Geometric construction activities on the Euclidean plane using Geogebra

PhD seminars – May 29th, 2023

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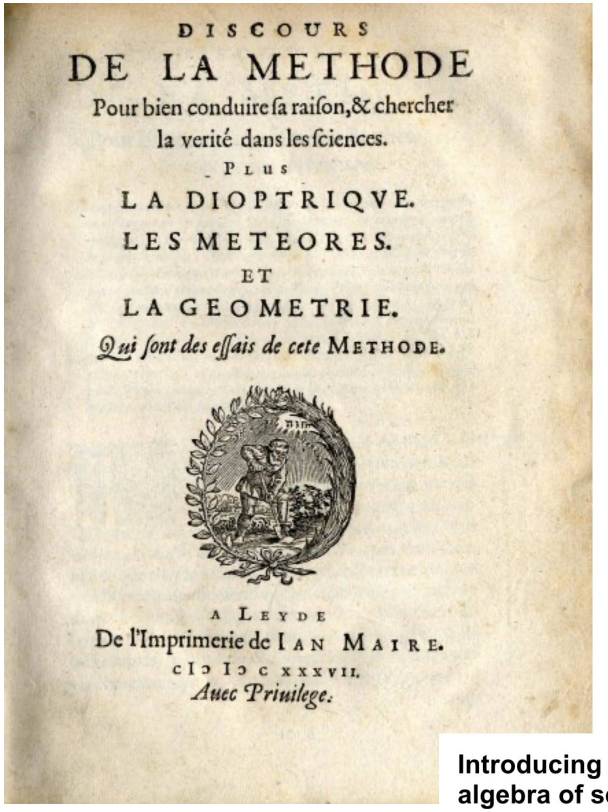
“La Géométrie” (1637) by René Descartes (1596-1650)

(with apologies to the historians of mathematics)



DISCOURS
DE LA METHODE
Pour bien conduire ſa raiſon, & chercher
la verité dans les ſciences.
PLUS
LA DIOPTRIQVE.
LES METEORES.
ET
LA GEOMETRIE.
Qui ſont des eſſais de cete METHODE.

A LEYDE
De l'Imprimerie de IAN MAIRE.
CIↃ IↃ C XXXVII.
Auec Priuilege.

appendix of *Discours de la Méthode Pour bien conduire sa raison, et chercher la vérité dans les sciences* (1637)

“Just as arithmetic consists of only four or five operations […] so in geometry […] And I shall not hesitate to introduce these arithmetical terms into geometry, for the sake of greater clearness.” (Descartes, 1996)

(Imperi & Rogora, 2022)

Introducing the notion of function through Descartes’ algebra of segments

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Dipartimento di matematica, Università degli studi di Roma La Sapienza, Italia

Abstract. *In his Géométrie (1637) Descartes introduces the algebra of segments. That’s a fundamental step in the mathematical treatment of variable quantities before the creation of the differential calculus. It is an algebra with symbols but without numbers, in which the covariation between geometric variables, constrained by ruler and compass constructions or with other geometric constructions, can be expressed with symbolic equations. By using algebraic manipulations, it is possible to easily deduce the properties of the corresponding geometric constructions, including those that produce graphs of rational functions. We believe that the study of functions through Descartes's algebra can be didactically effective in teaching and learning the concept of function in secondary school. Firstly, it avoids the reference to real numbers; secondly, the interpretation of formulas as geometric constructions and vice versa facilitates the “transition” from functions understood as processes to functions understood as objects.*

Main ideas

- Some (simple) constructions can be «interpreted as operations», i.e. that, starting from some line segments, we can construct others (with straightedge and compass) in such a way that these constructions respect the properties we would expect from operations.
- In other words, we can define operations on line segments (on points) of a half-line.

