Zeta regularization and Casimir effect for a scalar field with singular background potentials

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Plan of the talk

- Motivations and basic ideas:
  - Casimir effect, delta-type potentials and zeta regularization.
- General formalism:
  - functional spaces, Schrödinger-type operators, integral kernels;
  - zeta regularization of a Wightman scalar field.
- Delta-type potentials:
  - singular perturbations of self-adjoint operators;
  - an explicitly solvable example.
- Further developments and outlook.

References

- S. Albeverio *et al.*, *Solvable Models in Quantum Mechanics* ('88).
- S. Albeverio, G. Cognola, M. Spreafico, S. Zerbini, *J.Math.Phys.* ('10).
- A. Mantile, A. Posilicano, M. Sini, *J.Diff.Eq.* ('16).
- D.F., *PhD thesis* ('16).

Motivations and basic ideas.

Casimir effect

- Generalized terminology: *physical phenomena related to the vacuum state of a quantum field confined by classical boundaries/potentials or living on curved/topologically non-trivial background spacetimes*.

  [Bordag, Dowker, Elizalde, Fulling, Kirsten, Milton, Moretti, Zerbini, ...
  ... Dappiaggi, Nosari, Pinamonti]

  - *No* self-interaction of the field.
  - *No* back-reaction effects.

  $\Rightarrow$ *Free* theory: effective interaction with a fixed classical background.

  Conceptual relevance: it involves the *renormalization of UV divergences*.

- Delta-type potentials are likely to give a more realistic mathematical description of the physical confinement (*semi-trasparent boundaries*).
  - Rigorous formulation in terms of self-adjoint extensions of symmetric operators $\rightarrow$ explicit constructions using *resolvent techniques*.
    [Albeverio, Birman, Dell'Antonio, Exner, Høegh-Krohn, Posilicano, Yafaev, ...]
  - Could lead to a *possible solution* of the issue of *boundary divergences*.

Zeta regularization

- Basic idea: to *give meaning via analytic continuation* to ill-defined expressions appearing in mathematics and physics.
  - To study regularity properties of pseudo-diff. operators and geometric invariants. [Minakshisundaram, Pleijel, Seeley, Ray, Singer]
  - To define renormalized (Vacuum) Expectation Values of local/global observables in QFT (effective action, total energy, stress-energy tensor).
    [Dowker, Critchley ('75); Hawking ('77); Zimerman *et al.* ('77); Wald ('79)]
    [Albeverio, Actor, Cognola, Elizalde, Kirsten, Moretti, Spreafico, Zerbini, ...]

- Constructive zeta approach in the framework of Wightman quantization:
  - introduce a zeta-reg. Wightman **scalar** field using *complex powers* of the Schrödinger-type operator which determines the Klein-Gordon eq.
    $\hookrightarrow$ *well-def. pointwise evaluation* of the field operator;
  - generate a zeta-reg. algebra of polynomial observables
    $\hookrightarrow$ define renormalized VEVs via *analytic continuation*;
  - use the *resolvent operator* to obtain the analytic continuation.

Abstract functional framework.

Basic elements

- $(\mathcal{H}, \langle\,|\,\rangle, \overline{}\,) \equiv \mathcal{H}$ = separable Hilbert space with conjugation.
- $\mathcal{A} : \mathrm{Dom}\mathcal{A} \subset \mathcal{H} \to \mathcal{H}$ *strictly* positive, self-adjoint, s.t. $\mathcal{A}\,\overline{f} = \overline{\mathcal{A}\,f}$.
- $\mathcal{A}^{-s}$ $(s \in \mathbb{C})$, $(\mathcal{A} - z)^{-n}$ $(z \in \rho(\mathcal{A}), n \in \mathbb{N})$, defined via *spectral theorem*.

◇ *In applications*:

- ⋄ $\mathcal{H}$ = *single-particle Hilbert space* in Fock space formulation;
- ⋄ $\overline{}$ = structure necessary to define a $\mathbb{C}$-*linear Wightman field*;
- ⋄ $\mathcal{A}$ = *Schrödinger-type operator giving rise to the Klein-Gordon eq.* ($\mathcal{A} > 0$ to avoid infrared divergences).

Functional spaces

- $\mathcal{H}^r :=$ completion of $\mathrm{Dom}\mathcal{A}^{r/2}$ w.r.t. $\langle g|f\rangle_r := \langle \mathcal{A}^{r/2} g|\mathcal{A}^{r/2} f\rangle$ $(r \in \mathbb{R})$.
  $\Rightarrow$ $\mathcal{H}^0 = \mathcal{H}$ and $\mathcal{H}^r \overset{dense}{\hookrightarrow} \mathcal{H}^u$ if $r \geqslant u$.
- $\mathcal{H}^{+\infty} := \bigcap_{r\in\mathbb{R}} \mathcal{H}^r$ with *Fréchet topology* induced by $\langle\,|\,\rangle_n$ $(n \in \mathbb{N})$.
- $\mathcal{H}^{-\infty} := \bigcup_{r\in\mathbb{R}} \mathcal{H}^r$ with *inductive limit topology*.

