Straightedge-compass vs Origami

PhD Seminars

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Both theories are used to solve geometric problems.

STRAIGHTEDGE-COMPASS THEORY

- It uses a ruler and a compass.
- New points are obtain by intersection between lines and circles.

ORIGAMI THEORY

- It uses a piece of paper by folding it.
- New points are obtain by intersection of folds, or by moving points with folds.

Example

Both theories can:

- construct an equilateral triangle starting from an edge;
- find the bisector of an angle given by two intersecting lines.

Who is better?

Who is better at solving many different geometric problems?

Go to **www.menti.com** and use the code **4777 1549**.

Motivation

The Origami theory has many applications:

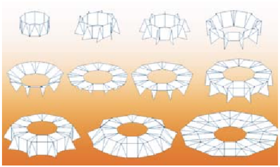


Figure: Eyeglass

Figure: Veins enlarger

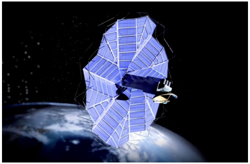


Figure: Solar panel

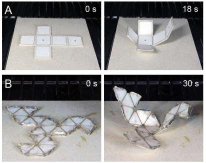


Figure: Self-folding paper

Straightedge-compass theory

The straightedge-compass theory is based on the following axioms:

- given two points A and B, we can draw a line thought them;
- given three points A, B, and C, we can draw the circle with center A, and radius $\overline{BC}$.

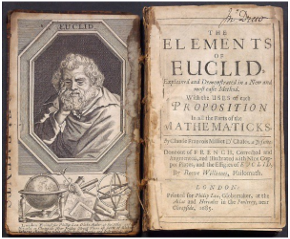


Figure: Euclid's elements

Possible constructions:

- division of segment in *n* equal parts;
- sum of numbers;
- product of two numbers.

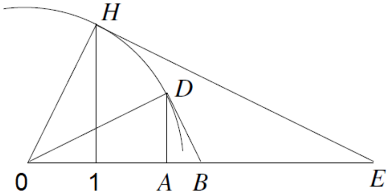


Figure: $\overline{0A} * \overline{0B} = \overline{0D}^2 = \overline{0E}$

Question: are all constructions possible?

Example

The heptagon, and the trisection of a general angle are not possible.

Figure: Heptagon

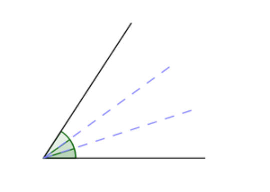


Figure: Angle trisector

Modern mathematics has joined the party

We call $\mathcal{P}$ a set of points. Certainly $\mathcal{P}$ has at least two points.
We can construct a Cartesian plane.

Definition

Given a set of point $\mathcal{P}$, we name $K_{\mathcal{P}}$ the minimal field which contains the coordinates of the points in $\mathcal{P}$.

Example

If $\mathcal{P}$ consists of exactly two points then $K_{\mathcal{P}} = \mathbb{Q}$.

Constructible points

Let $\mathcal{P}$ be a set of points.

Definition

We say that a point Q is *constructible in one step* if Q is:

- the intersection point of two lines drawn from the points in $\mathcal{P}$, or
- the intersection point of two circles drawn from the points in $\mathcal{P}$, or
- the intersection point of a line and a circle drawn from the points in $\mathcal{P}$;

Definition

We say that a point Q is *constructible* if there exists a sequence of points $(Q_1, \dots, Q_n)$ s.t. $Q_n = Q$, and Q_i is constructible in one step from $\mathcal{P} \cup \{Q_0, \dots, Q_{i-1}\}$.

Field extensions kick in

If Q is a constructible point, we obtain a sequence of field inclusions

$$K_{\mathcal{P}} \subset K_{\mathcal{P}\cup\{Q_0\}} \subset K_{\mathcal{P}\cup\{Q_0,Q_1\}} \subset \cdots \subset K_{\mathcal{P}\cup\{Q_0,\ldots,Q_{n-1},Q_n=Q\}}.$$

In particular, for every index i,

$$K_{\mathcal{P}\cup\{Q_0,\ldots,Q_{i-1}\}} \subset K_{\mathcal{P}\cup\{Q_0,\ldots,Q_i\}}$$

is a field extension!

Definition

A field exstension $K \subset L$ consists of two fields K, L such that K is a subfield of L. Moreover, L can be viewed as a K-vector space. $[L : K]$ is the *degree* and denotes the dimension of L as K-vector space.

Theorem

Let Q be a constructible point in one step from $\mathcal{P}$. Then

$$[K_{\mathcal{P}\cup\{Q\}} : K_{\mathcal{P}}] = 1, \text{ or } 2.$$

Theorem

Let $Q = (x, y)$ be a constructible point from $\mathcal{P}$. Then $[K_{\mathcal{P}\cup\{Q_0,\dots,Q_{n-1},Q\}} : K_{\mathcal{P}}]$ is a power of 2. Moreover, also $[K_{\mathcal{P}}(x) : K_{\mathcal{P}}]$ and $[K_{\mathcal{P}}(y) : K_{\mathcal{P}}]$ are powers of 2.

The trisection of a general angle:

An angle ϕ is determined uniquely by the numbers $\cos\phi$, and $\sin\phi$.

If ϕ is general, then the minimal polynomial of $\cos\frac{\phi}{3}$ is $T(x) = 4x^3 - 3x - \cos\phi$. Therefore, the construction has degree 3, and it is not possible.

The *n*-gon

A positive integer n is called *Fermat prime* if it is prime, and $n = 2^{2^m} + 1$ for some $m \in \mathbb{N}$. For example $3, 5, 17, 257, 65537$ are Fermat primes (and the only ones known today).

