-to-one correspondence between commutative even real spectral triples

$$\left(\mathcal{A} = C^\infty(M),\ \mathcal{H} = L^2(M, \mathcal{S}),\ D = \not\partial,\ J,\ \gamma\right)$$

(that respect the five conditions given in ***A. Connes (2008): On the spectral characterization of manifolds. (arXiv:0810.2088)****), and smooth oriented compact Riemannian spin manifolds M.*

$\longrightarrow$ Study of **Riemannian spin manifolds** through spectral triples.

$\longrightarrow$ Riemannian spin manifolds are important objects in physics.

$\longrightarrow$ The differential is $\delta(f)\psi = [-i\gamma^\mu\partial_\mu, f]\psi = -i\gamma^\mu\partial_\mu(f)\psi$ with $f \in C^\infty(M)$

What does this mean for Physicists?

- We are now free to forget the idea of a Riemannian (spin) manifolds with points as being **the primary structure** on which functions can take values as a secondary object.

$$f \in \mathcal{A} \quad \longrightarrow \quad f(x) \quad \longrightarrow \quad \hat{x}(f) \qquad \equiv \qquad \mathcal{A} \quad \longrightarrow \quad \mathcal{C}^{\infty}(M) \quad \longrightarrow \quad M$$

- The algebra of function together with the Dirac operator and the Hilbert space of bi-spinors can be taken as first, and the Riemannian (spin) manifold as second deduction.

  $\longrightarrow$ **Spectral description** of the Riemannian (spin) manifolds.

- This algebraic "view" permit to go beyond the classical geometrical (Riemannian) picture.

$\rightsquigarrow$ **What happens if we consider NC algebras $\mathcal{A}$?**

The matrix space:

- $(\mathcal{A}_F, \mathcal{H}_{\mathcal{A}_F}, D_{\mathcal{A}_F}, J_{\mathcal{A}_F}, \gamma_{\mathcal{A}_F})$
- Finite dimensional algebra $\mathcal{A}_F = M_n(\mathbb{C})$
- $\psi \in \mathcal{H}_{\mathcal{A}_F} \equiv (\psi_1, \ldots \psi_n)$
- $D_{\mathcal{A}_F}$ is a matrix acting on elements of $\mathcal{H}_{\mathcal{A}_F}$.
- Derivation $\delta(a) = [D_{\mathcal{A}_F}, a]$ for $a \in \mathcal{A}_F$.
- Axioms of spectral triples.

$\longrightarrow$ **Finite noncommutative space**

Distance Formula in Noncommutative Geometry

- **Points $\leftrightarrow$ pure states through Gelfand Naimark theorem.**

  $\longrightarrow$ Distance between states.

- Taking two states ψ_1 and ψ_2 with $\psi_{(1,2)} : \mathcal{A} \rightarrow \mathbb{C}$, we can define the distance between these two states:

$$d(\psi_1, \psi_2) = \sup\{|\psi_1(a) - \psi_2(a)|;\ a \in \mathcal{A},\ \ ||[D, a]|| \leq 1\}$$

- Consider the commutative algebra $\mathcal{A} = \mathcal{C}^\infty(M)$, and the usual Dirac operator $D_M = i\gamma^\mu\partial_\mu$.

  $\longrightarrow$ Pure states $\equiv$ points via Gelfand duality $\psi_x(f) = f(x)$ so that $\psi_{1,2} \longrightarrow x_{1,2} \in M$.

  $\longrightarrow$ We then we recover the usual distance corresponding to the metric g on M:

$$d_g(x_1, x_2) = \sup\{|f(x_1) - f(x_2)|;\ f \in \mathcal{C}^\infty(M),\ \ ||[D_M, f]|| \leq 1\}.$$

Why NCG is interesting?

- Reformulation and generalization of the geometric framework.

  $\longrightarrow$ Inspired by QM's formalism.

  $\longrightarrow$ Complete reformulation of Riemannian geometry in NCG.

- New way to see geometry starting from what we can observe (The algebra of Observables).

- Quantum gravitation's problem: **which variables to quantize?**

  $\longrightarrow$ NCG: start from the **observables** (of matter), which became quantized in QM.

  $\longrightarrow$ **The choice is given by physics** (non ad-hoc one).

- The full geometrization of all forces in one framework was (almost) done in NCG.

  (does not include Lorentzian spaces)

- Mathematical naturalness…

What is a Gauge Field Theory (GFT)?