We will not stress this aspect a lot in the following

The operations on the half-line

"To add or subtract […] lines; or else, taking one line which I shall call unity in order to relate it as closely as possible to numbers […] and having given two other lines, to find a fourth line which shall be to one of the given lines as the other is to unity (which is the same as multiplication); or, again to find a fourth line which is to one of the given lines as the unity is to the other (which is the same as division)"

(Descartes, 1996)

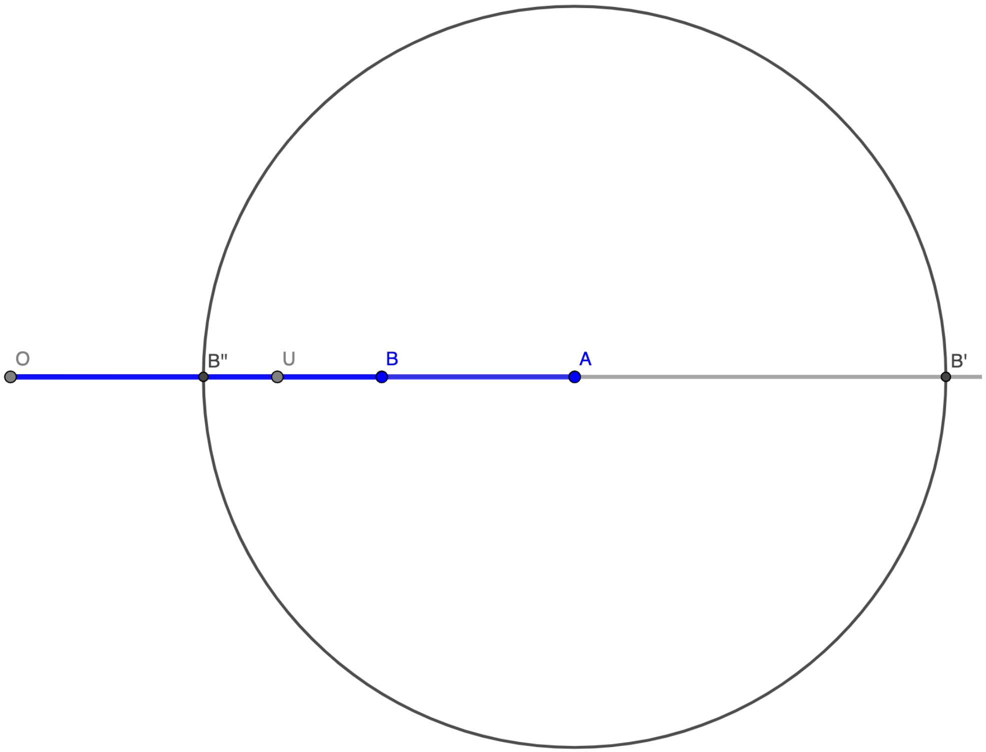
- In Descartes we find the algebra of line segments on a half-line, or equivalently, the algebra of equivalence classes of translation and rotation relations.
- Informally, two line segments on the plane are equivalent if they have the ‘same length’.
- In the plane, we fix a half-line to choose representatives, and the «unity» to define multiplication.
- We want to define an ‘algebra of the line segments on the plane’, or equivalently, algebra on the equivalence classes of the translation relation.
- In this case, we fix two points, the «origin» to choose representatives, and the «unity» to define multiplication.

The sum on the half-line

Let it be required to sum OA and OB.