Theorem (Gauss-Wantzel)

The n-gon is constructible if and only if $n = 2^m p_1 \dots p_s$, *with* $m \in \mathbb{N}$ *and* $p_1, \dots, p_s$ *are distinct Fermat primes.*

Example

The heptagon is not constructible because 7 is not Fermat prime and cannot be written as above.

Origami theory

The origami theory consists of seven axioms, called the *Huzita-Hatori axioms*, named after Huzita Humiaki (1924-2005) and Hatori Koshiro (1961). These axioms were first discovered by Jacques Justin.

Figure: Huzita Humiaki

The axioms are the following:

1. Given two distinct points p_1 and p_2, there is a unique fold that passes through both of them.
2. Given two distinct points p_1 and p_2, there is a unique fold that places p_1 onto p_2.
3. Given two lines l_1 and l_2, there is a fold that places l_1 onto l_2.
4. Given a point p_1 and a line l_1, there is a unique fold perpendicular to l_1 that passes through point p_1.
5. Given two points p_1 and p_2 and a line l_1, there is a fold that places p_1 onto l_1 and passes through p_2.
6. Given two points p_1 and p_2 and two lines l_1 and l_2, there is a fold that places p_1 onto l_1 and p_2 onto l_2.
7. Given one point p and two lines l_1 and l_2, there is a fold that places p onto l_1 and is perpendicular to l_2.

Bad paper-folding

The paper-foldings

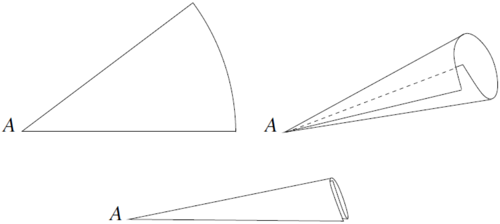


Figure: Angle trisection

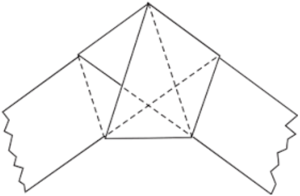


Figure: Knot-pentagon

are not allowed by the Huzita-Hatori axioms, but both the geometric constructions are possible!!!

Who is better?

Theorem

The straightedge-compass theory is equivalent to the first 5 axioms of the origami theory.

The sixth axiom allows to solve equations of degree 3.

The sixth axiom and...

Let p_1, p_2 be points and l_1, l_2 be lines.

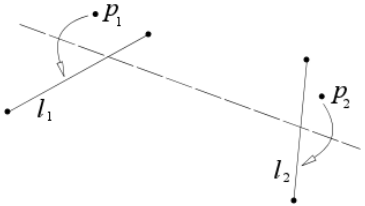


Figure: 6th axiom

There exists a fold that put p_1 into l_1, and p_2 into l_2 simultaneously (if it is possible). If the fold exists, then the new line is tangent to two parabolas: the first with focus p_1 and directrix l_1, and the second with focus p_2 and directrix l_2.

Eduard Lill in 1867 found a way to visualise roots of polynomials.

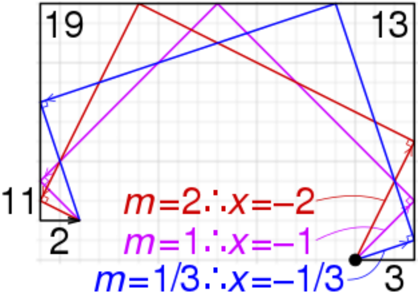


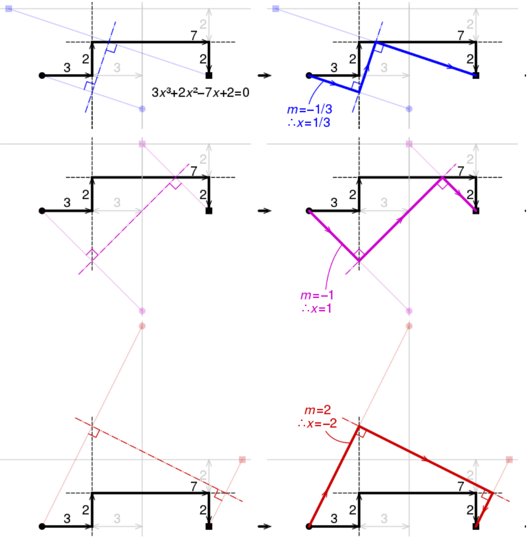
Figure: $p(x) = 3x^4 + 13x^3 + 19x^2 + 11x + 2$

In 1936 Margherita Piazzola Beloch showed that Lill's method can be applied to solve cubic equations.

Solving equations of degree three

In the example on the right, we want to find the roots of the polynomial

$$p(x) = 3x^3 + 2x^2 - 7x + 2.$$



The trisection of a general angle in origami theory

Recalling that finding $\frac{\phi}{3}$ is equivalent to finding the roots of $T(x) = 4x^3 - 3x - \cos\phi$, origami theory admits the trisection of a general angle.

The n-gon in origami theory

A positive integer n is called *Pierpont prime* if it is prime, and $n = 2^u 3^v + 1$ with $u, v \in \mathbb{N}$.

Theorem

The n-gon is constructible, in the origami theory, if and only if $n = 2^u 3^v p_1 \dots p_s$ where $u, v \in \mathbb{N}$, and $p_1, \dots, p_s$ are distinct Pierpont primes.

Example

The heptagon is constructible because 7 is Pierpont prime. But, the hendecagon is not constructible because 11 is not Pierpont prime.

Thank you for your attention.