$\Rightarrow$ Infinite scale: $\mathcal{H}^{+\infty} \overset{dense}{\hookrightarrow} \mathcal{H}^r \overset{dense}{\hookrightarrow} \mathcal{H} \overset{dense}{\hookrightarrow} \mathcal{H}^{-r} \overset{dense}{\hookrightarrow} \mathcal{H}^{-\infty}$ $(r > 0)$.

Extended structures

- $\exists!\ \langle\ |\ \rangle : \bigcup_{r\in\mathbb{R}} \mathcal{H}^{-r}\times\mathcal{H}^{r} \to \mathbb{C}$ extens. of the inner product on $\mathcal{H}$ s.t. the restrictions $\langle\ |\ \rangle \restriction \mathcal{H}^{-r}\times\mathcal{H}^{r}$ are continuous, sesqui-lin. Hermitian forms.

  $\Rightarrow\ \mathcal{H}^{-r} \overset{isom}{\simeq} (\mathcal{H}^{r})' =$ topol. dual of $\mathcal{H}^{r}$ $(r\in\mathbb{R}\cup\{+\infty\})$.
- $\exists!\ \overline{} : \mathcal{H}^{-\infty} \to \mathcal{H}^{-\infty}$ $\Rightarrow$ $\overline{} : \mathcal{H}^{r} \to \mathcal{H}^{r}$ is a conjugation, $\langle \overline{g} | f \rangle = \overline{\langle g | \overline{f} \rangle}$.
- $\exists!\ \mathcal{A}^{-s},\ (\mathcal{A}-z)^{-\ell} : \mathcal{H}^{-\infty} \to \mathcal{H}^{-\infty}$ continuous extensions

  $\Rightarrow\ \mathcal{A}^{-s} : \mathcal{H}^{r} \to \mathcal{H}^{r+2\Re s}$ $(r\in\mathbb{R})$ is an *Hilbertian isomorphism*.

  $\Rightarrow\ \{\Re s > \sigma\} \to \mathfrak{B}(\mathcal{H}^{r}, \mathcal{H}^{r+2\sigma})$, $s \mapsto \mathcal{A}^{-s}$ is analytic.

Cauchy's integral representation

$$\mathcal{A}^{-s} f = -\frac{(n-1)!}{2\pi i \prod_{\ell=1}^{n-1}(s-\ell)} \int_{\mathfrak{H}} dz\ z^{-s+n-1} (\mathcal{A}-z)^{-n} f\ .$$

- Gelfand-Pettis integral, holding in $\mathcal{H}^{u}$ for $r, u \in \mathbb{R}$, $\Re s, n > \frac{u-r}{2}$, $f \in \mathcal{H}^{r}$.
- Descending from Cauchy's formula:
  - z^{-s} with the determination $\arg z \in \mathbb{C} \setminus (-\infty, 0]$;
  - $\mathfrak{H} =$ Hankel contour around $\sigma(\mathcal{A})$ $(\sigma(\mathcal{A}) \subset [\varepsilon, +\infty),\ \varepsilon > 0)$;
  - regular at $s = 1, \ldots, n-1$ (no poles).

Integral kernels and their analytic continuation.

The case of Schrödinger-type operators

- $\mathcal{H} = L^2(\Omega)$, with $\Omega \subset \mathbb{R}^d$ *any* open domain (or Riemannian manifold).
- $\mathcal{A} = (-\triangle + V) \restriction \mathcal{D}_{\mathcal{A}}$, with $\triangle =$ Laplacian, $V \in C^\infty(\Omega)$,
  $\mathcal{D}_{\mathcal{A}} \subset L^2(\Omega) =$ *domain of self-adjointness* $(V, \mathcal{D}_{\mathcal{A}}$ s.t. $\mathcal{A} > 0)$.

$\Rightarrow$ $\mathcal{H}^r \hookrightarrow H^r_{loc}(\Omega) \hookrightarrow C^j(\Omega)$ for $j \in \mathbb{N}, r \in \mathbb{R}$ with $r > j + d/2$.

Integral kernels and regularity (*Note*: $\langle \; | \; \rangle =$ extens. of inner prod. on $\mathcal{H}$.)