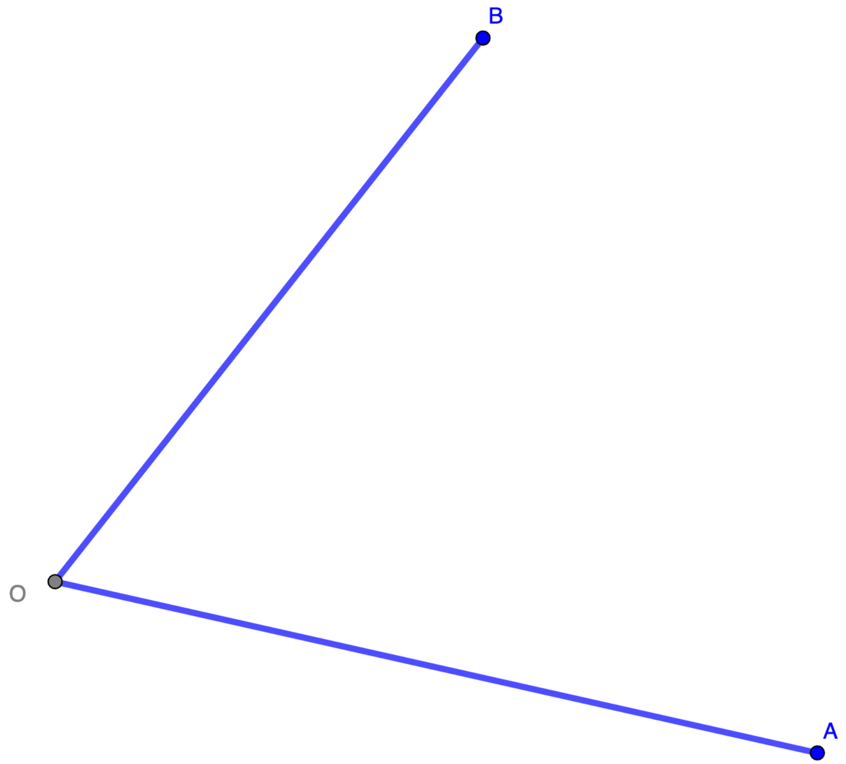
The sum on the half-line

Let it be required to sum OA and OB. I have to draw OB queued to OA; then OB’ is the sum of OA and OB… and OB’’ is the subtraction of OA and OB.



