Hadamard states for quantum Abelian duality

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Goals:

- Implement electromagnetic duality in a natural way
- Construct Hadamard states that unitarily implement duality

Outline:

1. Abelian duality via differential cohomology
2. Cauchy problem, observables and duality
3. Quantization and states
4. A sketch of the construction
5. Example: $M = \mathbb{R} \times S^1$

Abelian duality via differential cohomology

Electromagnetism (generalized): M g. h. spacetime, $m := \dim M$

Yang-Mills (bundle + connection)
Aharonov-Bohm effect

Differential cohomology
$h \in \hat{\mathrm{H}}^k(M;\mathbb{Z})$

Duality in Maxwell's equation:

$$F \longleftrightarrow *F$$

$$(h,\tilde{h}) \in \hat{\mathrm{H}}^k(M;\mathbb{Z}) \times \hat{\mathrm{H}}^{m-k}(M;\mathbb{Z})$$
$$h \longleftrightarrow \tilde{h}$$

Configuration space: Abelian group, no vector space!

$$\mathfrak{C}^k(M) = \Big\{(h,\tilde{h}) \in \hat{\mathrm{H}}^k(M;\mathbb{Z}) \times \hat{\mathrm{H}}^{m-k}(M;\mathbb{Z}) : \mathrm{curv}\, h = *\,\mathrm{curv}\,\tilde{h}\Big\}$$

Remark: $(h,\tilde{h})$ carries curvature + Chern class, occurring in *dual* copies.

Cauchy problem, observables and duality

This system has a well-posed Cauchy pbl. on a g. h. spacetime M:

$$\mathrm{curv}\, h = *\,\mathrm{curv}\, \tilde{h}, \qquad i_\Sigma^* h = h_0, \qquad i_\Sigma^* \tilde{h} = \tilde{h}_0$$

$i_\Sigma : \Sigma \to M$ Cauchy surface embedding, $(h_0, \tilde{h}_0) \in \hat{\mathrm{H}}^k(\Sigma;\mathbb{Z}) \times \hat{\mathrm{H}}^{m-k}(\Sigma;\mathbb{Z})$.

Symplectic structure: (Σ compact to avoid technicalities)

$$\sigma : \mathfrak{C}^k(M) \times \mathfrak{C}^k(M) \longrightarrow \mathbb{T}, \quad \sigma\big((h,\tilde{h}),(h',\tilde{h}')\big) = \int_\Sigma \tilde{h}\cdot h' - \tilde{h}'\cdot h$$

We regard $\sigma(\cdot,(h,\tilde{h}))$ as observables.

Duality: $\mathfrak{C}^k(M) \longrightarrow \mathfrak{C}^{m-k}(M), \qquad (h,\tilde{h}) \longmapsto (\tilde{h}, (-1)^{k(m-k)+1} h)$

It preserves σ, most interesting case: $m := \dim M = 2k$)

CCR-quantization: $W(h,\tilde{h})\, W(h',\tilde{h}') = e^{2\pi i \sigma((h,\tilde{h}),(h',\tilde{h}'))}\, W(h+h', \tilde{h}+\tilde{h}')$

Decomposition

$M = \mathbb{R} \times \Sigma$, Σ compact, $g = -\,\mathrm{d}\, t \otimes \mathrm{d}\, t + h_\Sigma$ ultrastatic.

$$
\begin{tikzcd}
\mathrm{Topf}^k(M)_1 \arrow[r, tail] \arrow[d, tail] & C \arrow[r, two heads] \arrow[d, tail] & \mathrm{Dyn}^k(M) \arrow[d, tail] \arrow[l, bend right] \\
A \arrow[r, tail] \arrow[d, two heads] & \mathfrak{C}^k(M) \arrow[r, two heads] \arrow[d, two heads] & B \arrow[d, two heads] \\
\mathrm{Topt}^k(M) \arrow[r, tail] \arrow[u, bend left] & D \arrow[r, two heads] & \mathrm{Topf}^k(M)_2 \arrow[ul]
\end{tikzcd}
$$

Symplectic decomposition:

$$\mathfrak{C}^k(M) \simeq \mathrm{Dyn}^k(M) \\ \oplus \mathrm{Topf}^k(M) \oplus \mathrm{Topt}^k(M)$$

Symplectic Abelian group $\mathfrak{C}^k(M)$ split into 3 symplectic sectors:

$\mathrm{Dyn}^k(M)$: Sector controlled by a PDE

$\mathrm{Topf}^k(M), \mathrm{Topt}^k(M)$: Characteristic classes and flat connections

Splitting compatible with duality!

Which states?

Symplectic decomposition $\implies$ C*-algebra factorization

To assign a state for the full theory, we assign one for each factor and combine them (with some care with the choice of tensor product). We aim at:

Hadamard Microlocal spectrum condition for the dynamical sector

Duality The state for each factor should be duality invariant
$\implies$ Duality unitarily implemented on the GNS triple

Remarks:

- Finitely generated topological sectors (quantum mechanics-like)
- microlocal spectrum condition only for the dynamical sector

Oscillatory degrees of freedom, typical of QFT, are combined to QM-like topological degrees of freedom (electric/magnetic fluxes, Aharonov-Bohm configurations).

How to get such states?