- $\exists!$ *Dirac delta* $\delta_{\mathbf{x}} \in \mathcal{H}^{-\infty}$ $(\mathbf{x} \in \Omega)$ s.t. $\langle \delta_{\mathbf{x}} | f \rangle = f(\mathbf{x})$ $(f \in \mathcal{H}^r, r > d/2)$.
- $\mathcal{A}^{-s}(\mathbf{x}, \mathbf{y}) := \langle \delta_{\mathbf{x}} | \mathcal{A}^{-s} \delta_{\mathbf{y}} \rangle =$ Dirichlet ker., well-def. for $\Re s > d/2$.
  $\Rightarrow$ $s \mapsto \mathcal{A}^{-s}(\mathbf{x}, \mathbf{y})$ is analytic, for any $\mathbf{x}, \mathbf{y} \in \Omega$.
  $\Rightarrow$ $\mathcal{A}^{-s}(\;,\;) \in C^j(\Omega \times \Omega)$ if $\Re s > (j + d)/2$.
  $\hookrightarrow$ It relates to *propagator's diagonal evaluation*.
- $(\mathcal{A} - z)^{-n}(\mathbf{x}, \mathbf{y}) := \langle \delta_{\mathbf{x}} | (\mathcal{A} - z)^{-n} \delta_{\mathbf{y}} \rangle =$ resolvent ker., well-def. for $n > d/2$.
  $\Rightarrow$ $\rho(\mathcal{A}) \ni z \mapsto (\mathcal{A} - z)^{-n}(\mathbf{x}, \mathbf{y})$ is analytic, for any $\mathbf{x}, \mathbf{y} \in \Omega$.
  $\Rightarrow$ $(\mathcal{A} - z)^{-n}(\;,\;) \in C^j(\Omega \times \Omega)$ if $n > (j + d)/2$.

The spectral kernel

- The abstract Cauchy's representation implies, for $\Re s, n > d/2$,

$$\mathcal{A}^{-s}(\mathbf{x},\mathbf{y}) = -\frac{(n-1)!}{2\pi i \prod_{\ell=1}^{n-1}(s-\ell)} \int_{\mathfrak{H}} dz \; z^{-s+n-1} (\mathcal{A}-z)^{-n}(\mathbf{x},\mathbf{y}) \,.$$

  - It follows from the def. of Gelfand-Pettis int. and integral kernels.
  - Valid for *any* $\mathbf{x},\mathbf{y} \in \Omega$: *also for* $\mathbf{y} = \mathbf{x} \in \Omega$.

- Introduce the *spectral kernel of order n*

$$E^n(\lambda;\mathbf{x},\mathbf{y}) := \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \Big[(\mathcal{A}-\lambda+i\varepsilon)^{-n}(\mathbf{x},\mathbf{y}) - (\mathcal{A}-\lambda-i\varepsilon)^{-n}(\mathbf{x},\mathbf{y}) \Big] \,.$$

  - A "*limiting absorption principle*" type computation
    $\hookrightarrow$ evaluation of the jump discontinuity of the resolvent on $\sigma(\mathcal{A})$.
  - $z \mapsto (\mathcal{A}-z)^{-n}(\mathbf{x},\mathbf{y})$ analytic on $\rho(\mathcal{A})$ $\Rightarrow$ $\operatorname{supp} E^n(\;;\mathbf{x},\mathbf{y}) \subset \sigma(\mathcal{A})$.
  - For $\Re s, n > d/2$, the Cauchy's repres. gives $(\lambda_0 := \inf \sigma(\mathcal{A}) > 0)$

$$\Rightarrow \quad \mathcal{A}^{-s}(\mathbf{x},\mathbf{y}) = \frac{(n-1)!}{\prod_{\ell=1}^{n-1}(s-\ell)} \int_{\lambda_0}^{+\infty} d\lambda \; \lambda^{-s+n-1} \, E^n(\lambda;\mathbf{x},\mathbf{y}) \,.$$

Analytic continuation of the diagonal Dirichlet kernel

- *Assume* there holds the large λ asymptotic expansion, for some $K \in \mathbb{N}$,

$$E^n(\lambda; \mathbf{x}, \mathbf{x}) = \lambda^{d/2-n} \sum_{k=1}^{K-1} e_k(\mathbf{x})\, \lambda^{-k/2} + R_K(\lambda; \mathbf{x}) \,, \quad \text{where}$$

  - $\mathbf{x} \in \Omega$ is fixed (diagonal case of interest for Casimir-type applications);
  - $e_1, \dots, e_{K-1} : \Omega \to \mathbb{R}$ (continuous functions);
  - $R_K(\lambda; \mathbf{x}) = O(\lambda^{-K/2})$ for $\lambda \to +\infty$ (remainder function).

  ◇ Often fulfilled in the case of elliptic operators [Agmon, Hörmander, ...]
  $\hookrightarrow$ more general cases handled similarly.