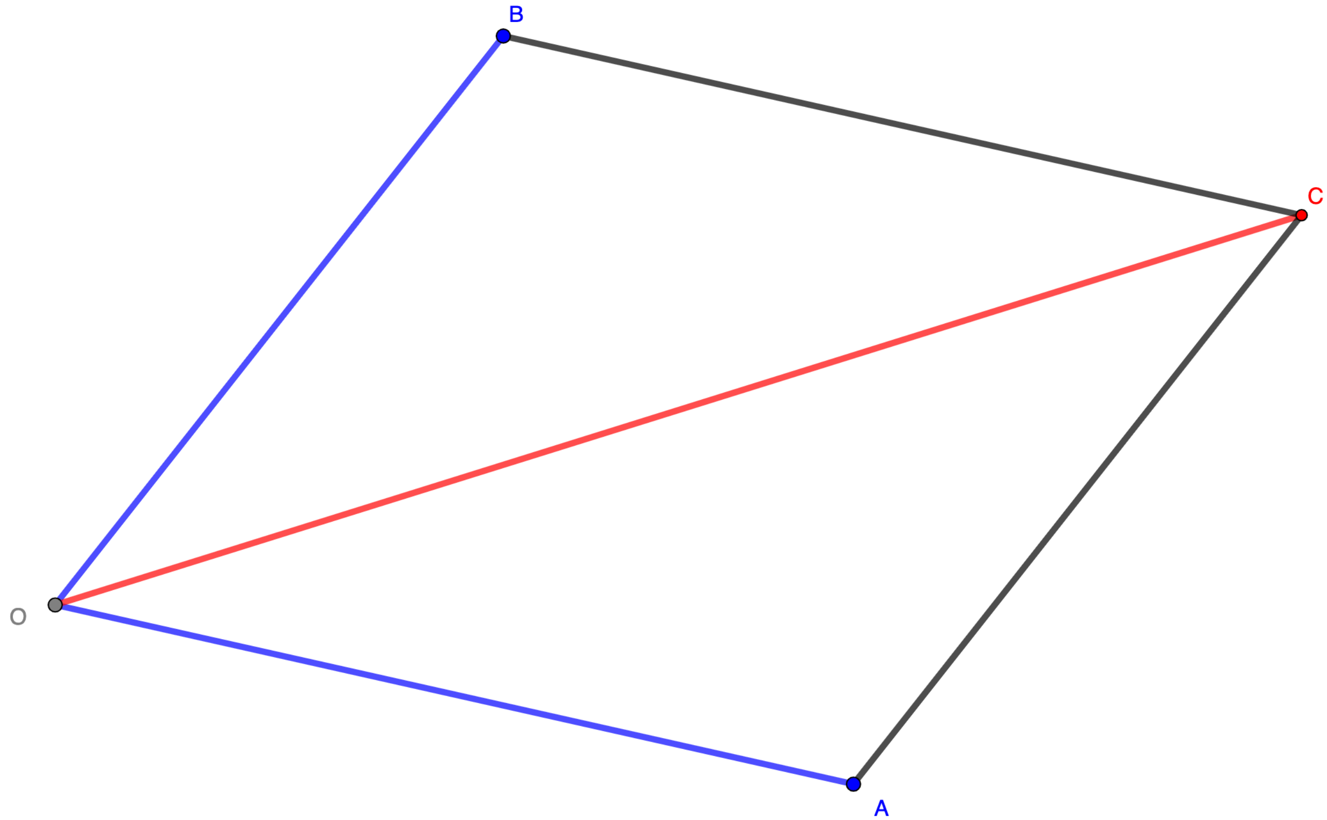
The sum on the plane

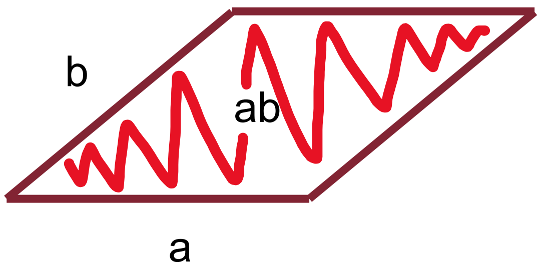
- Let it be required to sum OA and OB…



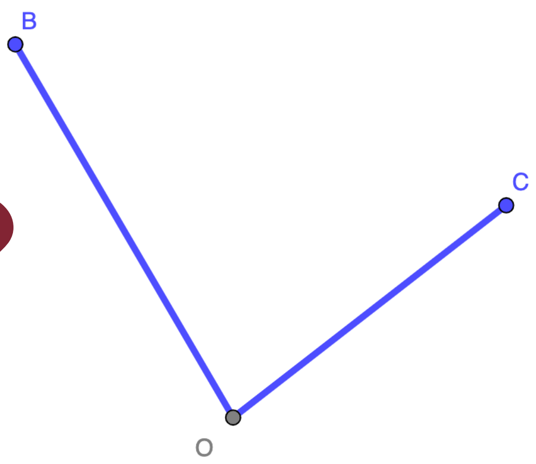
The sum on the plane

- Tip to tail method – parallelogram law works!





Vector product?



What about multiplication?

Any ideas?

- Descartes understood the usefulness of thinking of the multiplication (and division) of line segments (of the plane) as **a line segment (of the plane)**.

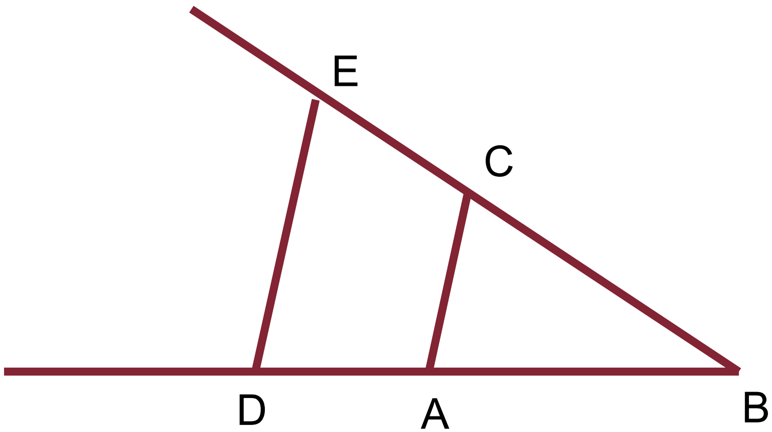
- Recall that Descartes required the unity!

The multiplication and the division on the half-line

- "For example, let AB be taken as unity, and let it be required to multiply BD by BC.
- I have to join the points A and C and draw DE parallel to CA; then BE is the product of BD and BC.
- If it be required to divide BE by BD, I join E and D and draw AC parallel to DE; then BC is the result of the division [of BE by BD]"

$$BE : BC = BD : AB$$

$$BE = \frac{BD}{AB} BC$$



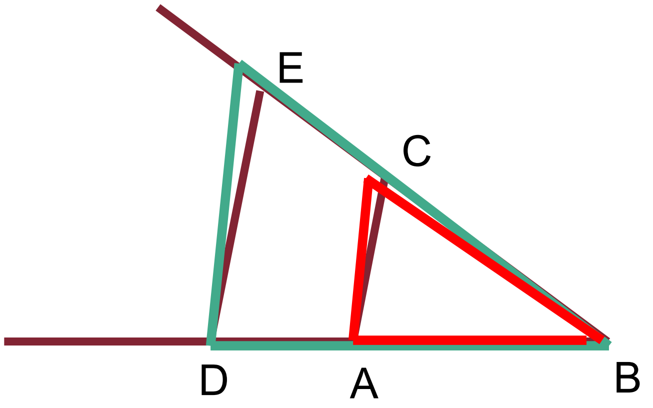
$$BC = \frac{AB}{BD} BE$$

(Descartes, 1996)