- "**Force**" is the concept that explain the **trajectories** of material objects.
  $\longrightarrow$ The trajectories are determined by the forces the material object undergo.
- For **fundamental particles** (described by states ψ), trajectories are determined by the **interference pattern** of the particle's state ψ in space-time.
  $\longrightarrow$ The particle's trajectory is where $\psi\psi^\dagger$ is high.
- This interference pattern is derived from the knowledge of the **phase** $\theta(x)$ the physical state takes over all space i.e. $\psi(x) \propto \exp(i\theta(x))$.
- The **evolution of the phase** is given by the **gauge field** $\boldsymbol{A_\mu(x)}$:

$$\hat{\psi}(x + dx^\mu) = \exp(i(e/\hbar)\boldsymbol{A_\mu(x)}dx^\mu)\psi(x)$$

$\longrightarrow$ **The gauge field encode fundamental interactions/forces.**

- General covariant derivative: $\boldsymbol{D_\mu = \partial_\mu + (ie/\hbar)A_\mu}$.

What is a Gauge Field Theory (GFT)?

- Phase accumulation around an infinitesimal loop given by the square of sides (dx^{μ}, dx^{ν}):

$$\hat{\psi}(x) = (1 + (ie/\hbar)\boldsymbol{F}_{\mu\nu}dx^{\mu}dx^{\nu})\psi(x)$$

With $\boldsymbol{F}_{\mu\nu} = (\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}) - i(e/\hbar)[A_{\mu}, A_{\nu}]$ the curvature of the gauge potential.

$\longrightarrow$ This fundamental quantity is invariant under natural symmetries of the theory.

- Let $\mathcal{G}$ be a **finite-dimensional compact Lie group ≡ symmetry group.**
- $\psi(x)$ is a multiplet representation of $\mathcal{G}$.
- Fundamental quantities such as the Lagrangian must be invariant under the action of $\mathcal{G}$:

$$\mathcal{L}(\psi, D\psi) = \mathcal{L}(g\psi, gD\psi) \qquad \forall g \in \mathcal{G}$$

This is the gauge principle.

Yang–Mills theory

- $g \in SU(n) \to$ gauge transformations $\boldsymbol{A \to \tilde{A} = gAg^{-1} + g\partial g^{-1}}$.
- $\psi(x) = \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_n(x) \end{pmatrix}$; $A_\mu(x) = A_\mu^a(x) T^a$; $g(x) = e^{i\alpha^a(x) T^a}$.
- The Lagrangian of the SMPP is the following:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\overline{\psi} \not{D} \psi + \overline{\psi}_i y_{ij} \Phi \psi_j + h.c. + |D^\mu \Phi|^2 + \lambda \Phi^2 + \mu \Phi^4$$

with $\overline{\psi} = \psi^\dagger \gamma_0$, $\not{D} = i\gamma^\mu D_\mu$, y_{ij} the Yukawa coupling matrix, Φ the Higgs field.

= Yang–Mills Higgs theory

$\to$ **Full understanding of particles physics and fundamental interactions.**

NCG and Gauge Field Theory

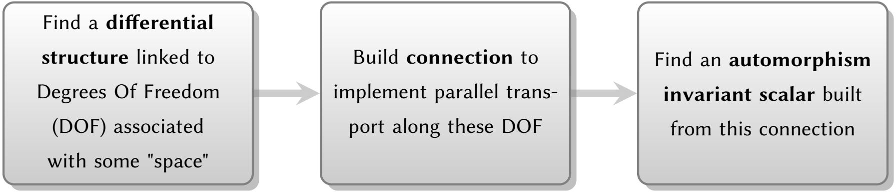
Usual framework for Yang–Mills Higgs theory: **Vector bundles.**

Theorem (Serre-Swan theorem (1962))

Any ***vector bundle*** *on a smooth compact manifold M defines a $C^\infty(M)$-**projective module of finite type** on the algebra $C^\infty(M)$ by considering the set of smooth sections of this vector bundle.*

$\longrightarrow$ Possibility to naturally implement the gauge principle in NCG.

Strategy to create Non-Commutative Gauge Field Theories (NCGFTs):



How to model a NCGFT?

Strategy to create Noncommutative Gauge Field Theories (NCGFTs):

The basic ingredient is an associative algebra $\mathcal{A}$. Then:

Representation theory: a (projective finitely generated) module $\mathcal{M}$ over $\mathcal{A}$.

Gauge group: $\mathcal{U}(\mathcal{A})$ or $\mathrm{Aut}(\mathcal{M})$.

