The Effective Topos

PhD seminar

Enrico Ghiorzi

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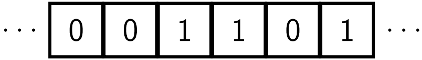
Computability: Turing Machines

A Turing Machine is made by

- a tape (a row of cells, containing symbols)



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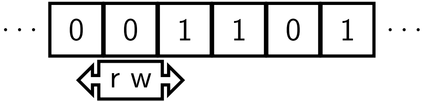
  - rewrite cell's content



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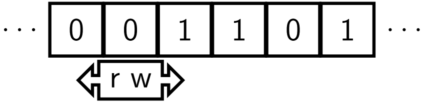

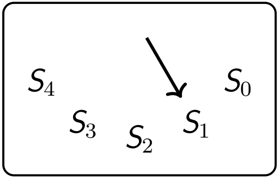

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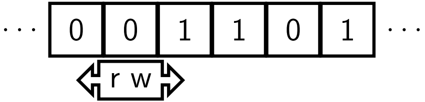
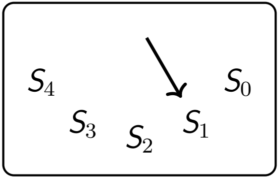
- a tape (a row of cells, containing symbols)
- a head, which
  - runs along the tape (left/right)
  - reads cell's content
  - rewrite cell's content
- states, one of which is the current one
- a list of instructions, each of them made by
  - initial state and symbol
  - new state and symbol
  - a direction (left/right)

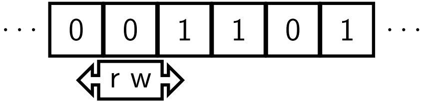


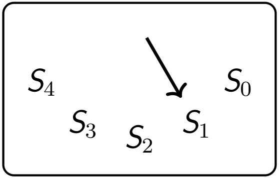


Computability: Turing Machines

Each step:

- head reads current symbol
- look for an instruction such that
  - initial state = current state
  - initial symbol = current symbol

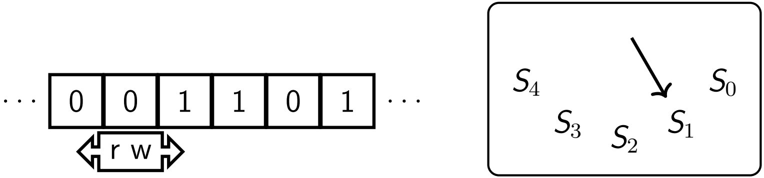




Computability: Turing Machines

Each step:

- head reads current symbol
- look for an instruction such that
  - initial state = current state
  - initial symbol = current symbol
- cell ⟵ new symbol
- state ⟵ new state
- the head moves left/right

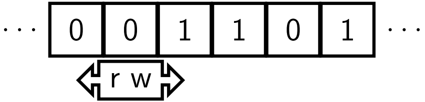


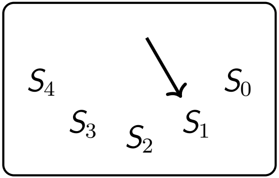
Computability: Turing Machines

Each step:

- head reads current symbol
- look for an instruction such that (∗)
  - initial state = current state
  - initial symbol = current symbol
- cell ⟵ new symbol
- state ⟵ new state
- the head moves left/right

Repeat till (∗) is possible, then halt





Computability: Partial Recursive Functions

PRF are functions $\mathbb{N}^k \to \mathbb{N}$ generated by

constants $\mathbb{N}^0 = \{ * \} \to \mathbb{N}$

projections $\pi_i \colon \mathbb{N}^k \to \mathbb{N}$

successor $s \colon \mathbb{N} \to \mathbb{N}$

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successor $s \colon \mathbb{N} \to \mathbb{N}$

and are closed under

composition $f_1 \colon \mathbb{N}^k \to \mathbb{N}, \dots\ f_n \colon \mathbb{N}^k \to \mathbb{N},\ g \colon \mathbb{N}^n \to \mathbb{N}$

$$\boxed{g(f_1, \dots f_n) \colon \mathbb{N}^k \to \mathbb{N}}$$

recursion $g \colon \mathbb{N}^k \to \mathbb{N}, h \colon \mathbb{N}^{k+2} \to \mathbb{N}$ induces $f \colon \mathbb{N}^{k+1} \to \mathbb{N}$

$$\boxed{f(0, \vec{x}) = g(\vec{x}) \qquad f(n+1, \vec{x}) = h(n, f(n, \vec{x}), \vec{x})}$$

minimization $f \colon \mathbb{N}^{k+1} \to \mathbb{N}$ induces $g \colon \mathbb{N}^k \to \mathbb{N}$ where

$$\boxed{g(\vec{x}) = \min \{ c \in \mathbb{N} \mid f(c, \vec{x}) = 0 \}}$$

partial!