The multiplication on the plane

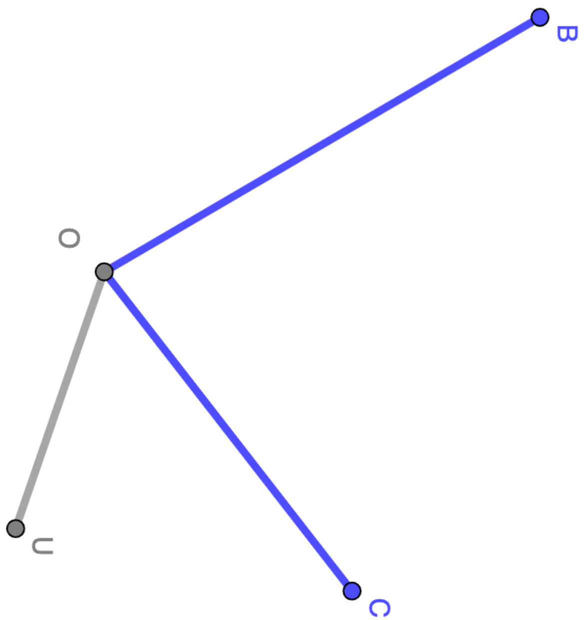
- Let AB be taken as unity, and let it be required to multiply BD by BC.

- We saw two similar triangles!



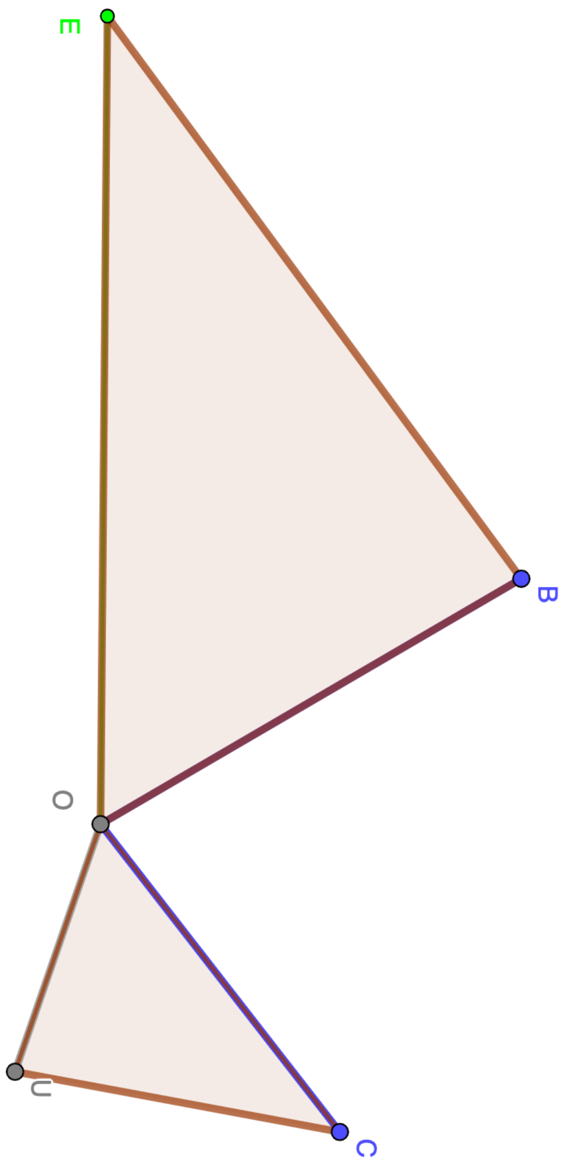
The multiplication on the plane

- Let AB be taken as unity, and let it be required to multiply BD by BC.



The multiplication on the plane

- Let AB be taken as unity, and let it be required to multiply BD by BC.
- We construct the triangle OBE similar to OUC where BOE corresponds to UOE, and OU corresponds to OB



Let's move on

GeoGebra

How we did design the laboratory

Thanks ER, LF, AB, ADZ!

Stage DIMA 2023

Who

- Address students of secondary school last years (age 16-18) thinking of enrolling in Mathematics degree course

Why

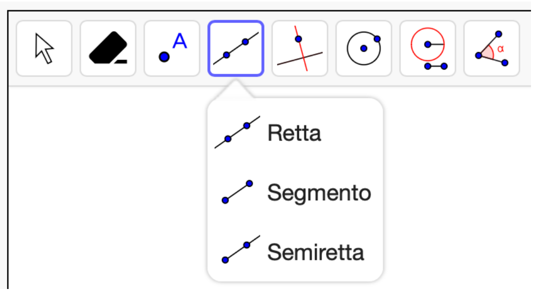
- The main aim is to make students close to different aspects of mathematics, pointing out how “**doing mathematics**” could be fun, inspiring and useful in applications, while being challenging.