- *Then*, Cauchy's representation can be re-expressed as

$$\mathcal{A}^{-s}(\mathbf{x}, \mathbf{x}) = \frac{(n-1)!}{\prod_{\ell=1}^{n-1}(s-\ell)} \left[\sum_{k=1}^{K-1} \frac{\lambda_0^{-s+(d-k)/2} e_k(\mathbf{x})}{s-(d-k)/2} + \int_{\lambda_0}^{+\infty} d\lambda \;\, \lambda^{-s+n-1} R_K(\lambda; \mathbf{x}) \right].$$

  - Derived for $\Re s > d/2$ $\to$ makes sense for $\Re s > d/2 - K/2$.
  $\hookrightarrow$ analytic continuation to a *meromorphic function*
  with possible simple poles at $s = (d-k)/2$ $(k \in \{1, ..., K-1\})$.

Zeta-regularized Wightman scalar field.

Fock space quantization

- $\mathfrak{F}^{\vee}(\mathcal{H}) := \bigoplus_{n=0}^{+\infty} \mathcal{H}^{\vee n} =$ bosonic Fock space ($\vee :=$ symm. tens. prod.).
  - $\mathfrak{F}_0^{\vee}(\mathcal{H}) :=$ finite particle subspace ($\mathbf{f} \in \mathfrak{F}_0^{\vee}(\mathcal{H})$).
  - $\mathbf{v} := 1 \in \mathbb{C} \equiv \mathcal{H}^{\vee 0}$ vacuum state.
- Time zero Wightman field

$$\hat{\varphi}(h) := \frac{1}{\sqrt{2}} \left(\hat{a}^{-}(\overline{\mathcal{A}^{-1/4} h}) + \hat{a}^{+}(\mathcal{A}^{-1/4} h) \right).$$

  - $\hat{a}^{\pm}(h) =$ creation/annihilation operators on $\mathfrak{F}_0^{\vee}(\mathcal{H})$, def. for $h \in \mathcal{H}$ $([\hat{a}^{\pm}(h), \hat{a}^{\pm}(k)] \subset \mathbb{O},\ [\hat{a}^{-}(h), \hat{a}^{+}(k)] \subset \langle h|k \rangle\, \mathbb{I};\ \hat{a}^{-}(h)\,\mathbf{v} := \mathbf{0})$.
    $\Rightarrow$ $\hat{\varphi}(h)$ well-defined for test-elements $h \in \mathcal{H}^{-1/2}$.
  - $\overline{} =$ conjugation s.t. $\mathcal{A}\overline{f} = \overline{\mathcal{A} f}$ $\Rightarrow$ $h \mapsto \hat{\varphi}(h)$ $\mathbb{C}$-linear.
- Time evolution via *second quantization* of $e^{-it\sqrt{\mathcal{A}}}$ ($t \in \mathbb{R} =$ time):

$$\hat{\varphi}_t(h) := \Gamma(e^{it\sqrt{\mathcal{A}}})\, \hat{\varphi}(h)\, \Gamma(e^{-it\sqrt{\mathcal{A}}}) \qquad (h \in \mathcal{H}^{-1/2}).$$

  - $\mathbb{R} \to \mathfrak{F}_0^{\vee}(\mathcal{H})$, $t \mapsto \hat{\varphi}_t(h)\,\mathbf{f}$ is of class C^n, for $h \in \mathcal{H}^{-\frac{1}{2}+n}$.
  - Strong form of the Klein-Gordon eq., for $h \in \mathcal{H}^{3/2}$ ($c = \hbar = 1$):

$$\left[\partial_{tt} \hat{\varphi}_t(h) + \hat{\varphi}_t(\mathcal{A}\, h) \right] \mathbf{f} = \mathbf{0}.$$

Zeta-regularized field

- Of interest for applications: pointwise evaluation "$\hat{\varphi}(\mathbf{x}) := \hat{\varphi}(\delta_{\mathbf{x}})$"
  $\hookrightarrow$ *ill-defined* since $\delta_{\mathbf{x}} \notin \mathcal{H}^{-1/2}$ $(\delta_{\mathbf{x}} \in \mathcal{H}^{-r},\ r > d/2)$.

- Consider the zeta-regularized Dirac delta $(\mathcal{A}^{-s}\mathcal{H}^{-r} \overset{isom}{\simeq} \mathcal{H}^{2\Re s - r})$

$$\delta_{\mathbf{x}}^{u} := (\mathcal{A}/\kappa^2)^{-u/4}\,\delta_{\mathbf{x}} \in \mathcal{H}^{-\rho} \quad \text{for } \Re u > d - 2\rho\,,$$
$$\kappa \in \mathbb{R} = \text{renormalization mass parameter}\,.$$

  Define the zeta-regularized field at $x \equiv (t, \mathbf{x}) \in \mathbb{R} \times \Omega$

$$\hat{\varphi}^{u}(x) := \hat{\varphi}_t(\delta_{\mathbf{x}}^{u}) \qquad \text{for } \Re u > d - 1\,.$$

  - $u = 0 \leftrightarrow$ non-regularized theory.
  - Generate an algebra of zeta-reg. polynomial observables on $\mathfrak{F}_0^{\vee}(\mathcal{H})$.