Differential structure: any differential calculus defined on top of $\mathcal{A}$.

- The derivation-based differential calculus canonically associated to $\mathcal{A}$.
- **Spectral triple $(\mathcal{A}, \mathcal{H}, D)$: need supplementary structures, and $\mathcal{M} = \mathcal{H}$.**

Covariant derivative: A NC connection on $\mathcal{M}$ based on the differential calculus.

Action functional: An automorphism invariant scalar built from this connection.

- Derivations: Hodge star operator and NC-integration $\longrightarrow$ action functional.
- **Spectral triples: Spectral and Fermionic actions.**

Gauge Theory in the Spectral Triples Framework

- The $\mathcal{A}$-bimodule of Connes's differential **one-forms** is given by

$$\Omega^1_D(\mathcal{A}) := \left\{ \sum_k a_k \, [D, b_k] \, : \, a_k, b_k \in \mathcal{A} \right\}.$$

$\delta : \mathcal{A} \to \Omega^1(\mathcal{A})$ with $\delta(.) = [D, .]$ and $\delta^2 \neq 0$.

- $u \in \mathcal{U}(\mathcal{A})$ defines the unitary $U = \pi(u) J \pi(u) J^{-1} : \mathcal{H} \to \mathcal{H}$.
  - $\omega \in \Omega^1_D(\mathcal{A}) \to$ Fluctuated Dirac operator:

$$\boldsymbol{D_\omega = D + \omega + \epsilon' J \omega J^{-1}}.$$

  - **Inner fluctuation:** $UDU^\dagger = (D_\omega)^u = U D_\omega U^*$.
  - Equivalence with gauge transformation: $\omega^u = u \omega u^* + u \mathrm{d}_U u^* \leftrightarrow \boldsymbol{(D_\omega)^u = D_{\omega^u}}$.

Spectral and Fermionic Actions

- Lets consider the spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$.
- The fluctuated Dirac operator $D_\omega = D + \omega + \epsilon' J \omega J^{-1}$ with $\omega \in \Omega^1_D(\mathcal{A})$.
- The fermionic action is defined by:

$$\mathcal{S}_f[D_\omega, \widetilde{\psi}] = \frac{1}{2}\langle J\widetilde{\psi}, D_\omega \widetilde{\psi}\rangle_{\widetilde{\mathcal{H}}}.$$

  with $\widetilde{\psi}$ an element of the Grassmann vector space $\widetilde{\mathcal{H}}$ associated with $\mathcal{H} = L^2(M, \mathcal{S})$.
- The **spectral action** is given by:

$$\mathcal{S}[D] = \operatorname{Tr} f(D_\omega D_\omega^\dagger / \Lambda^2)$$

- The spectral action can be expanded using heat kernel expansion:

$$\mathcal{S}[D] = \lim_{\Lambda\to\infty} \operatorname{Tr} \exp(-D^2/\Lambda^2) \simeq \sum_{n\geq 0} \Lambda^{2m-n} a_n(D^2)$$

  with a_n the Seeley-de Witt Coefficients.

Spectral Action for the Manifold

- If we take the spectral triple $(\mathcal{A} = C^\infty(M),\ \mathcal{H} = L^2(M, \mathcal{S}),\ D = \not{\partial},\ J,\ \gamma)$, we obtain:

$$D^2 = \Delta^{\mathcal{S}} + \frac{1}{4}s \qquad \text{with the scalar curvature} \qquad s = R_{\mu\nu}g^{\mu\nu}.$$

- The non-zero contributions are given by the Seeley-de Witt Coefficients:

$$a_0(D^2) = \frac{1}{(2\pi)^m}\int_M dvol$$

$$a_2(D^2) = -\frac{1}{12(2\pi)^m}\int_M s\, dvol \quad \sim \quad \textbf{Einstein–Hilbert action}$$

$$a_4(D^2) = \frac{1}{360(2\pi)^m}\int_M \left(\frac{5}{4}s^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} + 30\Omega^{\mathcal{S}}_{\mu\nu}(\Omega^{\mathcal{S}})^{\mu\nu}\right)dvol$$