When

- From Monday 13th to Thursday 16th February 2023

https://dima.unige.it/per_le_scuole/stage_al_dima

Design of a Geogebra classroom



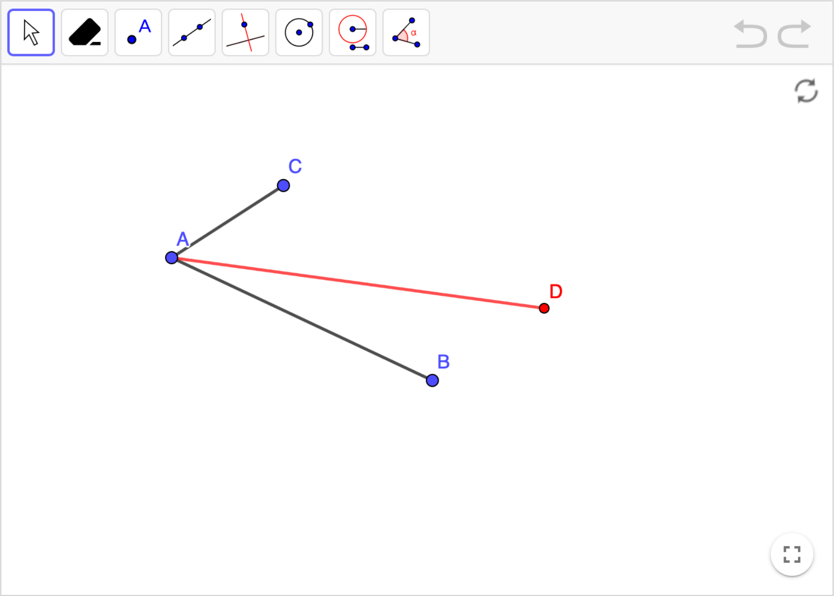
Two qualities that characterise the activity of the mathematician are curiosity and the desire to engage in research. Let us try to use this attitude to solve this laboratory.

- (Part 0: be familiar with GeoGebra)
- Part 1 and 2:
  - Questions (3/4): **construct, interpret, explore, observe**
  - Dinamic window
  - Question: **recognize**
- Part 3:
  - Questions (5): **explore, interpret**
  - Dinamic window
  - Question (3): **recognize**

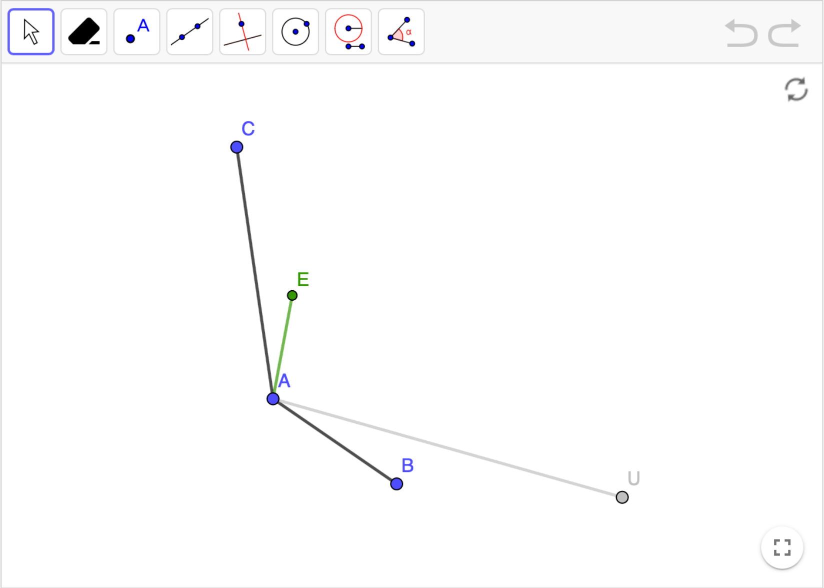
Design of Geogebra classroom

- Dragging B and C (also A, U) while observing **D=D(A,B,C)** and **E=E(A,B,C,U)** allows one to investigate relative movement and gain sensory experiences, and then answer questions.