- The zeta-reg. two-point function is ($\mathbf{v}$ = vacuum state)

$$\langle\, \mathbf{v} \,|\, \hat{\varphi}^{u}(x)\,\hat{\varphi}^{u}(y)\,\mathbf{v}\,\rangle = \frac{\kappa^u}{2}\,\langle\, \delta_{\mathbf{x}} \,|\, \big(e^{-i\sqrt{\mathcal{A}}\,(t-t')}\mathcal{A}^{-\frac{u+1}{2}}\big)\,\delta_{\mathbf{y}}\,\rangle\,.$$

  - It determines any polynomial obs. VEV (free theory $\rightarrow$ Wick's thm).
  - Differentiable function of $x, y \in \mathbb{R} \times \Omega$: of class C^n for $\Re u > 2n + d - 1$
    $\Rightarrow$ *evaluation along the diagonal $y = x$ makes sense*.

Renormalization by analytic continuation (a case study)

- The zeta-reg. vacuum polarization is given by

$$\langle\, \mathbf{v} \,|\, \hat{\varphi}^u(t,\mathbf{x})^2\, \mathbf{v}\,\rangle \;=\; \frac{\kappa^u}{2}\, \mathcal{A}^{-\frac{u+1}{2}}(\mathbf{x},\mathbf{x})\,.$$

  - *No normal ordering* to make connection with the *Casimir effect*.
  - $u \mapsto \langle \mathbf{v}|\hat{\varphi}^u(t,\mathbf{x})^2 \mathbf{v}\rangle \equiv \langle \mathbf{v}|\hat{\varphi}^u(\mathbf{x})^2 \mathbf{v}\rangle$ is analytic for $\Re u > d-1$ (fixed $\mathbf{x}\in\Omega$).

- Express $\mathcal{A}^{-\frac{u+1}{2}}(\mathbf{x},\mathbf{x})$ in terms of $E^n(\lambda;\mathbf{x},\mathbf{x})$ via *Cauchy's identity* and use asymptotics of the spectral function $E^{(n)}(\lambda;\mathbf{x},\mathbf{x})$ for $\lambda\to+\infty$.

  $\Rightarrow$ Explicit analytic extension of $u \mapsto \langle \mathbf{v}|\hat{\varphi}^u(\mathbf{x})^2 \mathbf{v}\rangle$ (fixed $\mathbf{x}\in\Omega$), *meromorphic* near $u=0$ (*possible pole singularity*).

- The renormalized vacuum polarization is ($\mathbf{x}\in\Omega$)

$$\langle \mathbf{v}|\hat{\varphi}(\mathbf{x})^2\, \mathbf{v}\rangle_{ren} \;:=\; RP\Big|_{u=0} \langle \mathbf{v}|\hat{\varphi}^u(\mathbf{x})^2\, \mathbf{v}\rangle\;,$$

$$RP\Big|_{u=0} \;:=\; \textit{regular part} \text{ of the Laurent expansion at } u=0\,.$$

  - *No pole at* $u=0$ $\Rightarrow$ finite result *independent of* κ, *without subtraction of divergent expressions*.
  - *Pole at* $u=0$ $\Rightarrow$ *minimal subtraction scheme* [Wald, Blau, Visser, Wipf, ...] $\leftrightarrow$ addition of local counter-terms (*explicit dependence on* κ).

Delta-type potentials.

Preliminary remarks

- Study Casimir-type configurations with "semi-trasparent boundaries": likely, a more realistic mathematical description of field confinement.
  [Mamaev, Trunov ('81), Bordag *et al.* ('92), Milton ('04), ... : *local obs., 1D models*]
  [Spreafico, Zerbini ('09), Albeverio *et al.* ('10-'15), ... : *global obs., 3D models*]

- Heuristic formulation:

$$\mathcal{A} \text{ ``=''} -\triangle_m + \alpha\,\delta_\Gamma \quad \text{on} \quad \mathcal{H} = L^2(\mathbb{R}^d)\,,$$

  - $-\triangle_m := -\triangle + m^2$: distributional Laplacian ($m > 0$ to avoid IR diverg.);
  - $\alpha \in \mathbb{R}$: coupling constant ($\sim$ inverse scattering length in QM);
  - δ_Γ : Dirac delta supported on a smooth hypersurface $\Gamma \subset \mathbb{R}^d$.
    - ◇ *Note: δ_Γ not a regular perturbation of $-\triangle_m$* ($\nexists$ Dom$\mathcal{A} \subset \mathcal{H}$).
      (δ_Γ as "limit" of sharply peaked potentials $\rightarrow$ nontrivial subject.)