1. Decompose the theory into dynamical and topological sectors via a symplectically orthogonal and duality compatible splitting

   (non-unique, non-canonical choice of splitting)

2. Assign a duality-invariant state to each sector

   Dynamical sector: Hodge decomposition on Σ (compact) and (a bit of) microlocal analysis

   Topological sectors: several possibilities (e.g. particles on a circle)

3. State on the full theory as tensor product of states on sectors

   (some care with tensor products of C^*-algebras required)

4. Associated GNS triple and unitary operator implementing duality

2D example

$M = \mathbb{R} \times S^1$, $g = -\mathrm{d}\,t \otimes \mathrm{d}\,t + d\theta \otimes d\theta$

$$\mathfrak{C}^1(M) = \left\{(h, \tilde{h}) \in C^\infty(M; \mathbb{T})^2 : d\log h = *d\log \tilde{h}\right\}$$
$$\simeq \underbrace{\mathrm{d}\,C^\infty(M) \cap *\,\mathrm{d}\,C^\infty(M)}_{\textit{dynamical}} \oplus \underbrace{\mathbb{T}^2 \oplus \mathbb{Z}^2}_{\textit{topological}}$$

$$h(t,\theta) = h_0 + z\,\theta - \Big(\tilde{z}\,t + \sum_{k\geq 1}\big(c_k^+\cos(2\pi k(t+\theta)) + c_k^-\cos(2\pi k(t-\theta))$$
$$-s_k^+\sin(2\pi k(t+\theta)) - s_k^-\sin(2\pi k(t-\theta))\big) \quad \text{mod } \mathbb{Z}\Big)$$
$$\tilde{h}(t,\theta) = \tilde{h}_0 + \tilde{z}\,\theta - \Big(z\,t + \sum_{k\geq 1}\big(-c_k^+\cos(2\pi k(t+\theta)) + c_k^-\cos(2\pi k(t-\theta))$$
$$+s_k^+\sin(2\pi k(t+\theta)) - s_k^-\sin(2\pi k(t-\theta))\big) \quad \text{mod } \mathbb{Z}\Big)$$

$k \geq 1 \implies$ *no zero-modes in the dynamical sector!*

2D example: state on the dynamical sector

$M = \mathbb{R} \times S^1$, $g = -\mathrm{d}\, t \otimes \mathrm{d}\, t + d\theta \otimes d\theta$

Formula for the two-point function:

$$\omega_2(\rho, \rho') = \sum_{k \geq 1} (4\pi k)^{-1} \Big(\widehat{\delta\,\rho}(k,k)\,\widehat{\delta\,\rho'}(-k,-k) + \widehat{\delta\,\rho}(k,-k)\,\widehat{\delta\,\rho'}(-k,k) \Big)$$

$$= \sum_{k \geq 1} \pi k \big(c_k^+ c_k'^+ + c_k^- c_k'^- + s_k^+ s_k'^+ + s_k^- s_k'^- \big)$$

Remarks:

Line 1 Observables labelled by 1-forms $\rho \in \Omega^1_c(M)$.
ω_2 fulfils the microlocal spectrum condition.

Line 2 Fourier coefficients $c_k^\pm, s_k^\pm$ from the previous slide.

0-modes None! (treated separately in the topological sector).

0-modes are evil!

2D example: topological sector

$M = \mathbb{R} \times S^1 \implies \mathrm{Topf}^1(M) \simeq \mathrm{H}^1(M;\mathbb{T})^2 \oplus \mathrm{H}^1(M;\mathbb{Z})^2 \simeq \mathbb{T}^2 \otimes \mathbb{Z}^2$

Symplectic structure:

$$\sigma_T((u, \tilde{u}, z, \tilde{z}), (u', \tilde{u}', z', \tilde{z}')) = \tilde{u}\, z' - u'\, \tilde{z} - \tilde{u}'\, z + u\, \tilde{z}' \in \mathbb{T}$$

State: *two particles on a circle, initial positions $u, \tilde{u}$, discrete momenta $z, \tilde{z}$.*

$$\omega_T(W(u, \tilde{u}, z, \tilde{z})) = \begin{cases} 1, & \text{if } z = 0 = \tilde{z}; \\ 0, & \text{otherwise.} \end{cases}$$

Momenta label a basis $\{|z, \tilde{z}\rangle\}$ of the GNS-Hilbert space.

Unitary operator for duality: $U|z, \tilde{z}\rangle = |\tilde{z}, z\rangle$ (flipping momenta)

Self-adj. momentum observables: $P|z, \tilde{z}\rangle = z|z, \tilde{z}\rangle, \tilde{P}|z, \tilde{z}\rangle = \tilde{z}|z, \tilde{z}\rangle$
(spectra given by the values of momenta)

Conclusions

- Differential cohomology encodes Chern classes (topological fluxes), flat connections (Aharonov-Bohm effect), ...
- Abelian duality is implemented naturally.
- Splitting the model into dynamical and topological sectors leads to a factorization of the observable algebra.
- States can be obtained by assigning a state on each sector:
  Dynamical sector Ground state, microlocal spectrum condition.
  Topological sector Quantum mechanics-like (e.g. particles on S^1).
- The GNS triple implements Abelian duality as a unitary operator.

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Thank you for your attention!