Prima finestra interattiva: Dopo aver interagito con la seguente immagine, proviamo a rispondere alle domande precedenti. Iniziamo a esplorare con il tasto muovi.

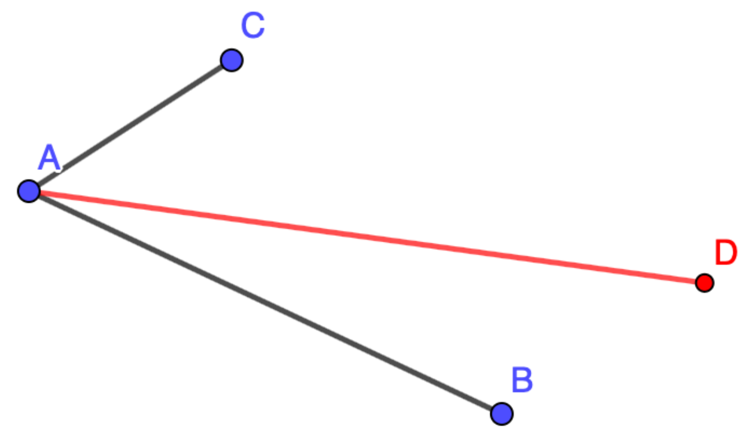


Seconda finestra interattiva: Dopo aver interagito con la seguente immagine, proviamo a rispondere alle domande precedenti. Iniziamo a esplorare con il tasto muovi.



PART 1: Questions

- Let's try to understand: how is the AD segment constructed? What does it represent? Let's try to explore and describe it.
- Let's try to understand: what happens when segment AB is in the line containing segment AC? Let's try to explore and describe this.
- Note the observations we make as we explore and try to answer the questions.
- Can we identify properties of an operation that we know and that describe what segment AD represents? Let us try to make at least two of them explicit and explain what kind of exploration we are doing to convince ourselves.



What «discourse» does it graft?

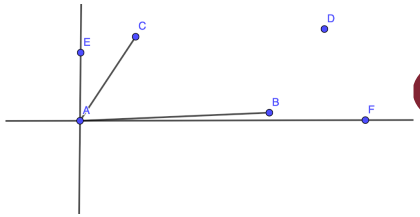
It is not that of complex numbers!

- Main directions (also hybrid):
  - «**Vectors**» (magnitude, direction, vector, components, resulting (vector), vector sum, vector product, tip to tail method, translation)
  - «**Parallelogram**» (vertex, side, congruent, diagonal, parallelogram, rectangle, angles)
  - Euclidean geometry (parallel lines, line segments, intersections, endpoint)
  - Trigonometry (sines formula)

AD is the resultant vector of AC and AB in magnitude and direction. If we construct a parallelogram with AC and AB as the two congruent sides, we find that AD is its diagonal. (parallelogram ABCD). We can find the **magnitude of AD** by knowing the angles [CAD and BAD] formed in A, **calculating the components in x and y (establishing a reference point)** and using the Pythagorean theorem.(G 4-8)

Taking the points $B(x_B, y_B)$ e $C(x_C, y_C)$, we can observe that $D(x_C, y_B)$. Then, we can deduce that the **mathematical definition** of the line segment $=\sqrt{(x_C - x_B)^2 + (y_C - y_B)^2}$

(G 4-8)



«Analytical»/measurement goals, «analytical experience» of Geogebra?

What characteristics of doing mathematics/being a mathematician do you think you have found in this laboratory?

Doing research

Reasoning

The fact that ideas and generalisations about universal laws come together

Before finding a solution to a problem, we can search for the solution by answering a few questions that we ask ourselves intelligently, analysing specific points that will help us work out the solution.

Listening and comparing

Tirelessness

What characteristics of doing mathematics/being a mathematician do you think you have found in this laboratory?

Starting from a seemingly unrelated situation and arriving at a meaning that is consistent with everything.

Links and logic between different areas of mathematics

Never underestimate anything, pay attention to every detail

Combining multiple skills

Trying to analyse an event/phenomenon/problem by making many observations

Ideas

Making figures move moves ideas

(E. Castelnuovo)

Thanks for your attention!

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Descartes, R. (1996). The geometry of René Descartes (D. E. Smith & M. L. Latham, Trans.). Dover Publications.

Imperi, N., & Rogora, E. (2022). Introdurre la nozione di funzione con l’algebra dei segmenti di Cartesio. La matematica e la sua didattica, 30(1–2), 33–52.