- Rigorous results:
  - $\mathcal{A}$ = self-adjoint extension of the symmetric op. $(-\triangle_m) \restriction H^2_0(\mathbb{R}^d \setminus \Gamma)$.
  - co-dim.$(\Gamma) \leqslant 3$ ($(-\triangle_m) \restriction H^2_0(\mathbb{R}^d \setminus \Gamma)$ ess. self-adj. for co-dim. > 3).
  - Explicit Krein-type relations for resolvent operators.

A Krein-type approach

- Von Neumann theory: $-\triangle_m > 0 \;\Rightarrow\; \exists$ self-adj. extensions
  $\hookrightarrow$ non-trivial construction if deficiency subspaces are $\infty-$dim.
  $\Rightarrow$ Equiv. characterization using a Krein-type approach [Posilicano].
  Case study: co-dim.$(\Gamma) = 1$ (generalization to co-dim.$(\Gamma) = 2, 3$).

- Basic elements:
  - $\mathcal{A}_0 := (-\triangle_m) \restriction H^2(\mathbb{R}^d)$, free Laplacian (ess. self-adj. on $L^2(\mathbb{R}^d)$);
  - $\gamma : H^2(\mathbb{R}^d) \to H^{3/2}(\Gamma)$, trace on Γ (Γ regular enough, $\gamma f = f|_\Gamma$);
  - $\mathcal{S} := \mathcal{A}_0 \restriction H^2_0(\mathbb{R}^d \setminus \Gamma)$, closed, densely defined, symmetric operator.

  $\hookrightarrow$ Self-adj. extensions of $\mathcal{S}$ as restrictions of the adjoint $\mathcal{S}^\dagger$.
  $(\mathrm{Dom}\mathcal{S}^\dagger = \{f \in L^2(\mathbb{R}^d \setminus \Gamma) \,|\, \triangle_m f \in L^2(\mathbb{R}^d \setminus \Gamma)\} = H^2(\mathbb{R}^d \setminus \Gamma)\,.)$

- Single layer operator $(z \in \rho(\mathcal{A}_0) = \mathbb{C} \setminus [m^2, +\infty))$:

$$G_z := (\gamma(\mathcal{A}_0 - \bar{z})^{-1})^\dagger : H^{-3/2}(\Gamma) \to L^2(\mathbb{R}^d)$$

  - $(\mathcal{A}_0 - z) G_z$ "=" δ_Γ (indeed, by def., $\langle (\mathcal{A}_0 - z) G_z g, f \rangle_{\mathbb{R}^d} = \langle g, \gamma f \rangle_\Gamma$);
  - Related operators: $(G_{\bar{z}})^\dagger : L^2(\mathbb{R}^d) \to H^{3/2}(\Gamma)$,
    $\gamma G_z : \mathrm{Dom}(\gamma G_z) \subset L^2(\Gamma) \to L^2(\Gamma)$.

  $\hookrightarrow$ *All explicitly determined in terms of* $(\mathcal{A}_0 - z)^{-1}$ *and* γ.

Delta potentials as self-adjoint extensions

- The self-adj. extensions of $\mathcal{S}$ corresponding to delta-type potentials are:

$$\mathcal{A} := \mathcal{S}^\dagger \restriction \left\{ u + G_{\lambda_0} q \;\middle|\; u \in H^2(\mathbb{R}^d),\, q \in \mathrm{Dom}(\gamma G_{\lambda_0}),\; \gamma(u + G_{\lambda_0} q) = -\alpha^{-1} q \right\},$$

$$\mathcal{A}(u + G_{\lambda_0} q) = \mathcal{A}_0 u + \lambda_0 G_{\lambda_0} q \,.$$

  - $\lambda_0 \in \rho(\mathcal{A}_0)$ fixed arbitrarily $\to$ def. of $\mathcal{A}$ independent of λ_0.
  - Heuristic computation $\Rightarrow$ $\mathcal{A}$ "$=$" $\mathcal{A}_0 + \alpha \langle \delta_\Gamma , \ \rangle \delta_\Gamma$ on $\mathrm{Dom}\mathcal{A}$.
  - The parameter $\alpha \in \mathbb{R}$: $\alpha < 0$ $\to$ delta-well ("attractive potential");
    $\alpha = 0$ $\to$ $\mathcal{A} = \mathcal{A}_0$ (free Laplacian);
    $\alpha = +\infty$ $\to$ Dirichlet b.c. on Γ.