PRF are numerable: $n(m) = n$-th p. r. function evaluated in m

Computability: Church-Turing thesis

Theorem

PRF $\equiv$ TM

Church-Turing thesis

All effectively computable functions are PRF (eq. TM)

Relizability

$\Sigma = \mathcal{P}(\mathbb{N})$ non-standard truth values

Σ^X non-standard predicates on set X

$$\bot_x = \emptyset$$

$$\top_x = \mathbb{N}$$

$$(s = t) = \{\, s \,\} \cap \{\, t \,\}$$

$$(\phi \wedge \psi)_x = \{\, \langle n, m \rangle \mid n \in \phi_x \text{ and } m \in \psi_x \,\}$$

$$(\phi \vee \psi)_x = \{\, \langle 0, n \rangle \mid n \in \phi_x \,\} \cup \{\, \langle 1, m \rangle \mid m \in \psi_x \,\}$$

$$(\phi \implies \psi)_x = \{\, e \mid \text{if } n \in \phi_x \text{ then } e(n){\downarrow} \text{ and } e(n) \in \psi_x \,\}$$

$$(\forall y \colon Y .\, \phi)_x = \bigcap \{\, \phi_{(x,y)} \mid y \in Y \,\}$$

$$(\exists y \colon Y .\, \phi)_x = \bigcup \{\, \phi_{(x,y)} \mid y \in Y \,\}$$

ϕ is valid ($\vdash_X \phi$) if $\bigcap \{\, \phi_x \mid x \in X \,\} \neq \emptyset$

Relizability: Excluded middle

Let s, t variables of type $\mathbb{N}$

$$\begin{aligned}\big((s = t) \vee (s \neq t)\big)_{s,t} &= \{ \langle 0, n \rangle \mid n \in (s = t)_{s,t} \} \\ &\quad \cup \{ \langle 1, n \rangle \mid n \in (s \neq t)_{s,t} \} \\ &= \{ \langle 0, n \rangle \mid n \in \{ s \} \cap \{ t \} \} \\ &\quad \cup \{ \langle 1, n \rangle \mid n \in \mathbb{N}, \{ s \} \cap \{ t \} = \emptyset \} \\ &= \{ \langle 0, s \rangle \mid s = t \} \cup \{ \langle 1, n \rangle \mid n \in \mathbb{N}, s \neq t \}\end{aligned}$$

and $\bigcap_{s,t \in \mathbb{N}} \{ \langle 0, s \rangle \mid s = t \} \cup \{ \langle 1, n \rangle \mid n \in \mathbb{N}, s \neq t \} = \emptyset$

so $\not\vdash_{s,t} (s = t) \vee (s \neq t)$

The Effective Topos

Definition (Objects of $\mathcal{E}$ff)

$(X, =_X)$ with

- X set
- $=_X : X \times X \to \Sigma$ such that

$$\begin{array}{rl} \text{Simmetry} & \vdash_X (x =_X x') \implies (x' =_X x) \\ \text{Transitivity} & \vdash_X (x =_X x') \wedge (x' =_X x'') \implies (x =_X x'') \\ \text{Reflexivity} & \text{No!} \end{array}$$

The Effective Topos

Definition (Morphisms)

$(X, =_X) \xrightarrow{[F]} (Y, =_Y)$ given by $\boxed{F \colon X \times Y \to \Sigma}$ fn. relation

$$\text{Relational} \vdash F(x, y) \wedge (x = x') \wedge (y = y') \implies F(x', y')$$

$$\text{Strict} \vdash F(x, y) \implies (x = x) \wedge (y = y)$$

$$\text{Single valued} \vdash F(x, y) \wedge F(x, y') \implies (y = y')$$

$$\text{Total} \vdash (x = x) \implies \exists y : Y.\ F(x, y)$$

$[F] = [G]$ when $\vdash F(x, y) \iff G(x, y)$.

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\text{Total} &\vdash (x = x) \implies \exists y : Y.\ F(x, y)
\end{aligned}$$

$[F] = [G]$ when $\vdash F(x, y) \iff G(x, y)$.

Definition (Composition)

Let $(X, =_X) \xrightarrow{[F]} (Y, =_Y) \xrightarrow{[G]} (Z, =_Z)$ then

$$\begin{aligned}
GF \colon X \times Z &\longrightarrow \Sigma \\
(x, z) &\longmapsto \exists y : Y.\ F(x, y) \wedge G(y, z)
\end{aligned}$$

Examples: $\mathcal{S}\mathrm{et}$, $\mathbb{1}$, $\mathbb{2}$

$$\begin{aligned} \Delta\colon \mathcal{S}\mathrm{et} &\longrightarrow \mathcal{E}\mathrm{ff} \\ S &\longmapsto (S, \delta) \end{aligned} \qquad \delta(s, s') = \begin{cases} \mathbb{N} & \text{if } s = s' \\ \emptyset & \text{otherwise} \end{cases}$$

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$\mathbb{1}_{\mathcal{E}\mathsf{ff}} = \Delta(\mathbb{1})$ initial object $(\mathbb{1} = \{ * \})$