→ **Spectral invariant associated with the geometry of the manifold.**

The Almost Commutative Manifold

- NC extention of the manifold inside the framework of even and real spectral triples.
- Simple choice $\longrightarrow$ **almost commutative manifold** $M \times F$ with F a **finite space.**
- The corresponding spectral triple is $(\widehat{\mathcal{A}}, \mathcal{H}_{\widehat{\mathcal{A}}}, D_{\widehat{\mathcal{A}}}, J_{\widehat{\mathcal{A}}}, \gamma_{\widehat{\mathcal{A}}})$ with:

$$\widehat{\mathcal{A}} = \mathcal{C}^\infty(M) \otimes \mathcal{A}_F \qquad J_{\widehat{\mathcal{A}}} = J_M \otimes J_{\mathcal{A}_F}$$
$$\mathcal{H}_{\widehat{\mathcal{A}}} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_{\mathcal{A}_F} \qquad \gamma_{\widehat{\mathcal{A}}} = \gamma_M \otimes \gamma_{\mathcal{A}_F}$$
$$D_{\widehat{\mathcal{A}}} = D_M \otimes \mathbb{1} + \gamma_M \otimes D_{\mathcal{A}_F} \qquad .$$

$\longrightarrow$ Computation of the fluctuated Dirac operator: $D_{\widehat{\mathcal{A}},\omega} = D_M \otimes 1 + \gamma^\mu \otimes A_\mu + \gamma_M \otimes \Phi$.

$\longrightarrow$ Computation of the associated Spectral and Fermionic actions $\equiv$ NCGFT$_{\widehat{\mathcal{A}}}$'s Lagrangian.

$\longrightarrow$ **Yang–Mills–Higgs type theory coupled to Gravitation.**

$\longrightarrow$ Example: $\mathcal{A}_F = M_n(\mathbb{C})$ so that $\dim(F) = n$ is the dimension of the fiber.

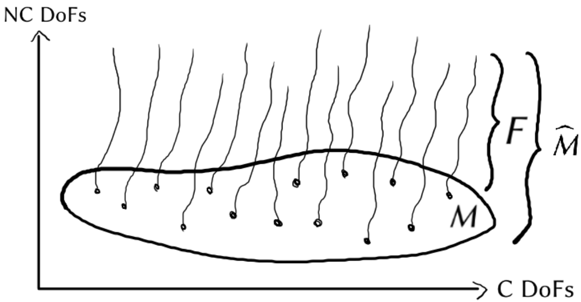
The NonCommutative Standard Model (NCSM)

- NCGFT which reproduces the standard model Lagrangian coupled to Gravity (Chamseddine, Connes, Lott, Marcolli…1996 ⟶ 2012)

  ⟶ Based on the model of AC Manifold $M \times F$.

  ⟶ Rely on a good choice of an even real Spectral triple.

- **Gives the Lagrangian of the SM coupled to gravity from pure geometry.**



$$\hat{M} = M \times F \;\longrightarrow\; \widehat{\mathcal{A}} = C^{\infty}(M) \otimes \mathcal{A}_F$$

$M \longrightarrow$ C DoFs

(along spacetime directions)

$F \longrightarrow$ NC DoFs

(along algebraic directions)

⟶ All interactions arise from an underlying NCG.

The NonCommutative Standard Model (NCSM)

- The crucial role is played by $\mathcal{A}_F = \mathcal{A}_{SM} := \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$.
- The symmetry group of the spectral action is $\mathcal{G} = Map(M, G) \rtimes \mathrm{Diff}(M)$.

  $\longrightarrow$ $\mathrm{Diff}(M)$ the diffeomorphism group.

  $\longrightarrow$ $Map(M, G)$ the gauge group of second kind with $G = U(1) \times SU(2) \times SU(3)$.
- In this way, NCG provides a unified framework to **describe both Einstein-Hilbert gravity (in Euclidean signature) and classical gauge theories.**
- It gives an elegant description of the SMPP, including Higgs mechanism and neutrino mixing, **as "gravity" on an AC-manifold.**
- The fermionic masses are encoded into $D_{\mathcal{A}_F}$, so that the masses of the Higgs boson and the ones of fermions became related, **offering a prediction for Higgs mass.**

Missing Points and Outlook

- **Main advantage of the NCSM:** very constrained description of the SMPP

  $\longrightarrow$ *Go beyond the SMPP (GUT, ...).*

- **The signature problem:** the model is inherently Euclidean (positive definite signature)

  $\longrightarrow$ *Find a "Lorentzian" approach to NCG.*

- **The Higgs mass problem:** wrong prediction for the Higgs mass.

  $\longrightarrow$ *Attempts to modify the spectral triple axioms to obtain the correct Higgs mass.*

- **The fermion doubling problem:** too many degrees of freedom for fermionic fields

Thank you