- The resolvent of $\mathcal{A}$ is given by the Krein formula

$$(\mathcal{A} - z)^{-1} = (\mathcal{A}_0 - z)^{-1} - G_z (\alpha^{-1} + \gamma G_z)^{-1} (G_{\bar{z}})^\dagger \,.$$

  - $z \in \rho(\mathcal{A}) \supset \{ z \in \rho(\mathcal{A}_0) \text{ s.t. } (\alpha^{-1} + \gamma G_z)^{-1} \text{ is bounded} \}$.
  - $\sigma_{a.c.}(\mathcal{A}) = \sigma_{ess}(\mathcal{A}) = \sigma(\mathcal{A}_0) = [m^2, +\infty)$.
  - $\sigma_p(\mathcal{A}) = \{ \lambda \in \mathbb{R} \setminus \sigma(\mathcal{A}_0) \text{ s.t. } 0 \in \sigma_p(\alpha^{-1} + \gamma G_\lambda) \}$ $\to$ possible *resonances*.

  $\Rightarrow$ $\mathcal{A} > 0$ *not granted a priori* (can be enforced with m large enough).

An example: delta-potential supported on a plane in $\mathbb{R}^3$.

Description of the model

- Case study: computation of $\langle \mathbf{v} | \hat{\varphi}(\mathbf{x})^2 \mathbf{v} \rangle_{ren}$ for a massive scalar field on $\mathbb{R}^3$ with background delta potential on the plane $\pi = \{x^1 = 0\}$.

  [Mamaev, Trunov ('81), Bordag *et al.* ('92), Milton ('04), Khusnutdinov ('06), ...]

- Field theory determined by $\mathcal{A} = \mathcal{A}_1 \otimes \mathbb{I}_2 \dot{+} \mathbb{I}_1 \otimes \mathcal{A}_2$ on $L^2(\mathbb{R}) \otimes L^2(\mathbb{R}^2)$:
  - $\mathcal{A}_1$"$=$"$\mathcal{A}_0 + \alpha \langle \delta_0, \ \rangle \delta_0$ ($\mathcal{A}_0 = (-\partial_{11} + m^2) \restriction H^2(\mathbb{R}), \alpha \in \mathbb{R}, \delta_0 =$ delta at $x^1 = 0$)
    $\hookrightarrow$ rigorous def. of $\mathcal{A}_1$ as self-adj. extension of $\mathcal{A}_0 \restriction H^2_0(\mathbb{R} \setminus \{0\})$;
  - $\mathcal{A}_2 := (-\partial_{22} - \partial_{33}) \restriction H^2(\mathbb{R}^2)$ $\to$ free theory on $\mathbb{R}^2 \equiv \mathbb{R}^3/\pi$.

- General identities for Dirichlet kernels [D.F., L. Pizzocchero ('14)]:

$$\mathcal{A}^{-s}(\mathbf{x}, \mathbf{x}) = \frac{\Gamma(s-1)}{4\pi\, \Gamma(s)}\, \mathcal{A}_1^{-(s-1)}(x^1, x^1) \quad (\mathbf{x} \in \mathbb{R}^3, x^1 \in \mathbb{R}, \Re s > 3/2)$$

$$\Rightarrow \quad \langle \mathbf{v} \,|\, \hat{\varphi}^u(x^1)^2\, \mathbf{v} \rangle = \frac{\kappa^u\, \Gamma(\frac{u-1}{2})}{8\pi\, \Gamma(\frac{u+1}{2})}\, \mathcal{A}_1^{-\frac{u-1}{2}}(x^1, x^1) \qquad (\Re u > 2)\,.$$

$\Rightarrow$ *It suffices to consider the 1D problem determined by $\mathcal{A}_1$ on $L^2(\mathbb{R})$.*

The reduced $1D$ problem

- Let $\mathcal{S} := \mathcal{A}_0 \restriction H_0^2(\mathbb{R}\setminus\{0\})$ and consider $\left(G_z = \int_{\mathbb{R}} \frac{dk}{\sqrt{2\pi}} \frac{|k\rangle}{k^2+m^2-z},\, z\in\mathbb{C}\setminus[m^2,+\infty)\right)$

$$\mathcal{A}_1 := \mathcal{S}^\dagger \restriction \left\{u+G_0 q \,\middle|\, u\in H^2(\mathbb{R}),\, q\in\mathbb{C},\, u(0)+\frac{q}{2m} = -\frac{q}{\alpha}\right\}, \quad \mathcal{A}_1(u+G_0 q) = \mathcal{A}_0 u\,.$$

  - $\sigma_{a.c.}(\mathcal{A}_1) = [m^2,+\infty)$,
    $$\sigma_p(\mathcal{A}_1) = \begin{cases} \emptyset & \text{if } \alpha>0 \\ \left\{m^2 - \left(\frac{\alpha}{2}\right)^2\right\} & \text{if } \alpha<0 \end{cases} \quad \Rightarrow \quad \begin{array}{c} \mathcal{A}_1 > 0 \text{ iff } (\alpha>0 \wedge m>0) \\ \text{or } (\alpha<0 \wedge m>|\alpha|/2)\,. \end{array}$$