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$$\mathbb{1}_{\mathcal{E}\mathrm{ff}} = \Delta(\mathbb{1}) \text{ initial object } (\mathbb{1} = \{ * \})$$

$$\mathbb{2}_{\mathcal{E}\mathrm{ff}} = \mathbb{1}_{\mathcal{E}\mathrm{ff}} + \mathbb{1}_{\mathcal{E}\mathrm{ff}} \neq \Delta(\mathbb{2}) \text{ indeed } (i =_{\mathbb{2}} j) = \{ i \} \cap \{ j \}$$

$\boxed{(i =_{\mathbb{2}} 0) \vee (i =_{\mathbb{2}} 1)}$ is not valid:

$$\{ (0 =_{\mathbb{2}} 0) \vee (0 =_{\mathbb{2}} 1) \} \cap \{ (1 =_{\mathbb{2}} 0) \vee (1 =_{\mathbb{2}} 1) \}$$
$$= \{ \langle 0,0 \rangle \} \cap \{ \langle 1,1 \rangle \} = \emptyset$$

$\boxed{\neg(i =_{\mathbb{2}} 0) \implies (i =_{\mathbb{2}} 1)}$ is valid:

$$\{ \neg(0 =_{\mathbb{2}} 0) \implies (0 =_{\mathbb{2}} 1) \} \cap \{ \neg(1 =_{\mathbb{2}} 0) \implies (1 =_{\mathbb{2}} 1) \}$$
$$= \{ \emptyset \implies \emptyset \} \cap \{ \mathbb{N} \implies \{ 1 \} \}$$
$$= \{ c_1 \colon \mathbb{N} \to \{ 1 \} \}$$

Examples: Natural Number Object

$$\mathbb{N}_{\mathcal{E}\mathrm{ff}} = \Big(\mathbb{N},\ (n =_{\mathbb{N}} m) = \{\, n \,\} \cap \{\, m \,\}\Big)$$

$$0\colon \mathbb{1}_{\mathcal{E}\mathrm{ff}} \to \mathbb{N}_{\mathcal{E}\mathrm{ff}} \qquad 0(*, n) = \{\, n \,\} \cap \{\, 0 \,\}$$

$$\mathsf{succ}\colon \mathbb{N}_{\mathcal{E}\mathrm{ff}} \to \mathbb{N}_{\mathcal{E}\mathrm{ff}} \quad \mathsf{succ}(n, m) = \{\, n + 1 \,\} \cap \{\, m \,\}$$

We can write arithmetic propositions!

Theorem

An arithmetic propositition is realized (= valid) iff it is true for $\mathbb{N}_{\mathcal{E}\mathrm{ff}}$ in $\mathcal{E}\mathrm{ff}$

Examples: products and equalizers

Products

Let $(X, =_X)$ and $(Y, =_Y)$ objects in $\mathcal{E}$ff

$$X \times Y = \big(X \times Y, =_{X \times Y}\ \big)$$

$$\big(\langle x, y\rangle =_{X \times Y} \langle x', y'\rangle\big) = \boxed{\big(x =_X x'\big) \wedge \big(y =_Y y'\big)}$$

Examples: products and equalizers

Products

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Equalizers

Let $(X, =_X) \underset{[G]}{\overset{[F]}{\rightrightarrows}} (Y, =_Y)$ morphisms in $\mathcal{E}$ff

$$E = \big(X, =_E\big)$$
$$\big(x =_E x'\big) = \boxed{\big(x =_X x'\big) \wedge \exists y : Y.\ F(x, y) \wedge G(x, y)}$$

Examples: Power set and Exponentials

Power sets

Let $(X, =_X)$ object in $\mathcal{E}$ff

$$\mathcal{P}(X) = \left(X \to \Sigma, =_{\mathcal{P}(X)}\ \right)$$

$$\left(U =_{\mathcal{P}(X)} U'\right) = \boxed{\forall x : X.\ U(x) \iff U'(x)}$$

Examples: Power set and Exponentials

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Exponentials

Let $(X, =_X)$ and $(Y, =_Y)$ objects in $\mathcal{E}$ff

$$Y^X = \big(\,\mathcal{P}(X \times Y), =_{Y^X}\ \big)$$

$$\big(f =_{Y^X} g\big) = \boxed{\begin{array}{l} \big(f =_{\mathcal{P}(X \times Y)} g\big) \wedge \forall x : X.\ \exists y : Y.\ f(x, y) \\ \qquad \wedge \big(\exists y' : Y.\ f(x, y') \implies (y =_Y y')\big) \end{array}}$$

Toposes

Definition ((Elementary) Topos)

A topos is a category with

- Terminal object, binary products and equalizers
- Exponentials
- Subobject classifier

Question

Is $\mathcal{E}$ff a topos?

Toposes

Definition ((Elementary) Topos)

A topos is a category with

- ✓ Terminal object, binary products and equalizers
- ✓ Exponentials
- ✓ Subobject classifier (Σ)

Question

Is $\mathcal{E}$ff a topos? Yes!

Informal note

$\mathcal{E}$ff is an elementary topos because it looks like

- $\mathcal{S}$et
- (Intuitionistic) logic

but not a Grothendieck topos as it's not a category of sheaves

That's all!

(thanks for attending)