- $d=1 \Rightarrow$ consider the diagonal resolvent kernel $(z\in\rho(\mathcal{A}_1),\, x^1\in\mathbb{R})$:

$$(\mathcal{A}_1 - z)^{-1}(x^1,x^1) = \frac{i}{2\sqrt{z-m^2}} + \frac{\alpha\, e^{2i|x^1|\sqrt{z-m^2}}}{2\sqrt{z-m^2}\,(2\sqrt{z-m^2}+i\alpha)}\,.$$

- The spectral kernel is determined by the jump discontinuities of $\sqrt{z-m^2}$:

$$E^1(\lambda;x^1,x^1) = e_0(\lambda) + e_p(\lambda;x^1) + e_{a.c.}(\lambda;x^1)\,,$$

$$e_0(\lambda) := \frac{\chi_{(m^2,+\infty)}(\lambda)}{2\pi\sqrt{\lambda-m^2}}\,, \quad e_p(\lambda;x^1) := \frac{|\alpha|-\alpha}{4}\, e^{-|\alpha x^1|}\, \delta\left(\lambda - m^2 + \left(\frac{\alpha}{2}\right)^2\right),$$

$$e_{a.c.}(\lambda;x^1) := \frac{2\alpha\sqrt{\lambda-m^2}\sin(2|x^1|\sqrt{\lambda-m^2}) - \alpha^2\cos(2|x^1|\sqrt{\lambda-m^2})}{8\pi\sqrt{\lambda-m^2}\left(\lambda-m^2+\left(\frac{\alpha}{2}\right)^2\right)}\,\chi_{(m^2,+\infty)}(\lambda)\,.$$

$\hookrightarrow$ Use their large λ asymptotics to anaytically continue $s \mapsto \mathcal{A}_1^{-s}(x^1,x^1)$.

The renormalized vacuum polarization

- Previous results $\Rightarrow$ by analytic continuation at $u = 0$ one gets

$$\langle \mathbf{v} | \hat{\varphi}(x^1)^2 \mathbf{v} \rangle_{ren} = F_0 + F_p(x^1) + F_{a.c.}(x^1) \,,$$

$$F_0 := \frac{m^2}{8\pi^2}\left[\ln\left(\frac{m}{2\kappa}\right) - 1\right], \qquad F_p(x^1) := \frac{|\alpha| - \alpha}{16\pi}\sqrt{m^2 - \frac{\alpha^2}{4}}\, e^{-|\alpha x^1|} \,,$$

$$F_{a.c.}(x^1) := \frac{\alpha^2}{16\pi^2}\left[\sinh(y)\,\mathcal{I}_S(y) - \cosh(y)\,\mathcal{I}_C(y) - \frac{2m}{\alpha}\,K_1(y)\right]_{y\,=\,2m|x^1|} +$$

$$- \int_{m^2}^{+\infty} d\lambda\, \frac{\sqrt{\lambda}}{4\pi}\left[e_{a.c.}(\lambda; x^1) - \frac{\alpha \sin(2|x^1|\sqrt{\lambda - m^2})}{4\pi\lambda} + \frac{\alpha^2 \cos(2|x^1|\sqrt{\lambda - m^2})}{8\pi\lambda^{3/2}}\right]$$

$$\left(\mathcal{I}_C(y) := \gamma_{EM} + \log y + \int_0^y dw\, \frac{\cosh w - 1}{w} \,, \quad \mathcal{I}_S(y) := \int_0^y dw\, \frac{\sinh w}{w}\right).$$

- F_0 = free massive theory contribution (by *subtraction of pole singularity*); $F_{p/a.c.}$ = point/continuous spectrum contribution (by *pure an. cont.*). ($F_{a.c.}$ by explicit integration of first terms in asymptotic expansion of $e_{a.c.}$.)

- Asymptotics:
  - $\langle \mathbf{v} | \hat{\varphi}(x^1)^2 \mathbf{v} \rangle_{ren} = -\frac{\alpha}{16\pi^2|x^1|} + O(\log(m|x^1|))$ for $x^1 \to 0^{\pm}$;
  - $\langle \mathbf{v} | \hat{\varphi}(x^1)^2 \mathbf{v} \rangle_{ren} = F_0 + O(|x^1|^{-\infty})$ for $x^1 \to \pm\infty$.

Summary and outlook.

Summary:

- functional analytic framework;
- constructive ZR approach in the framework of Wightman quantization;
- Casimir effect for delta-type background potentials.

Further developments:

- explicit analysis of other configurations (e.g., point-interaction in $\mathbb{R}^3$);
- investigation of boundary divergences in relation with singular potentials.

Thanks a lot for your attention!